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# On Bounded Compact-Weak Approximate Identities

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Abstract. In this paper, we provide examples which show that there exists a type of approximate identity between the class of bounded weak approximate identities and bounded approximate identities.

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## 1. Introduction

In 1977, Jones and Lahr introduced a new notion of an approximate identity for a commutative Banach algebra $A$ called "bounded weak approximate identity (b.w.a.i. in short)" and gave an example of a semisimple and commutative semigroup algebra with a b.w.a.i. which has no approximate identity, bounded or unbounded. Indeed, a bounded net $\{e_\alpha\}$ in $(A, \|\cdot\|)$ is a b.w.a.i., if for each $a \in A$ and $\phi \in \Delta(A)$, $|\widehat{ae_\alpha}(\phi) - \widehat{a}(\phi)| \longrightarrow 0$. Since for each compact subset $K$ of $\Delta(A)$ and $\varphi \in K$ we have

$$|\varphi(a)| \leq \|\widehat{a}\|_K = \sup_{\psi \in K} |\psi(a)| \leq \|a\|,$$

one may ask the following question:

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What happens if the net $\{e_\alpha\}$ satisfies the following condition and, does it differ from the definition of a bounded weak approximate identity or they are equivalent?

For each $a \in A$ and compact subset $K$ of $\Delta(A)$,

$$\sup_{\phi \in K} |\widehat{ae_\alpha}(\phi) - \widehat{a}(\phi)| \longrightarrow 0.$$

In this paper, we investigate the above condition for commutative Banach algebras with non-empty character spaces. This study provides us a new kind of approximate identity between bounded approximate identity and bounded weak approximate identity.

## 2. Preliminaries

To deal with the main examples and to have a better understanding of what will be given, we need some basic definitions in which we are going to mention them in this section.

Throughout the paper, suppose that $A$ is a commutative Banach algebra and suppose $\Delta(A)$ is the character space of $A$, that is, the space consisting of all non-zero homomorphisms from $A$ into $\mathbb{C}$.

A bounded net $\{e_\alpha\}$ in $A$ is called a bounded approximate identity (b.a.i.) if for all $a \in A$, $\lim_\alpha ae_\alpha = a$. The notion of a bounded approximate identity first arose in Harmonic analysis; see [1, Section 2.9] for a full discussion of approximate identity and its applications.

For $a \in A$, we define $\widehat{a} : \Delta(A) \longrightarrow \mathbb{C}$ by $\widehat{a}(\phi) = \phi(a)$ for all $\phi \in \Delta(A)$. Then $\widehat{a}$ is in $C_0(\Delta(A))$ and $\widehat{a}$ is called the Gel'fand transform of $a$. Note that $\Delta(A)$ is equipped with the Gel'fand topology which turns $\Delta(A)$ into a locally compact Hausdoroff space; see [7, Definition 2.2.1, Theorem 2.2.3(i)]. Since for each $\phi \in \Delta(A)$, $\|\phi\| \leq 1$; see [7, Lemma 2.1.5], we have $\|\widehat{a}\|_{\Delta(A)} \leq \|a\|$, where $\|\cdot\|$ denotes the norm of $A$.

Suppose that $\phi \in \Delta(A)$. We denote by $A_c$ the space of all $a \in A$ such that supp $\widehat{a}$ is compact, and by $J_\phi$ the space of all $a \in A_c$ such that $\phi \notin$ supp $\widehat{a}$. Also, let $M_\phi = \ker(\phi) = \{a \in A : \phi(a) = 0\}$.

Let $X$ be a non-empty locally compact Hausdoroff space. A subalgebra $A$ of $C_0(X)$ is called a function algebra if $A$ separates strongly the points of $X$, that is, for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$, and for each $x \in X$, there exists $f \in A$ with $f(x) \neq 0$. A function algebra $A$ is called a Banach function algebra if $A$ has a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach algebra. A Banach function algebra $A$ is called natural if $X = \Delta(A)$, that is, every character of $A$ is an evaluation functional on some $x \in X$, or $x \longrightarrow \phi_x$ is a homeomorphism. In the latter case $M_{\phi_x}$ is denoted by $M_x$. In the rest of the paper, sometimes, we use the uniform norm on $X$ for which it is defined as

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \quad (f \in C_0(X)).$$

Let $S$ be a non-empty set. By $c_{00}(S)$, we denote the space of all functions on $S$ of finite support. A Banach sequence algebra on $S$ is a Banach function algebra $A$ on $S$ such that $c_{00}(S) \subseteq A$.

A net $\{e_\alpha\}$ in $A$ is called a bounded weak approximate identity (b.w.a.i.) if there exists a non-negative constant $C$ such that for each $\alpha$, $\|e_\alpha\| \leq C$ and for all $a \in A$ and $\phi \in \Delta(A)$

$$\lim_{\alpha} |\phi(ae_\alpha) - \phi(a)| = 0, \tag{2.1}$$

or equivalently, $\lim_\alpha \phi(e_\alpha) = 1$ for each $\phi \in \Delta(A)$. To see more applications and recent works on this topic, one can see [3, 4, 5, 6, 8].

If $\Delta(A) = \emptyset$, every bounded net in $A$ is a b.w.a.i. for $A$. So, to avoid trivialities we will assume that $A$ is a Banach algebra with $\Delta(A) \neq \emptyset$.

In the case that $A$ is a natural Banach function algebra, a bounded weak approximate identity $\{u_\alpha\}$ is called a bounded pointwise approximate identity (BPAI); see [2, Definition 2.11].

Let $A$ be a Banach function algebra. We say that $A$ has bounded relative approximate units (BRAUs) of bound $m$ if for each non-empty compact subset $K$ of $\Delta(A)$ and $\epsilon > 0$, there exists $f \in A$ with $\|f\| \leq m$ and $|1 - \varphi(f)| < \epsilon$ for all $\varphi \in K$; see [2, Section 2.2].

In the next sections, first, we give the definition of the main object of the paper. Then, we present some illuminating examples and finally we provide some hereditary properties. The main reference that lead us to this notion and its results is a profound paper by Dales and Ülger; see [2].

## 3. Main result

In what follows, we suppose that $\mathcal{K}(\Delta(A))$ denotes the collection of all compact subsets of $\Delta(A)$. We start off this section with the following definitions.

**Definition 3.1.** A *compact-weak approximate identity* (c-w.a.i.) for $A$, is a net $\{e_\alpha\}$ in $A$ such that for each $a \in A$ and $K \in \mathcal{K}(\Delta(A))$

$$\|\widehat{ae_\alpha} - \widehat{a}\|_K = \sup_{\phi \in K} |\widehat{ae_\alpha}(\phi) - \widehat{a}(\phi)| \longrightarrow 0. \tag{3.1}$$

If the net $\{e_\alpha\}$ is bounded, we say that it is a bounded compact-weak approximate identity (b.c-w.a.i.) for Banach algebra $A$.

Note that relation 2.1 tells us that $\widehat{ae_\alpha}$ tends to $\widehat{a}$ in the pointwise convergence topology of $C_0(\Delta(A))$, but relation 3.1 says $\widehat{ae_\alpha}$ tends to $\widehat{a}$ in the compact convergence topology of $C_0(\Delta(A))$. On the other hand, if $ae_\alpha$ tends to $a$ in $A$, then $\widehat{a}$ is the uniform limit of $\{\widehat{ae_\alpha}\}$ in the space $C_0(\Delta(A))$. Nevertheless, there are a lot of examples in the literature of mathematical analysis about the difference(s) among these three topologies(pointwise, compact and uniform convergence), in the rest of this section, we are going to present two illuminating examples.

Like an approximate identity that has a version of approximate units, the compact-weak approximate identities also have the same version in which we give it in the following definition.

**Definition 3.2.** The Banach algebra $A$ has *compact-weak approximate units* (c-w.a.u.), if for each $a \in A$, $K \in \mathcal{K}(\Delta(A))$ and $\epsilon > 0$, there exists $e \in A$ such that $\|\widehat{ae} - \widehat{a}\|_K < \epsilon$. The approximate units have bound $m$, if $e$ can be chosen with $\|e\| \leq m$. In this case we say $A$ has b.c-w.a.u.

Clearly, if $A$ has a b.c-w.a.i., then it has b.c-w.a.u. Also, one can check easily that every approximate identity is a c-w.a.i., and as well, every c-w.a.i. is a w.a.i. But the following sophisticated examples show us that the converse of these statements are not valid in general.

Note that if a Banach function algebra $A$ has a b.c-w.a.i., then it has BRAUs. The following example gives a Banach function algebra with a BPAI, while it does not have any BRAUs. So, two concepts of BPAI (or b.w.a.i.) and b.c-w.a.i. are different.

**Example 3.3.** Let $\mathbb{I} = [0,1]$ and let $A = \{f \in C(\mathbb{I}) : I(f) < \infty\}$, where

$$I(f) = \int_0^1 \frac{|f(t) - f(0)|}{t} dt.$$

For each $f \in A$, define $\|f\| = \|f\|_{\mathbb{I}} + I(f)$. We know that $(A, \|\cdot\|)$ is a natural Banach function algebra. Now, takes the Banach algebra $M_0 = \{f \in A : f(0) = 0\}$ into account. It has been shown that $M_0$ does not have any BRAUs, but it has a BPAI; see [2, Example 5.1]. Indeed, the BPAI of $M_0$ is obtained as follows: take a finite subset $F = \{t_1, t_2, \ldots, t_k\}$ of $(0,1]$. Suppose that $f^F_{n,i}$ is a function on $\mathbb{I}$ such that $f^F_{n,i}(t_i) = 1$, for every $x$ outside of the interval $[t_i - \frac{1}{n}, t_i + \frac{1}{n}]$, $f^F_{n,i}(x) = 0$ and on the intervals $[t_i - \frac{1}{n}, t_i]$ and $[t_i, t_i + \frac{1}{n}]$, $f^F_{n,i}$ is linear. Now, consider

$$\{f^F_{n,i} : n \in \mathbb{N}, F \in \mathcal{F}(0,1]\},$$

where $\mathcal{F}(0,1]$ denotes the collection of all finite subsetes of $(0,1]$. One can check that $\{f^F_{n,i}\}_{(n,F)}$ is a BPAI for $M_0$. On the other hand, to see that $M_0$ does not have a BRAUs, for every $n \in \mathbb{N}$ with

$$|1 - f(x)| < \frac{1}{2}, \quad \forall x \in [\frac{1}{n}, 1],$$

we have $\|f\| \geq \frac{\log n}{2}$, and this implies that the possible relative approximate units of $M_0$ can not be chosen with a specific bound.

Therefore, $M_0$ does not have any b.c-w.a.i., while it has a b.w.a.i.

The following example shows the difference between b.a.i. and b.c-w.a.i.

**Example 3.4.** Let $\alpha = (\alpha_k) \in \mathbb{C}^{\mathbb{N}}$ and for each $n \in \mathbb{N}$, set

$$p_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} k|\alpha_{k+1} - \alpha_k|, \quad p(\alpha) = \sup_{n\in\mathbb{N}} p_n(\alpha).$$

Put $A = \{\alpha \in c_0 : p(\alpha) < \infty\}$. By [1, Example 4.1.46], $A$ is a natural Banach sequence algebra on $\mathbb{N}$ for the norm given by $\|\alpha\| = \|\alpha\|_{\mathbb{N}} + p(\alpha)$. For each $K \in \mathcal{K}(\mathbb{N})$, there exists $\alpha_K \in A$ such that $\alpha_K(k) = 1$ for all $k \in K$ and $\|\alpha_K\| \leq 4$. So, $\{\alpha_K : K \in \mathcal{K}(\mathbb{N})\}$ is a b.c-w.a.i. for $A$. But by parts (iii) and (v) of [1, Example 4.1.46], $A^2$ has infinite codimension in $A$ where $A^2 = \text{linear span}\{ab : a, b \in A\}$. Therefore, $A$ has no b.a.i.

*Remark* 3.5. By Cohen's factorization theorem, we know that if a Banach algebra $A$ has a b.a.i., then $A$ factors, that is, for each $a \in A$, there exists $b, c \in A$ such that $a = bc$. One may ask this question: whether the statement of the Cohen theorem remains valid if we replace b.a.i. by b.c-w.a.i.? Example 3.4 gives a negative answer to this question. Indeed, if $A$ factors, then $A^2 = A$ which is a contradiction.

## 4. Hereditary Properties

In this section we will show that for some certain closed ideals $I$ of a Banach algebra $A$ with some conditions, $I$ has a b.c-w.a.i. if and only if $A$ has a b.c-w.a.i. First, we give the following result which shows the relation between a b.c-w.a.u. and a b.c-w.a.i., and it is a key tool in what follows.

**Proposition 4.1.** *Let $A$ be a commutative Banach algebra. Then $A$ has a b.c-w.a.u. if and only if $A$ has a b.c-w.a.i.*

*Proof.* Let $A$ has a b.c-w.a.u. of bound $M > 0$. Suppose that $\mathcal{F}$ is a finite subset of $A$, $K \in \mathcal{K}(\Delta(A))$ and $\epsilon > 0$. Similar to the proof of [7, Proposition 1.1.11], there exists $e_{(\mathcal{F},K,\epsilon)} \in A$ such that $\|e_{(\mathcal{F},K,\epsilon)}\| \leq M$ and

$$\|\widehat{ae_{(\mathcal{F},K,\epsilon)}} - \widehat{a}\|_K < \epsilon \quad (a \in \mathcal{F}).$$

Now, consider the following net,

$$\mathfrak{U} = \{e_{(\mathcal{F},K,1/n)} : K \in \mathcal{K}(\Delta(A)), \mathcal{F} \subseteq A \text{ is finite and } n \in \mathbb{N}\}.$$

It is a routin calculation that $\mathfrak{U}$ is a b.c-w.a.i. for $A$. □

Let $I$ be a closed ideal of $A$. Suppose that $I$ and $A/I$; the quotient Banach algebra, have b.a.i. of bound $m$ and $n$, respectively. Then $A$ has a b.a.i. of bound $m+n+mn$; see [7, Lemma 1.4.8 (ii)]. In the setting of b.c-w.a.i. we have the following version of the mentioned assertion.

**Lemma 4.2.** *Suppose that $I$ has a b.c-w.a.i. of bound $m$ and $A/I$ has a b.a.i. of bound $m$. Then $A$ has a b.c-w.a.i. of bound $m+n+mn$.*

*Proof.* With a slight modification in the proof of [7, Lemma 1.4.8 (ii)], we can see the proof. □

There exists a Banach function algebra $A$ such that $A$ has a b.c-w.a.i., but one of its closed ideals has no b.c-w.a.i.; see [8, Example 5.6]. Indeed, $A = C^1[0,1]$; the algebra of all functions with continuous derivation, has a b.c-w.a.i., but for each $t_0 \in [0,1]$, $M_{t_0}$ has no BPAI and hence no b.c-w.a.i. So, the converse of Lemma 4.2 is not valid in general. Although we have the following result.

**Theorem 4.3.** *Let $A$ be a Banach algebra, $\phi_0 \in \Delta(A)$ and $\overline{J_{\phi_0}} = M_{\phi_0}$. Suppose that there exists $n > 0$ such that for each neighborhood $U$ of $\phi_0$, there exists $a \in A$ with $\phi_0(a) = 1$, $\|a\| \leq n$ and $\operatorname{supp} \widehat{a} \subseteq U$. Then $M_{\phi_0}$ has a b.c-w.a.i. if and only if $A$ has a b.c-w.a.i.*

*Proof.* Suppose that $M_{\phi_0}$ has a b.c-w.a.i. Clearly, $M_{\phi_0}$ has codimension 1, that is, $A/M_{\phi_0}$ is generated by one vector. Therefore, $A/M_{\phi_0}$ has a b.a.i. and hence by Lemma 4.2, $A$ has a b.c-w.a.i.

Conversely, suppose that $A$ has a b.c-w.a.i. Let $a \in M_{\phi_0}$ and $\varepsilon > 0$. Since $\overline{J_{\phi_0}} = M_{\phi_0}$, there exists $a_1 \in J_{\phi_0}$ such that

$$\|a - a_1\| < \varepsilon.$$

Therefore, for each $W \in \mathcal{K}(\Delta(A))$, there exists $m > 0$ and $b \in A$ such that

$$\|\widehat{a_1} - \widehat{a_1 b}\|_W < \varepsilon, \quad \|b\| \leq m.$$

There exists a neighborhood $U$ of $\phi_0$ in $\Delta(A)$ such that $U \cap \operatorname{supp} \widehat{a_1} = \emptyset$, because $a_1 \in J_{\phi_0}$ and $\Delta(A)$ is Hausdorff. Now, by the hypothesis, there exists $c \in A$ such that $\|c\| \leq n$, $\phi_0(c) = 1$ and $\operatorname{supp} \widehat{c} \subseteq U$. Therefore, $\widehat{a_1}\widehat{c} = 0$, $b - bc$ is in $M_{\phi_0}$ and $\|b - bc\| \leq m(n+1)$. So, for $W \in \mathcal{K}(\Delta(A))$ we have

$$\begin{aligned}
\|\widehat{a} - \widehat{a}(\widehat{b - bc})\|_W &\leq \|\widehat{a} - \widehat{a_1}\|_W + \|\widehat{a_1} - \widehat{a_1}\widehat{b}\|_W + \|\widehat{a_1}\widehat{b} - \widehat{a}(\widehat{b - bc})\|_W \\
&\leq \|a - a_1\| + \varepsilon + \|\widehat{a_1}\widehat{b} - \widehat{a}(\widehat{b - bc}) + \widehat{b}\widehat{a_1}\widehat{c}\|_W \\
&\leq 2\varepsilon + \|(\widehat{a_1} - \widehat{a})(\widehat{b - bc})\|_W \\
&\leq 2\varepsilon + \varepsilon m(1+n).
\end{aligned}$$

Hence, by Proposition 4.1, $M_{\phi_0}$ has a b.c-w.a.i., which completes the proof. □

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