Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **9** (2)(2020), 243-255

High Order Quadrature Based Iterative Method for Approximating the Solution of Nonlinear Equations

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Abstract. In this paper, weight function and composition techniques are utilized to speeds up the convergence order and increase the efficiency of an existing quadrature based iterative method. This results in the proposition of its improved form from a two-point quadrature based method of convergence order $\rho = 3$ with efficiency index $EI = 1.3161$ to a three-point method of convergence order $\rho = 8$ with $EI = 1.5157$ at the cost of one additional function evaluation. The method is used to approximate the solution of some nonlinear equations and the results generated are compared with that of some existing methods. Numerical results shows that method developed here in is very efficient in approximation of solution of nonlinear equations.

Keywords: Nonlinear equation; Quadrature formula; Iterative method; weight function.

2010 Mathematics subject classification: 34K28; 74H15.

1. Introduction

Nonlinear equations are used in describing real life phenomena. For example problems in population growth, electric circuit, chemical engineering, economics are modeled in nonlinear equations.

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Received: 17 July 2020
Revised: 9 September 2020
Accepted: 9 September 2020

Consider the nonlinear equation,

$$g(x) = 0 \tag{1.1}$$

where $x \in R$, $g : D \subset R \to R$ is functional.
Over the years, several multi-point quadrature based iterative methods for the approximation of the solution α of (1.1) have been proposed in literature, amongst these are methods developed in [1-6] and some references there in. Most of these iterative methods are considered computationally expensive because they involve the evaluation of high number of distinct functions and its derivatives at each step of iteration. Due to these high computation cost, quadrature based iterative methods for approximating the solution of (1.1) are less attractive in practical application. An important rule in developing iterative methods is to develop iterative methods that can attain high convergence order by utilizing the evaluation of few numbers of distinct functions $g(\cdot)$ and derivative of functions $g'(\cdot)$ per iteration. Consequently, in this paper, we accelerate the convergence order and improve the computational efficiency of a two step quadrature based iterative method for solving (1.1) developed in [5] . The basic idea used in developing the iterative method is the composition and weight function technique. The technique involves the composition of the Newton method with weighted quadrature based iterative function as the compound predictor function and a corrector function with an economic weight function.
The rest part of this paper is arranged in the following format. Section 2 of this paper, presents the method development. In Section 3, the convergence of the method is established, while in Section 4, the numerical implementation of the method is presented. Finally, the conclusion and suggestions areas for further research targeted at improving the developed method are presented in Section 5.

2. Method Development

Suppose $x_{k+1} = \psi(x_k)$ is an iterative function.

Definition 1. The solution of a nonlinear equation (1.1) is a scalar α such that

$$g(\alpha) = 0 \tag{2.1}$$

Definition 2. [1] Let $e_k = |x_k - \alpha|$ be the error in the kth iteration of an iterative method $\psi(x_k)$, then the equation

$$e_{k+1} = N e_k^{\rho} + O\left(e_k^{\rho+1}\right) \tag{2.2}$$

is called the error equation of $\psi(x_k)$, where N is constant and the value ρ is its order of convergence.

Definition 3. [7] The efficiency of an iterative method $\psi(x_k)$ is measured by efficiency index (EI) defined by

$$EI = \rho^{\frac{1}{T}} \tag{2.3}$$

where T is the total number of distinct functional evaluations per iteration.

Definition 4. [8] An iterative function $\psi\left(x_k\right)$ such that

$$\psi\left(x_k\right)=\psi^{(1)}\left(\psi^{(2)}\left(\left(\cdots\left(\psi^{(r)}\left(x_k\right)\right)\cdots\right)\right)\right.$$

is the composition of the iterative functions

$$\psi^{(1)}\left(x_k\right), \psi^{(2)}\left(x_k\right), \cdots, \psi^{(r)}\left(x_k\right).$$

Let $\psi_\rho\left(x_k\right)$ be an iterative method of convergence order ρ. Consider the family of two point quadrature based iterative method for approximating the solution α of (1.1) developed in [5], given as:

$$\begin{aligned}\psi_{3^{rd}NR}(x_k) =&\psi_{2^{nd}NM}(x_k) - \frac{g\left(\psi_{2^{nd}NM}(x_k)\right)}{\sum_{k=1}^{q}\mu_i g'\left(x_k+\theta_i\left(\psi_{2^{nd}NM}(x_k)-x_k\right)\right)},\\ &k=1,2,\cdots\end{aligned} \tag{2.4}$$

where $\psi_{2^{nd}NM}(x_k) = x_k - \frac{g(x_k)}{g'(x_k)}$ is the classical Newton method of convergence order $\rho = 2$, θ_i and $\mu_i, i = 1, 2, \cdots, q$ are knots and weights respectively such that

$$\sum_{i=1}^{q}\mu_i=1 \tag{2.5}$$

and

$$\sum_{i=1}^{q}\mu_i\theta_i=\frac{1}{2} \tag{2.6}$$

The equations in (2.5) and (2.6) are consistency conditions [9]. It is proven that the family of iterative method in (2.4) is of convergence order $\rho = 3$ [5]. For some μ_i and θ_i satisfying the conditions in equations (2.5) and (2.6), where $\theta_i \neq 0, i = 1, 2, \cdots$, methods developed from the family of iterative methods in equation (2.4) requires the evaluation of two functions $g(\cdot)$ and many number of function derivatives $g'(\cdot)$ per iteration step as q (the quadrature points) becomes large. This is setback because evaluation of functions and functions derivatives are computationally expensive. By Definition 3, the most efficient member of the family of iterative method in (2.4) is the method obtained that will require minimum number function evaluation per iteration. One of such member of (2.4) is the one obtained when $q = \mu = 1$. By setting $q = \mu = 1$, and $\theta = \frac{1}{2}$, a two point special member of the method in equation (2.4) is obtained in [5] as:

$$\psi_{3^{rd}NR}(x_k) = \psi_{2^{nd}NM}(x_k) - \frac{g\left(\psi_{2^{nd}NM}(x_k)\right)}{g'\left(\frac{x_k + \psi_{2^{nd}NM}(x_k)}{2}\right)}, k = 1, 2, \cdots \tag{2.7}$$

The method (2.7) requires four functions evaluation per iteration to achieve order three convergence. By Definition 3, its efficiency index (EI) is 1.3161. The question is; can the method in equation (2.7) achieve convergence order four without additional function evaluation? To answer this question, a slight modification on the method $\psi_{3^{rd}NR}(x_k)$ in equation (2.7) is made by proposing a two-point fourth order iterative method $\psi_{4^{th}OM}(x_k)$ in [10] given as:

$$\psi_{4^{th}OM}(x_k) = \psi_{2^{nd}NM}(x_k) - \frac{g\left(\psi_{2^{nd}NM}(x_k)\right)}{2g'\left(\frac{x_k + \psi_{2^{nd}NM}(x_k)}{2}\right) - g'\left(x_k\right)}, k = 1, 2, \cdots \tag{2.8}$$

It is important to note that the order of convergence of the method in equation (2.8) is increased by one unit without additional function evaluation. However, we can achieve higher order of convergence at the cost of one additional function evaluation. This is made possible by the introduction of additional iteration point and using economic weight function. Recently, the order of convergence of many existing iterative methods have been improved using weight function (see [11-16]). In fact Babajee in [16] developed a technique for improving the order of convergence of old iterative methods. The technique is stated in Theorem 1.

Theorem 1. [16] *Assume $f : D \subset R \to R$ is sufficiently differentiable in D (an open interval). If $\psi_{old}(x)$ is an iterative function of order ρ, then the iterative function defined as $\psi_{new}(x) = \psi_{old}(x) - G(\psi_{old}(x))$ is of convergence order $\rho + q$ so long the weight function G satisfies the error equation*

$$G = \tfrac{1}{g'(\alpha)}(1 + C_G e^q + O(e^{q+1})),$$

where C_G is a constant.
Further, suppose the error equation of the old iterative function is given by

$$e_{old} = \psi_{old}(x) - \alpha = C_{old}e^{\rho} + \cdots,$$

then, the error equation of the new iterative function is given by

$$e_{new} = \psi_{new}(x) - \alpha = -C_{old}e^{\rho+q} - c_2 {C_{old}}^2 e^{2\rho} + \cdots,$$

where $c_k = \frac{g^{(k)}(\alpha)}{k!g'(\alpha)}, k = 1, 2, 3, \cdots$

Consequently, we improve the convergence order of the two-point iterative method $\psi_{3^{th}OM}(x_k)$ in equation (2.7) to obtain a three-point eighth order convergence iterative method $\psi_{8^{th}OM}(x_k)$ as follows:

$$\begin{cases} \psi_{4^{th}OM}(x_k) = \psi_{2^{nd}NM}(x_k) - \dfrac{g\left(\psi_{2^{nd}NM}(x_k)\right)}{2g'\left(\frac{x_k+\psi_{2^{nd}NM}(x_k)}{2}\right) - g'\left(x_k\right)}, \\ \psi_{8^{th}OM}(x_k) = \psi_{4^{th}OM}(x_k) - \Omega\left(x_k\right) G\left(\beta\left(x_k\right)\right) H\left(\eta\left(x_k\right)\right), k = 1, 2, \cdots \end{cases} \tag{2.9}$$

where

$$\Omega\left(x_k\right) = \frac{g\left(\psi_{4^{th}OM}(x_k)\right)\left(\psi_{4^{th}OM}(x_k) - \psi_{2^{nd}NM}(x_k)\right)}{g(\psi_{4^{th}OM}(x_k)) - g(\psi_{2^{nd}NM}(x_k))},$$

$$\beta\left(x_k\right) = \frac{g\left(\psi_{4^{th}OM}(x_k)\right)}{g(x_k)}, \qquad \eta\left(x_k\right) = \frac{g\left(\psi_{2^{th}NM}(x_k)\right)}{g(x_k)}$$

Furthermore, $G\left(\beta\left(x_k\right)\right)$ and $H\left(\eta\left(x_k\right)\right)$ are suitable weight functions to be determined.

The weight function $G\left(\beta\left(x_k\right)\right)$ and $H\left(\eta\left(x_k\right)\right)$ are built in the second step the method in equation (2.9) such a way that no additional function derivative is required so that it accelerates the convergence order to eight. One iteration cycle of the developed three point, eight order method $(\psi_{8^{th}OM}(x_k))$ requires five functions evaluation (that is, three functions $g(\cdot)$ and two derivative of functions $(g'(\cdot))$). This implies that by Definition 3, its $EI = 1.5157$. This is computationally more efficient than the special case of the method in equation (2.4) obtained by setting $q = \mu = 1$, and $\theta = \frac{1}{2}$ developed in [5] with $EI = 1.3161$, the fourth order convergence method $(\psi_{4^{th}OM}(x_k))$ in equation (2.8) with $EI = 1.4142$ and the three points, fifth order quadrature based method $(\psi_{5^{th}DKR}(x_k))$ developed in [16] with $EI = 1.4953$ given as:

$$\psi_{5^{th}DKR}(x_k) = x_k - H\left(\tau\right) \frac{6g\left(x_k\right)}{g'(x_k) + 4g'\left(\frac{x_k+\psi_{2^{nd}NM}(x_k)}{2}\right) + g'\left(\psi_{2^{nd}NM}(x_k)\right)}, \tag{2.10}$$

where $\quad H(\tau) = 1 + \frac{1}{4}(\tau-1)^2 - \frac{3}{8}(\tau-1)^3, \quad \tau = \frac{g'(\psi_{2^{nd}NM}(x_k))}{g'(x_k)}.$

3. Convergence Analysis of the Developed Method

In this section, the convergence of the developed iterative method (2.9) is established.

Theorem 2. *Suppose the function $g : D \subset R \to R$ is continuously differentiable and $g'(\cdot)$ is nonzero in the neighborhood $D \in R$ containing α. If x_0 is an initial guess in the neighborhood of α , then the sequence of approximations $\{x_k\}_{k\geq 0}, (x_k \in D)$ generated by the method in equation (2.9) converges to α with order $\rho = 8$ provided the weight functions satisfy the conditions*:

$$G(0) = H(0) = 1, \quad G'(0) = 0, \quad H'(0) = \tfrac{4}{3}, \quad G''(0) = 2, \quad G'''(0) = 16$$

Proof. Let $e_k = |x_k - \alpha|$ be the error in the kth iteration point and $c_k = \frac{g^{(k)}(\alpha)}{k!g'(\alpha)}$, $k \geq 2$. Using Taylor series to expand $g(x)$ and $g'(x)$ about α respectively and set $x{=}x_k$, the following equations are obtained.

$$\begin{aligned} g(x_k) &= g(\alpha + e_k) \\ &= g'(\alpha)\left[e_k + \sum_{n=2}^{8} c_n e_k^n + O\left(e_k^9\right)\right], k{=}0,1,2,\cdots \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} g'(x_k) &= g'(\alpha + e_k) \\ &= g'(\alpha)\left[1 + \sum_{n=2}^{8} c_n e_k^{n-1} + O\left(e_k^9\right)\right], k{=}0,1,2,\cdots \end{aligned} \tag{3.2}$$

Using equations (3.1) and (3.2), the error equation of the Newton methods is:

$$\psi_{2^{nd}NM}(x_k) = \alpha + c_2 e_k^2 + \left(-2c_k^2 + 2c_3\right) e_k^3 + \left(4c_2^3 - 7c_2c_3 + 3c_4\right) e_k^4 + O\left(e_k^5\right) \tag{3.3}$$

Now

$$g(\psi_{2^{nd}NM}(x_k)) = g'(\alpha)\left[c_2e_k^2 + \left(-2c_2^2 + 2c_3\right) e_k^3 + \left(4c_2^2 - 7c_2c_3 + 3c_4\right) e_k^4 \right. \\ \left. + O\left(e_k^5\right)\right] \tag{3.4}$$

and

$$g'\left(\frac{x_k+\psi_{2^{nd}NM}(x_k)}{2}\right)=g'(\alpha)\Big[1+c_2e_k+\left(c_2^2+\frac{3c_3}{4}\right)e_k^2$$
$$+\left(\frac{3c_2c_3}{2}+2c_2\left(-c_2^2+c_3\right)+\frac{c_4}{2}\right)e_k^3$$
$$+\left(3c_3\left(-\frac{3c_2^2}{4}+c_3\right)+\frac{3c_2c_4}{2}+c_2\left(4c_2^3-7c_2c_3+3c_4\right)+\frac{5c_5}{16}\right)e^4$$
$$+O\left(e_k^5\right)] \tag{3.5}$$

Using equations (3.2) and (3.5),

$$\frac{1}{2g'\left(\frac{x_k+\psi_{2^{nd}NM}(x_k)}{2}\right)-g'(x_k)}=\left(g'(\alpha)\right)^{-1}\Big[1+\left(-2c^22+\frac{3c_3}{2}\right)e_k^2$$
$$+\left(4c^32-7c_2c_3+3c_4\right)e_k^3$$
$$+\left(-4c^42+\frac{25c_2c_3}{2}-9c_2c_4-\frac{5\left(6c_3-7c_5\right)}{8}\right)e_k^4+O\left(e_k^5\right)] \tag{3.6}$$

From equations (3.3), (3.4) and (3.6),

$$\psi_{4^{th}OM}(x_k)=\left(c_2^3-\frac{3c_2c_3}{2}\right)e_k^4+\left(-4c_2^4+10c_2^2c_3-3c^23-3c_2c_4\right)e_k^5+O\left(e_k^6\right) \tag{3.7}$$

Now, from equation (3.7) the following is obtained,

$$g\left(\psi_{4^{th}OM}(x_k)\right)=g'(\alpha)\left[\left(c_2^3-\frac{3c_2c_3}{2}\right)e_kk^4+\left(-4c_2^4+10c_2^2c_3-3c_3^2-3c_2c_4\right)e^5\right.$$
$$+O\left(e_k^6\right)] \tag{3.8}$$

From equations (3.3), (3.4), (3.7) and (3.8), we obtain

$$\Omega(x_k)=(c_2^3-\frac{3c_2c_3}{2})e_k^4+(-4c_2^4+10c_2^2c_3-3c_3^2-3c_2c_4)e_k^5$$
$$+\frac{1}{8}(72c_2^5-268c_2^3c_3+194c_2c_3^2+120c_2^2c_4-84c_3c_4-35c_2c_5)e_k^6$$
$$+(-14c_2^6+75c_2^4c_3+\frac{33c_3^3}{2}-41c_2^3c_4-9c_4^2-\frac{c_2^2(382c_3^2-77c_5)}{4}$$
$$-\frac{59c_3c_5}{4}+c_2(69c_3c_4-\frac{45c_6}{8}))e_k^7+O(e_k^8) \tag{3.9}$$

The Taylor expansion of the weight functions $G\left(\beta\left(x_k\right)\right)$ and $H\left(\eta\left(x_k\right)\right)$ about 0 are,

$$G\left(\beta\left(x_k\right)\right) = G(0)+\left(\beta\left(x_k\right)\right)G'(0)+\frac{1}{2}\left(\beta\left(x_k\right)\right)^2 G''(0)+\frac{1}{6}\left(\beta\left(x_k\right)\right)^3 G'''(0)+\cdots \tag{3.10}$$

and

$$H\left(\eta\left(x_k\right)\right) = H(0)+\left(\eta\left(x_k\right)\right)H'(0)+\frac{1}{2}\left(\eta\left(x_k\right)\right)^2 H''(0)+\frac{1}{6}\left(\eta\left(x_k\right)\right)^3 H'''(0)+\cdots \tag{3.11}$$

where

$$\begin{aligned}\beta(x_k) &= \frac{g(\psi_{4^{th}OM}(x_k))}{g(x_k)}\\ &= (3c_2^3 - c_2c_3)e_k^3 + (-5c_2^4 + \frac{23c_2^2c_3}{2} - 3c_3^2 - 3c_2c_4)e_k^4\\ &+ \frac{1}{8}(120c_2^5 - 380c_2^3c_3 + 144c_2^2c_4 - 84c_3c_4 + 5c_2(46c_3^2 - 7c_5))e_k^5\\ &+ \frac{1}{8}(-280c_2^6 + 1140c_2^4c_3 - 504c_2^3c_4 - 189c_2^2(6c_3^2 - c_5)\\ &+ 2(78c_3^3 - 36c_4^2 - 59c_3c_5) + c_2(672c_3c_4 - 45c_6))e_k^6 + O(e_k^7)\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\eta(x_k) &= \frac{g(\psi_{2^{nd}NM}(x_k))}{g(x_k)})\\ &= c_2e_k + (-3c_2^2 - 2c_3)e_k^3 + (8c_2^3 - 10c_2c_3 + 3c_4)e_k^3\\ &+ (-20c_2^4 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4 + 4c_3)e_k^4\\ &+ (45c_2^5 - 118c_2^3c_3 + 55c_2c_3^2 + 51c_2^2c_4 - 22c_3c_4 - 18c_2c_5 + 5c_6)e_k^5\\ &+ (-112c_2^6 + 344c_2^4c_3 + 26c_3^3 - 163c_2^3c_4 - 28c_3c_5 + c_2^2(-252c_3^2 + 65c_5)\\ &+ 2c_2(75c_3c_4 - 11c_6) + 6c_7)e_k^6) + O(e_k^7)\end{aligned} \tag{3.13}$$

By inserting the equations in (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11) into the last step of the method in equation (2.9) yield

$$\begin{aligned}
\psi_{8^{th}OM}(x_k) &= \alpha - (c_2(2c_2^2 - 3c_3)(-1 + G(0)H(0)))e_k^4 + (3c_3^2(-1\\
&+ G(0)H(0)) + 3c_2c_4(-1 + G(0)H(0)) + c_2^4(-4 + 4G(0)H(0) - G'(0)H(0))\\
&+ \frac{1}{2}(c_2^2c_3(20 - 20G(0)H(0) + 3G'(0)H(0))))e_k^5 + (83c_3c_4(-H(0)\\
&+ G(0)H(0)) - 24c_2^2c_4(-5 + 5G(0)H(0) - G'(0)H(0)) + c_2^5(80 - 72G(0)H(0)\\
&+ 56G'H(0) - 4G''(0)H(0)) + 2c_2^3(-140 + 134G(0)H(0) - 66G'(0)H(0)\\
&+ 3G''(0)H(0)) + c_2(35(-1 + G(0)H(0)) + c_3^2(194 - 194G(0)H(0)\\
&+ 48G'(0)H(0))))e_k^6 + (4c_2^3c_4(-88 + 82G(0)H(0) - 54G'(0)H(0)\\
&+ 3G''(0)H(0)) + c_2(45c_6(-1 + G(0)H(0)) - 24c_3c_4(-23 + 23G(0)H(0)\\
&- 7G'(0)H(0))) + 2(36c_4^2(-1 + G(0)H(0)) + 59c_3c_5(-1 + G(0)H(0))\\
&+ c_3^3(66 - 66G(0)H(0) + 24G'(0)H(0))) + \frac{1}{3}(4c_2^6(-120 + 84G(0)H(0)\\
&- 174G'(0)H(0) + 30G''(0)H(0) - G'''(0)H(0)\\
&- 6G(0)H'(0)) + 2c_2^4c_3(360 + 374G'(0)H(0) - 46G''(0)H(0)\\
&+ G'''(0)H(0) + 12(-25H(0) + H'(0))) + c_2^2(7c_5(22 - 22G(0)H(0)\\
&+ 5G'(0)H(0)) + 2c_3^2(-406 + 382G(0)H(0) - 273G'(0)H(0)\\
&+ 18G''(0)H(0) - 9G(0)H'(0)))))e_k^7 + O\left(e_k^8\right)
\end{aligned} \tag{3.14}$$

For $\psi_{8^{th}OM}(x_k)$ to be of convergence order $\rho = 8$, the coefficients of e_k^4, e_k^5, e_k^6 and e_k^7 in equation (3.14) must vanish. This is only achievable when $G(0) = H(0) = 1$, $G'(0) = 0$, $H'(0) = \frac{4}{3}$, $G''(0) = 2$, $G'''(0) = 16$. And it follows that the asymptotic error of the iterative method $\psi_{8^{th}OM}(x_k)$ is:

$$e_{k+1} = \alpha - \frac{1}{6}(c_2^2(2c_2^2 - 3c_2))(29c_2^3 - 16c_2c_3 + 3c_4)e_k^8 + O\left(e_k^9\right) \tag{3.15}$$

Equation (3.15) shows that the iterative method in equation (2.9) is of convergence order eight.

Remark 3.1. Any two weight functions that satisfies the conditions in Theorem 2 will produce an iterative method for solving (1.1) that is of convergence order eight. In this case, the following functions satisfies the conditions:

$$H(\eta_k) = 1 + \frac{4}{3}\eta_k \tag{3.16}$$

and

$$G(\beta_k) = 1 + \beta_k^2 + \frac{8}{3}\beta_k^3 \tag{3.17}$$

4. Numerical Experimentation

In this section, the developed iterative method is implemented on some standard numerical problems in literature, so as to illustrate its accuracy and effectiveness. The developed method ($\psi_{8^{th}OM}(x_k)$) performance is compared with the Newton method ($\psi_{2^{nd}NM}(x_k)$), the Noor et al. method ($\psi_{3^{rd}NR}(x_k)$) [5], the Ogbereyivwe and Muka method ($\psi_{4^{th}OM}(x_k)$) [10] and the Babajee method ($\psi_{5^{th}DKR}(x_k)$). All Numerical computations are done using PYTHON 2.7.12 with 1000 digits arithmetic precision on Intel Celeron(R) CPU 1.6 GHz with 2 GB of RAM processor. The stopping criteria $|g\,(x_{k+1})| \leq \epsilon$, where $\epsilon = 10^{-100}$ is error tolerance is used for computer programs. The computational results in terms of Number of Iterations (NIT) required by each method to converge to the solution α of (1.1) and the Petkovic computational local order of convergence ρ_{coc} [17] given as,

$$\rho_{coc} = \frac{log\,|g(x_{k+1})|}{log\,|g(x_k)|} \tag{4.1}$$

is presented in Table 1,2 and 3 for comparison. The following functions ($g_1(x) - g_7(x)$) and their roots α taken from [14] and [16] are used for numerical test.

$$g_1(x) = sin(2cosx) - 1 - x^2 + \exp(sin(x^3)),$$
$$\alpha = -0.7848959876612125352248560\cdots$$
$$g_2(x) = x^3 + 4x^2 - 10, \qquad \alpha = 1.3652300134140968457608068\cdots$$
$$g_3(x) = (x+2)\exp(x) - 1, \alpha = -0.4428544010023885831413279\cdots$$
$$g_4(x) = \sqrt{x} - cosx, \qquad \alpha = 0.6417143708728826583985653\cdots$$
$$g_5(x) = x^4 cos(x^2) - x^5 log(1 + x^2 - \pi) + \pi^2,$$
$$\alpha = 1.7724538509055160272981674\cdots$$
$$g_6(x) = \sqrt{x^2 + 2x + 5} - 2sinx - x^x + 3,$$
$$\alpha = 2.3319676558839640103080440\cdots$$
$$g_7(x) = ln(x^2 + x + 2) - x + 1, \ \alpha = 4.1525907367571582749969890\cdots$$

From the numerical results presented in Table 1,2 and 3, it is observed that the error margin of the developed method is less than that of the methods compared. Also, the developed method convergence rate and computational order of convergence not only supports the theoretical order of convergence obtained in Section 3, but also superior to the methods compared.

TABLE 1. Comparison of methods results for $g_1(x)$-$g_3(x)$.

Problem	Methods	x_0	NIT	$\lvert g(x_{k+1})\rvert$	ρ_{coc}	x_0	NIT	$\lvert g(x_{k+1})\rvert$	ρ_{coc}
$g_1(x)$	$\psi_{2^{nd}NM}(x_k)$	-0.7	7	2.4361e-148	2.0	-1.0	7	1.6999e-120	2.0
	$\psi_{3^{rd}NR}(x_k)$		5	8.8550e-282	3.0		5	1.4274e-226	3.0
	$\psi_{4^{h}OM}(x_k)$		4	2.9350e-270	4.0		4	1.6864e-198	4.0
	$\psi_{5^{th}DKR}(x_k)$		3	2.9335e-111	5.0		4	4.6040e-476	5.0
	$\psi_{8^{th}OM}(x_k)$		3	5.3554e-481	8.0		3	3.6933e-370	8.0
$g_2(x)$	$\psi_{2^{nd}NM}(x_k)$	2.5	8	8.3175e-105	2.0	1.6	7	4.9233e-124	3.0
	$\psi_{3^{rd}NR}(x_k)$		6	7.5659e-301	3.1		5	2.3091e-236	3.0
	$\psi_{4^{h}OM}(x_k)$		4	4.3895e-118	4.1		4	2.0449e-265	4.1
	$\psi_{5^{th}DKR}(x_k)$		4	1.6651e-172	5.1		4	4.3562e-481	5.0
	$\psi_{8^{th}OM}(x_k)$		3	1.3164e-184	8.4		3	1.8917e-455	8.1
$g_3(x)$	$\psi_{2^{nd}NM}(x_k)$	-0.8	8	2.1510e-145	2.0	-0.3	7	6.9432e-131	2.0
	$\psi_{3^{rd}NR}(x_k)$		5	5.4108e-135	3.0		5	1.4593e-248	3.0
	$\psi_{4^{h}OM}(x_k)$		4	7.1669e-309	4.0		4	1.3180e-345	4.0
	$\psi_{5^{th}DKR}(x_k)$		4	3.6732e-144	5.0		4	2.6562e-102	4.9
	$\psi_{8^{th}OM}(x_k)$		3	2.3970e–354	8.0		3	6.9686e-541	8.0

TABLE 2. Comparison of methods results for $g_4(x)$-$g_6(x)$.

Problem	Methods	x_0	NIT	$\lvert g(x_{k+1})\rvert$	ρ_{coc}	x_0	NIT	$\lvert g(x_{k+1})\rvert$	ρ_{coc}
$g_4(x)$	$\psi_{2^{nd}NM}(x_k)$	0.7	6	3.1392e-135	2.0	2.0	7	3.6294e-125	2.0
	$\psi_{3^{rd}NR}(x_k)$		4	2.9425e-171	3.0		5	1.4763e-226	3.0
	$\psi_{4^{th}OM}(x_k)$		3	1.6986e-122	4.0		4	6.7906e-150	4.0
	$\psi_{5^{th}DKR}(x_k)$		3	1.1157e-221	5.0		4	4.6666e-347	5.0
	$\psi_{8^{th}OM}(x_k)$		2	9.1158e-113	8.1		3	2.1935e-299	8.1
$g_5(x)$	$\psi_{2^{nd}NM}(x_k)$	1.6	7	2.8005e-175	2.0	2.0	7	2.7873e-181	2.0
	$\psi_{3^{rd}NR}(x_k)$		4	4.8857e-102	3.1		4	1.8241e-106	3.1
	$\psi_{4^{th}OM}(x_k)$		4	1.0207e-283	4.0		4	2.1672e-185	4.1
	$\psi_{5^{th}DKR}(x_k)$		4	8.9512e-494	5.0		3	5.2311e-114	5.2
	$\psi_{8^{th}OM}(x_k)$		3	1.8842e-424	8.2		3	1.2103e-511	8.1
$g_6(x)$	$\psi_{2^{nd}NM}(x_k)$	1.8	7	3.1032e-117	2.0	3.0	11	4.2474e-174	2.0
	$\psi_{3^{rd}NR}(x_k)$		6	4.5979e-223	3.0		7	1.7074e-189	3.0
	$\psi_{4^{th}OM}(x_k)$		4	1.7711e-158	4.1		4	5.3450e-101	4.0
	$\psi_{5^{th}DKR}(x_k)$		4	1.0926e-448	5.0		4	7.9640e-123	4.9
	$\psi_{8^{th}OM}(x_k)$		3	3.4105e-448	8.0		3	1.4563e-389	7.9

TABLE 3. Comparison of methods results for $g_7(x)$.

Problem	Methods	x_0	NIT	$\lvert g(x_{k+1})\rvert$	ρ_{coc}	x_0	NIT	$\lvert g(x_{k+1})\rvert$	ρ_{coc}
$g_7(x)$	$\psi_{2^{nd}NM}(x_k)$	3.5	9	1.1418e-126	2.0	-0.3	9	2.0010e-189	2.0
	$\psi_{3^{rd}NR}(x_k)$		3	1.1595e-109	3.0		6	4.4666e-246	3.0
	$\psi_{4^{th}OM}(x_k)$		4	3.1640e-298	4.0		6	2.0395e-188	3.8
	$\psi_{5^{th}DKR}(x_k)$		3	1.1372e-136	3.0		4	1.1710e-121	5.0
	$\psi_{8^{th}OM}(x_k)$		3	3.7467e-547	8.0		4	2.6631e-157	8.3

5. Conclusions and Future Work

In this paper, a three - point quadrature based iterative method of convergence order eight for the approximation of the solution of nonlinear equations is developed using the composition and weight function technique. The developed method is a modification of a two - point quadrature based iterative method of convergence order three and increases its efficiency index from $EI = 1.3161$ to $EI = 1.5157$ at the cost of one additional function evaluation. The developed method is also better than the classical Newton method in terms of efficiency index ($EI = 1.4142$). The numerical experimentation of the developed method revealed that it is effective and efficient for approximation of solution of nonlinear equations. Hence, the new method can be considered as a good competitor to existing methods.
The following area can be considered for future research. First, modifying the developed method to make it optimal in line with Kung-Traub conjecture. Secondly, generalization of the developed method to attain convergence order n and lastly, generalization of the method to Banach spaces.

Author contributions: The individual contributions of authors have been: Conceptualization, methodology, analysis, validation, writing, editing, supervision - O. Ogbereyivwe; numerical implementation, writing - V. Ojo-Orobosa.

Conflicts of interest: The authors absolutely declare that there is no conflict of interest.

Funding: The authors are grateful to Tertiary Education Trust Fund (TETFUND), Nigeria for supporting this research through Institutional Based Research (IBR) Fund.

Acknowledgements: The authors would like to thank the editors and anonymous reviewers for their constructive comments and useful suggestion that led to the improvement of this paper.

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