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A modified forward-backward splitting method for sum of monotone operators and demicontractive mappings

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Abstract. In this paper, by using a modified forward-backward splitting method, the author introduces and studies an iterative algorithm for finding a common element of the set of fixed points of demicontractive mappings and the set of solutions of variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings in real Hilbert spaces. The author proves that the sequence x_n which is generated by the proposed iterative algorithm converges strongly to a common element of two sets above. Finally, some applications are given.

Keywords: Fixed points; Set-valued operators; Variational inclusion problems, Forward-backward splitting method.

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1. Introduction

Let H be a real Hilbert space. For a multivalued map $A : H \to 2^H$, the domain of A, $D(A)$, the image of a subset S of H, $A(S)$, the range of A, $R(A)$, and the graph of A, $G(A)$, are defined as follows:

$$D(A) := \{x \in H : Ax \neq \emptyset\}, \; A(S) := \cup\{Ax : x \in S\},$$

$$R(A) := A(H), \; G(A) := \{[x, u] : x \in D(A), \; u \in Ax\}.$$

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A multivalued map $A : D(A) \subset H \to 2^H$ is called monotone if the inequality

$$\langle u - v, x - y \rangle \geq 0$$

holds for each $x, y \in D(A)$, $u \in Ax$, $v \in Ay$.

Let $A : H \to H$ be a single-valued nonlinear mapping is said *α-inverse strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay,\ x - y \rangle \geq \alpha \|Ax - Ay\|^2,\ \forall\ x, y \in H.$$

It is immediate that if A is α- inverse strongly monotone, then A is monotone and Lipschitz continuous.
Let K be a nonempty closed convex subset of a real Hilbert space H.

Let $A : H \to H$ be a single-valued nonlinear mapping and $B : H \to 2^H$ be a set-valued mapping. The variational inclusion problem is as follows: find $x \in H$ such that

$$0 \in B(x) + A(x). \tag{1.1}$$

The set of solution of (1.1) is denoted by $(A + B)^{-1}(0)$. If $A = 0$, then problem (1.1) becomes the inclusion problem introduced by Rockafellar [18]. Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two possibly simpler nonlinear operators. Splitting methods for linear equations were introduced by Peaceman and Rachford [16] and Douglas and Rachford [6]. A splitting method for (1.1) means an iterative method for which each iteration involves only with the individual operators A and B, but not the sum $A + B$. A popular method for solving problem (1.1) is the well-known forward-backward splitting method introduced by Passty [15] and Lions and Mercier [10].The method is formulated as

$$x_{n+1} = (I - \lambda_n B)^{-1}(I - \lambda_n A)x_n,\ \lambda_n > 0, \tag{1.2}$$

under the condition that $D(B) \subset D(A)$. It was shown, see for example [4], that weak convergence of (1.2) requires quite restrictive assumptions on A and B, such that the inverse of A is strongly monotone or B is Lipschitz continuous and monotone and the operator $A + B$ is strongly monotone on $D(B)$. Hence, the modification is necessary in order to guarantee the strong convergence of forward-backward splitting method (see, for example, [7, 20, 19] and the references contained in them).

Many problems arising in different areas of mathematics such as optimization, variational analysis, differential equations, mathematical economics can be modeled as fixed point equations of the form

$$x = Tx, \tag{1.3}$$

where T is a nonlinear mapping. Recently, studies on solutions of (1.3) were extensively carried out in Hilbert spaces and in certain Banach spaces; see, for example, [11, 12, 13] and the references therein.

Let X be a real normed space, K be a nonempty subset of X. A map $T : K \to K$ is said to be Lipschitz if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \ \ \forall x, y \in K, \tag{1.4}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive. The set of fixed points of the mapping T is denoted by $Fix(T) := \{x \in D(T) : x = Tx\}$. We assume that $Fix(T)$ is nonempty. A map T is called quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ holds for all x in K and $p \in Fix(T)$. The mapping $T : K \to K$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \forall x, y \in K.$$

A mapping $T : K \to H$ is called k-strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \quad \forall x, y \in K.$$

A map T is called k-demi-contractive if $Fix(T) \neq \emptyset$ and for $k \in [0, 1)$, we have

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2 \quad \forall x \in K, \ \ p \in Fix(T).$$

Remark 1.1. The class of demicontractive mappings is fundamental because it includes many types of nonlinear mappings arising in applied mathematics and optimization. We can see from the above definitions that the demicontractive mappings contains these mappings such as the directed mappings, the quasi-nonexpansive mappings, and the strictly pseudocontractive mappings with nonempty fixed point set.

Almost results existing for solving variational inclusion and fixed point problems by using forward-backward splitting method have been done for monotone operators and nonexpansive mappings.

In this paper, motivated by above results, the author introduces a new iterative algorithm and proves a strong convergence theorem for variational inclusion problem (1.1) and the fixed point problem (1.3) involving demicontractive mappings in Hilbert spaces without any compactness assumption. Finally, application to optimization problems with constraints is provided to support our main results.

2. Preliminaries

The demiclosedness of T usually plays an important role in dealing with the convergence of fixed point iterative algorithms.

Definition 2.1. Let H be a real Hilbert space and $T : D(T) \subset H \to H$ be a mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $\|x_n - Tx_n\|$ converges to zero, then $p \in Fix(T)$.

Let a set-valued mapping $B : H \to 2^H$ be a maximal monotone. We define a resolvent operator J_λ^B generated by B and λ as follows:

$$J_\lambda^B = (I + \lambda B)^{-1}(x),\ \forall x \in H,$$

where λ is a positive number. It is easily to see that the resolvent operator J_λ^B is single-valued, nonexpansive and 1-inverse strongly monotone and moreover, a solution of the problem 1.1 is a fixed point of the operator $J_\lambda^B(I - \lambda A)$ for all $\lambda > 0$ (see, for example, [9]).

Lemma 2.2. [10] *Let $B : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz and continuous monotone mapping. Then the mapping $B + A : H \to 2^H$ is maximal monotone.*

Lemma 2.3 ([5])**.** *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequalities hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2,\ \ \lambda \in (0, 1).$$

Lemma 2.4 ([20])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [12] *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping.*
(i) If T is a k-strictly pseudo-contractive mapping, then T satisfies the Lipschitzian condition

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k}\|x - y\|.$$

(ii) If T is a k-strictly pseudo-contractive mapping, then the mapping $I - T$ is demiclosed at 0.

Lemma 2.6 ([12], Proposition 2.1). *Assume K is a closed convex subset of a Hilbert space H. Let $T : K \to K$ be a self-mapping of C. If T is a k-demicontractive mapping, then the fixed point set $Fix(T)$ is closed and convex.*

Lemma 2.7. [17] *Let t_n be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence t_{n_i} of t_n such that t_{n_i} such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ t_n\} \leq t_{\tau(n)+1}.$$

Lemma 2.8. *Let H be a real Hilbert space and $A : H \to H$ be an α-inverse strongly monotone mapping. Then, $I - \theta A$ is nonexpansive mapping for all $x, y \in H$ and $\theta \in [0, 2\alpha]$.*

Proof. For all $x, y \in H$, we have

$$\begin{aligned} \|(I-\theta A)x - (I-\theta A)y\|^2 &= \|(x-y) - \theta(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\theta\langle Ax - Ay, x - y\rangle + \|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 + \theta(\theta - 2\alpha)\|Ax - Ay\|^2. \end{aligned}$$

□

3. Main results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone operator of K into H. Let $f : K \to K$ be a b-contraction mapping and B be a maximal monotone operator on H into 2^H such that the domain of B is included in K. Let $T : K \to K$ be a β-demicontractive mapping such that $Fix(T) \cap (A+B)^{-1}(0)$ is nonempty and $I - T$ is demiclosed at origin. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm:*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Big(\theta_n J^B_{\lambda_n}(I - \lambda_n A)x_n + (1-\theta_n) T J^B_{\lambda_n}(I - \lambda_n A)x_n\Big), \tag{3.1}$$

where $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lambda_n \in [a, b] \subset \big(0, min\{1, 2\alpha\}\big)$.

(ii) $\theta_n \in]\beta, 1[$ *and* $\lim\limits_{n\to\infty} \inf(1-\theta_n)(\theta_n - \beta) > 0$.

Then, the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $x^* \in Fix(T) \cap (A+B)^{-1}(0)$, *which is the unique solution of the following variational inequality:*

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0, \quad \forall p \in Fix(T) \cap (A+B)^{-1}(0). \tag{3.2}$$

Proof. From $(I-f)$ is strongly monotone and $Fix(T) \cap (A+B)^{-1}(0)$ is closed convex, then the variational inequality (3.2) has a unique solution in $Fix(T) \cap (A+B)^{-1}(0)$. Below we use x^* to denote the unique solution of (3.2). For each $n \geq 0$, we put $z_n := J^B_{\lambda_n}(I - \lambda_n A)x_n$ and $y_n = \theta_n z_n + (1-\theta_n)Tz_n$. Let $p \in \Gamma$. Then from Lemma 2.8, we have

$$\|z_n - p\| = \|J^B_{\lambda_n}(I - \lambda_n A)x_n - p\| \leq \|x_n - p\|, \;\; \forall n \geq 0.$$

By using (3.1), Lemma 2.3 and T is β-demicontractive, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \Big\|\theta_n(z_n - p) + (1-\theta_n)(Tz_n - p)\Big\|^2 \\
&= \theta_n\|z_n - p\|^2 + (1-\theta_n)\|Tz_n - p\|^2 - (1-\theta_n)\theta_n\|Tz_n - z_n\|^2 \\
&\leq \theta_n\|z_n - p\|^2 + (1-\theta_n)\Big(\|z_n - p\|^2 + \beta\|z_n - Tz_n\|^2\Big) - (1-\theta_n)\theta_n\|Tz_n - z_n\|^2.
\end{aligned}$$

Hence,

$$\|y_n - p\| \leq \|z_n - p\|^2 - (1-\theta_n)(\theta_n - \beta)\|Tz_n - z_n\|^2. \tag{3.3}$$

Since $\theta_n \in]\beta, 1[$, we obtain,

$$\|y_n - p\Big\| \leq \|z_n - p\Big\|. \tag{3.4}$$

Therefore

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{3.5}$$

By using (3.1) and inequality (3.5), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq [1 - (1-b)\alpha_n]\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-b}\}.
\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-b}\}, \quad n \geq 1.$$

Hence, $\{x_n\}$ is bounded.
Thus from inequality (3.3), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\leq \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|y_n-p\|^2+2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|\\ &\leq \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|y_n-p\|^2+2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|\\ &\leq \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|x_n-p\|^2-(1-\alpha_n)^2(1-\theta_n)(\theta_n-\beta)\|Tz_n-z_n\|^2\\ &+ 2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|x_n-p\|.\end{aligned}$$

Hence,

$$\begin{aligned}(1-\alpha_n)^2(1-\theta_n)(\theta_n-\beta)\|Tz_n-z_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\|f(x_n)-p\|^2\\ &+ 2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|x_n-p\|.\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, then there exists a constant $C>0$, we have

$$(1-\alpha_n)^2(1-\theta_n)(\theta_n-\beta)\|Tz_n-z_n\|^2\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nC. \tag{3.6}$$

Next, I prove that $x_n\to x^*$. To see this, let us consider two possible cases.

Case 1. Assume that the sequence $\{\|x_n-p\|\}$ is monotonically decreasing. Then $\{\|x_n-p\|\}$ must be convergent. Clearly, we have

$$\lim_{n\to\infty}\left[\|x_n-p\|^2-\|x_{n+1}-p\|^2\right]=0. \tag{3.7}$$

It then implies from (3.6) that

$$\lim_{n\to\infty}(1-\theta_n)(\theta_n-\beta)\|Tz_n-z_n\|^2=0. \tag{3.8}$$

Since $\lim_{n\to\infty}\inf(1-\theta_n)(\theta_n-\beta)>0$, we have

$$\lim_{n\to\infty}\left\|Tz_n-z_n\right\|=0. \tag{3.9}$$

From (3.1) and Lemma 2.8, it follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\leq \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|y_n-p\|^2+2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|\\ &\leq \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|z_n-p\|^2+2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|\\ &= \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|J_{\lambda_n}^B(I-\lambda_nA)x_n-J_{\lambda_n}^B(I-\lambda_nA)p\|^2\\ &+ 2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|\\ &\leq \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\Big[\|x_n-p\|^2+a(b-2\alpha)\|Ax_n-Ap\|^2\Big]\\ &+ 2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|.\end{aligned}$$

Therefore, we have

$$(1-\alpha_n)^2 a(2\alpha-b)\|Ax_n-Ap\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n(1-\tau\alpha_n)\|f(x_n)-p\|\|y_n-p\| +2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|.$$

Since, $\alpha_n \to 0$ as $n \to \infty$, inequality (3.7) and $\{x_n\}$ is bounded, we obtain

$$\lim_{n\to\infty}\|Ax_n-Ap\|^2=0. \tag{3.10}$$

Since $J^B_{\lambda_n}$ is 1-inverse strongly monotone and (3.1), we have

$$\begin{aligned}\|z_n-p\|^2 &= \|J^B_{\lambda_n}(I-\lambda_n A)x_n-J^B_{\lambda_n}(I-\lambda_n A)p\|^2\\ &\le \langle z_n-p,(I-\lambda_n A)x_n-(I-\lambda_n A)p\rangle\\ &= \frac{1}{2}\Big[\|(I-\lambda_n A)x_n-(I-\lambda_n A)p\|^2+\|z_n-p\|^2-\|(I-\lambda_n A)x_n-(I-\lambda_n A)p-(z_n-p)\|^2\Big]\\ &\le \frac{1}{2}\Big[\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n\|^2+2\lambda_n\langle z_n-p,Ax_n-Ap\rangle-{\lambda_n}^2\|Ax_n-Ap\|^2\Big].\end{aligned}$$

So, we obtain

$$\|z_n-p\|^2 \le \|x_n-p\|^2-\|x_n-z_n\|^2+2\lambda_n\langle z_n-p,Ax_n-Ap\rangle-{\lambda_n}^2\|Ax_n-Ap\|^2,$$

and thus

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\le \alpha_n^2\|f(x_n)-p\|^2+(1-\alpha_n)^2\|z_n-p\|^2+2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|\\ &\le \alpha_n^2\|f(x_n)-p\|^2+\|x_n-p\|^2-(1-\alpha_n)^2\|x_n-z_n\|^2-(1-\alpha_n)^2{\lambda_n}^2\|Ax_n-Ap\|^2\\ &+ 2\lambda_n(1-\alpha_n)^2\langle z_n-p,Ax_n-Ap\rangle+2\alpha_n(1-\alpha_n)\|f(x_n)-p\|\|y_n-p\|.\end{aligned}$$

Since, $\alpha_n \to 0$ as $n \to \infty$, inequalities (3.7) and (3.10), we obtain

$$\lim_{n\to\infty}\|x_n-z_n\|^2=0. \tag{3.11}$$

Next, i prove that $\limsup_{n\to+\infty}\langle x^*-f(x^*),x^*-x_n\rangle \le 0$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to x^{**} in K and

$$\limsup_{n\to+\infty}\langle x^*-f(x^*),\, x^*-x_n\rangle=\lim_{k\to+\infty}\langle x^*-f(x^*),\, x^*-x_{n_k}\rangle.$$

From (3.9) and $I-T$ is demiclosed, we obtain $x^{**}\in Fix(T)$. Let us show $x^{**}\in(A+B)^{-1}(0)$. Since A be an α-inverse strongly monotone, A is Lipschitz continuous monotone mapping. It follows from Lemma 2.2 that $B+A$ is maximal monotone. Let $(v,g)\in G(B+A)$, i.e., $g-Av\in B(v)$. Since $z_{n_k}=J^B_{\lambda_{n_k}}(x_{n_k}-\lambda_{n_k}Ax_{n_k})$, we have $x_{n_k}-\lambda_{n_k}x_{n_k}\in(I+\lambda_{n_k}B)z_{n_k}$, i.e., $\frac{1}{\lambda_{n_k}}(x_{n_k}-z_{n_k}-\lambda_{n_k}Ax_{n_k})\in B(z_{n_k})$. By maximal monotonicity of $B + A$, we have

$$\langle v - z_{n_k}, g - Av - \frac{1}{\lambda_{n_k}}(x_{n_k} - z_{n_k} - \lambda_{n_k} A x_{n_k})\rangle \geq 0$$

and so

$$\begin{aligned}\langle v - z_{n_k}, g\rangle &\geq \langle v - z_{n_k}, Av - \frac{1}{\lambda_{n_k}}(x_{n_k} - z_{n_k} - \lambda_{n_k} A x_{n_k})\rangle \\ &= \langle v - z_{n_k}, Av - Az_{n_k} + Az_{n_k} + \frac{1}{\lambda_{n_k}}(x_{n_k} - z_{n_k} - \lambda_{n_k} A x_{n_k})\rangle \\ &\geq \langle v - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle + \langle v - z_{n_k}, \frac{1}{\lambda_{n_k}}(x_{n_k} - z_{n_k})\rangle.\end{aligned}$$

It follows from $\|z_n - x_n\| \to 0$, $\|Az_n - Ax_n\| \to 0$ and z_{n_k} converges weakly to x^{**}, we get

$$\lim_{k\to+\infty}\langle v - z_{n_k}, g\rangle = \langle v - x^{**}, g\rangle \geq 0$$

and hence $x^{**} \in (A+B)^{-1}(0)$. Therefore, $x^{**} \in (A+B)^{-1}(0) \cap Fix(T)$. On other hand, the fact that x^* solves (3.2), we then have

$$\begin{aligned}\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \lim_{k\to+\infty}\langle x^* - f(x^*), x^* - x_{n_k}\rangle \\ &= \langle x^* - f(x^*), x^* - x^{**}\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to x^*$. From (3.1) and Lemma 2.3, we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - x^*\|^2 \\ &\leq \|\alpha_n(f(x_n) - f(x^*)) + (1-\alpha_n)(y_n - x^*)\|^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq \Big(\alpha_n\|f(x_n) - f(x^*)\| + \|(1-\alpha_n)(y_n - x^*)\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq \Big(\alpha_n b\|x_n - x^*\| + (1-\alpha_n)\|y_n - x^*\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq \Big((1-\alpha_n(1-b))\|x_n - x^*\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\ &\leq (1-\alpha_n(1-b))\|x_n - x^*\|^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle.\end{aligned}$$

From Lemma 2.4, its follows that $x_n \to x^*$.

Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Set $\Gamma_n = \|x_n - x^*\|^2$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ \Gamma_k \leq \Gamma_{k+1}\}$. We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. Let $i \in \mathbb{N}^*$, from (3.6), we have

$$(1-\alpha_{\tau(n)})^2(1-\theta_{\tau(n)})(\theta_{\tau(n)} - \beta)\|z_{\tau(n)} - Tz_{\tau(n)}\|^2 \leq \alpha_{\tau(n)} C.$$

Furthermore, we have

$$\lim_{n\to+\infty}(1-\alpha_{\tau(n)})^2(1-\theta_{\tau(n)})(\theta_{\tau(n)}-\beta)\|z_{\tau(n)}-Tz_{\tau(n)}\|^2=0.$$

Since $\theta_{\tau(n)}\in]\beta,1[$ and $\lim_{n\to\infty}\inf(1-\theta_{\tau(n)})(\theta_{\tau(n)}-\beta)>0$, we have

$$\lim_{n\to\infty}\|z_{\tau(n)}-Tz_{\tau(n)}\|=0. \tag{3.12}$$

By a similar argument as in case 1, we can show that $x_{\tau(n)}$ is bounded in K and $\limsup_{\tau(n)\to+\infty}\langle x^*-f(x^*),x^*-x_{\tau(n)})\rangle\le 0$. We have for all $n\ge n_0$,

$$0\le\|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2\le\alpha_{\tau(n)}[-(1-b)\|x_{\tau(n)}-x^*\|^2+2\langle x^*-f(x^*),x^*-x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)}-x^*\|^2\le\frac{2}{1-b}\langle x^*-f(x^*),x^*-x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2=0.$$

Therefore,

$$\lim_{n\to\infty}\Gamma_{\tau(n)}=\lim_{n\to\infty}\Gamma_{\tau(n)+1}=0.$$

Thus, by Lemma 2.7, we conclude that

$$0\le\Gamma_n\le\max\{\Gamma_{\tau(n)},\ \Gamma_{\tau(n)+1}\}=\Gamma_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty}\Gamma_n=0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

In the special case, where T is a strictly pseudo-contractive mapping, then Theorem 3.1 is reduced to the following:

Theorem 3.2. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone operator of K into H. Let $f:K\to K$ be a b-contraction mapping and B be a maximal monotone operator on H into 2^H such that the domain of B is included in K. Let $T:K\to K$ be a β-strictly pseudo-contractive mapping such that $Fix(T)\cap(A+B)^{-1}(0)$ is nonempty. For given $x_0\in K$, let $\{x_n\}$ be generated by the algorithm:*

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)\Big(\theta_nJ^B_{\lambda_n}(I-\lambda_nA)x_n+(1-\theta_n)TJ^B_{\lambda_n}(I-\lambda_nA)x_n\Big), \tag{3.13}$$

where $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$ *and* $\lambda_n\in[a,b]\subset\big(0,min\{1,\ 2\alpha\}\big)$.

(ii) $\theta_n\in]\beta,1[$ *and* $\lim_{n\to\infty}\inf(1-\theta_n)(\theta_n-\beta)>0$.

Then, the sequence $\{x_n\}$ *generated by* (3.13) *converges strongly to* $x^* \in Fix(T) \cap (A+B)^{-1}(0)$, *which is the unique solution of the following variational inequality* (3.2).

Proof. Since every strictly pseudo-contractive is demicontractive mapping, then, the proof follows Lemma 2.5 and Theorem 3.1. □

Finally, the following the minimization of composite objective function of the type

Problem 3.3.

$$\min_{x \in H} F(x) + g(x), \tag{3.14}$$

where $F : H \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous functional and $g : H \to \mathbb{R}$ is convex functional.

Many optimization problems from image processing [3], statistical regression, machine learning (see, e.g., [21] and the references contained therein), etc can be adapted into the form of (3.14).
Observe that problem 3.3 is equivalent to find $x \in H$ such that

$$0 \in \partial F(x) + \nabla g(x). \tag{3.15}$$

It is well known ∂F is maximal monotone (see, e.g., Minty [14]).

Lemma 3.4. *(Baillon and Haddad* [2]*) Let* H *be a real Hilbert space,* g *a continuously Fréchet differentiable, convex functional on* H *and* ∇g *the gradient of* g. *If* ∇g *is* $\frac{1}{\alpha}$*-Lipschitz continuous, then* ∇g *is* α*-inverse strongly monotone.*

Hence, one has the following result.

Theorem 3.5. *Let* H *be a real Hilbert space and* $g : H \to \mathbb{R}$ *a continuously Fréchet differentiable, convex functional on* H *and* ∇g *is* $\frac{1}{\alpha}$*-Lipschitz continuous. Let* $f : H \to H$ *be a b-contraction mapping and* $F : H \to \mathbb{R} \cup \{+\infty\}$ *is proper, convex and lower semi-continuous functional. For given* $x_0 \in H$, *let* $\{x_n\}$ *be generated by the algorithm:*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{\lambda_n}^{\partial F}(I - \lambda_n \nabla g)x_n, \tag{3.16}$$

where $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *be sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0,\ \sum\limits_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lambda_n \in [a,b] \subset \big(0, min\{1, 2\alpha\}\big)$

Suppose that Problem 3.3 is consistent. Then, the sequence $\{x_n\}$ *generated by* (3.16) *converges strongly to a solution of Problem 3.3, which is the unique solution of the following variational inequality:*

$$\langle x^* - f(x^*), x^* - p \rangle \le 0, \quad \forall p \in (\nabla g + \partial F)^{-1}(0). \tag{3.17}$$

Proof. We set $B = \partial F$, $\nabla g = A$, $K = H$ and $T = I$ into Theorem 3.1. Then, the proof follows Theorem 3.1. □

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