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Some Special Identities for Jacobsthal and Jacobsthal-Lucas Generalized Octonions

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Abstract. We study Jacobsthal and Jacobsthal-Lucas generalized octonions over the algebra $\mathbb{O}(a,b,c)$ where a,b and c are real numbers. We present Binet formulas for these types of octonions. Furthermore, we give some well-known identities such as Catalan's, Cassini's, d'Ocagne's identities, and other special identities for Jacobsthal and Jacobsthal-Lucas generalized octonions.

Keywords: Generalized octonion, Jacobsthal octonion, Jacobsthal-Lucas octonion.

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1. Introduction

Some special number sequences are of great importance in many areas of mathematics such as combinatorics, computer algorithms, and biological setting. Fibonacci,Lucas, Pell and Jacobsthal sequences are at the top of this large number sequences. Many researchers related to these series of numbers have made great contributions to this field with many studies. Before examining these studies, we take a look at the quaternions that form the basis of the number sequence we mentioned above.

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For arbitrary real constants a and b, the generalized quaternion algebra is $\mathbb{H}(a,b)$ with the basis $\{1, e_1, e_2, e_3\}$. The multiplication table for the basis of $\mathbb{H}(a,b)$ can be given as follows:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-a$	e_3	$-ae_2$
e_2	e_2	$-e_3$	$-b$	be_1
e_3	e_3	ae_2	$-be_1$	$-ab$

For $a = b = 1$, $\mathbb{H}(1,1)$ is the quaternion division algebra, for $a = 1, b = -1$, $\mathbb{H}(1,-1)$ is the algebra of split-quaternions or also called coquaternions, para-quaternions, anti-quaternions, pseudo-quaternions or hyperbolic quaternions.

The octonions constitute the largest normed division algebra over the real numbers and it is shown with the letter $\mathbb{O}$. The octonions have eight dimensions and they are alternative, flexible, power-associative, non-commutative and non-associative.

Let $\mathbb{O}(a,b,c)$ be the generalized octonion algebra over the $\mathbb{R}$ with the basis $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. The multiplication table for the basis of $\mathbb{O}(a,b,c)$ is given as follows:

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-a$	e_3	$-ae_2$	e_5	$-ae_4$	$-e_7$	ae_6
e_2	e_2	$-e_3$	$-b$	be_1	e_6	e_7	$-be_4$	$-be_5$
e_3	e_3	ae_2	$-be_1$	$-ab$	e_7	$-ae_6$	be_5	$-abe_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-c$	ce_1	ce_2	ce_3
e_5	e_5	ae_4	$-e_7$	ae_6	$-ce_1$	$-ac$	$-ce_3$	ace_2
e_6	e_6	e_7	be_4	$-be_5$	$-ce_2$	ce_3	$-bc$	$-bce_1$
e_7	e_7	$-ae_6$	be_5	abe_4	$-ce_3$	$-ace_2$	bce_1	$-abc$

If $\alpha \in \mathbb{O}(a,b,c)$, then we can write $\alpha = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7$. The conjugate of α is $\overline{\alpha} = \alpha_0 - \alpha_1 e_1 - \alpha_2 e_2 - \alpha_3 e_3 - \alpha_4 e_4 - \alpha_5 e_5 - \alpha_6 e_6 - \alpha_7 e_7$. The trace and the norm of α are, respectively

$$t(\alpha) = \alpha + \overline{\alpha} = 2\alpha_0$$

and

$$N(\alpha) = \alpha\overline{\alpha} = \alpha_0^2 + a\alpha_1^2 + b\alpha_2^2 + ab\alpha_3^2 + c\alpha_4^2 + ac\alpha_5^2 + bc\alpha_6^2 + abc\alpha_7^2.$$

Jacobsthal and Jacobsthal-Lucas numbers which are famous integer sequences satisfy the same recurrence relation except for initial conditions. Namely, Jacobsthal numbers satisfy the recurrence relation

$$J_n = J_{n-1} + 2J_{n-2}$$

with the initial conditions $J_0 = 0$ and $J_1 = 1$. Similarly, Jacobsthal-Lucas numbers satisfy the recurrence relation

$$JL_n = JL_{n-1} + 2JL_{n-2}$$

with the initial conditions $JL_0 = 2$ and $JL_1 = 1$.

Generating functions for the Jacobsthal sequence $\{J_n\}_{n=0}^{\infty}$ and Jacobsthal-Lucas sequence $\{JL_n\}_{n=0}^{\infty}$ are

$$\sum_{n=0}^{\infty} J_n x^n = \frac{x}{1-x-2x^2} \text{ and } \sum_{n=0}^{\infty} JL_n x^n = \frac{2-x}{1-x-2x^2}$$

respectively. The Binet formulas for the Jacobsthal and Jacobsthal-Lucas numbers are

$$J_n = \frac{\mu^n - \upsilon^n}{\mu - \upsilon}$$

and

$$JL_n = \mu^n + \upsilon^n$$

where $\mu = 2$ and $\upsilon = -1$ are solutions of the characteristic equation of $x^2 - x - 2 = 0$.

Horadam [4] defined Fibonacci and Lucas Quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$P_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3$$

respectively, where F_n is the nth Fibonacci number and L_n is the nth Lucas number.

Many researchers worked on these quaternions (for example [3, 5, 7]). Some authors studied on generalizations of Fibonacci and Lucas Quaternions (for example [1, 6, 10, 12]).

Kecilioglu and Akkus [8] defined the Fibonacci and Lucas octonions as

$$Q_n = \sum_{s=0}^{7} F_{n+s}e_s \textit{ and } T_n = \sum_{s=0}^{7} L_{n+s}e_s$$

where F_n and L_n are nth Fibonacci and Lucas numbers. They gave generating function, Binet formulas, and some identities for the Fibonacci and Lucas octonions. Also, they defined Split Fibonacci and Lucas octonions similarly in [2]. Savin [11] studied generalized Fibonacci and

Lucas octonions over the octonion algebras $\mathbb{O}_{\mathbb{R}}(a+1, 2a+1, 3a+1)$ where a is a real number and gave several basic properties for them.

Szynal-Liana and Wloch [13] introduced Jacobsthal quaternion JQ_n and Jacobsthal-Lucas quaternion JLQ_n and defined these quaternions as

$$JQ_n = J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3$$

and

$$JLQ_n = j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3$$

where J_n is the nth Jacobsthal number and j_n is the nth Jacobsthal-Lucas number.

Aydin and Yüce [14] investigated some properties of the Jacobsthal and Jacobsthal-Lucas quaternions. In [16], Tasci defined k-Jacobsthal and k-Jacobsthal-Lucas quaternions. Yasarsoy et. al. [17] introduced a new class of octonions of Jacobsthal and Jacobsthal-Lucas sequences. Furthermore, Aydin [15] gave the generalized Jacobsthal and generalized complex Jacobsthal and generalized dual Jacobsthal sequences.

Cimen and İpek [18] defined the nth Jacobsthal octonion and Jacobsthal Lucas octonion numbers, respectively, by the following recurrence relations;

$$\hat{J}_n = \sum_{s=0}^{7} J_{n+s}e_s$$

and

$$\hat{j}_n = \sum_{s=0}^{7} j_{n+s}e_s.$$

where J_n and j_n are the nth Jacobsthal and Jacobsthal-Lucas numbers.

In this paper, following Horadam, Kecilioglu and Akkus, and Cimen and İpek, we define the Jacobsthal and Jacobsthal-Lucas generalized octonions over the octonion algebra $\mathbb{O}(a, b, c)$. The nth Jacobsthal generalized octonion JGO_n is

$$JGO_n = \sum_{s=0}^{7} J_{n+s}e_s$$

and the nth Jacobsthal-Lucas generalized octonion $JLGO_n$ is

$$JLGO_n = \sum_{s=0}^{7} JL_{n+s}e_s.$$

where J_n is the nth Jacobsthal number and JL_n is the nth Jacobsthal-Lucas number.

2. Binet Formulas and Generalizations for Some Identities

There are three well-known identities for Jacobsthal and Jacobsthal-Lucas numbers, namely, Catalan's, Cassini's, and d'Ocagne's identities. The proofs of these identities are based on Binet formulas. We can obtain these types of identities for Jacobsthal and Jacobsthal-Lucas generalized octonions using the Binet formulas. The following theorem gives Binet formulas for the Jacobsthal and Jacobsthal-Lucas generalized octonions.

Theorem 2.1. *For any integer n, nth Jacobsthal generalized octonion is*

$$JGO_n = \frac{\mu^* \mu^n - \upsilon^* \upsilon^n}{\mu - \upsilon} \tag{2.1}$$

and nth Jacobsthal-Lucas generalized octonion is

$$JLGO_n = \mu^* \mu^n + \upsilon^* \upsilon^n \tag{2.2}$$

where $\mu = 2, \upsilon = -1, \mu^* = \sum\limits_{s=0}^{7} \mu^s e_s$ and $\upsilon^* = \sum\limits_{s=0}^{7} \upsilon^s e_s$.

Proof. Let us consider the following for eq. (2.1)

$$\mu JGO_n + JGO_{n-1} = \sum_{s=0}^{7} \left(\mu J_{n+s} + J_{n+s-1}\right) e_s.$$

By the help of the identity $\mu J_n + \mu^n J_{n-1}$, we get

$$\mu JGO_n + JGO_{n-1} = \mu^* \mu^n. \tag{2.3}$$

Similarly, using the identity $\upsilon^n = \upsilon J_n + J_{n-1}$, we have

$$\upsilon JGO_n + JGO_{n-1} = \upsilon^* \upsilon^n. \tag{2.4}$$

From the eqs. (2.3) and (2.4), we obtain

$$JGO_n = \frac{\mu^* \mu^n - \upsilon^* \upsilon^n}{\mu - \upsilon}.$$

By using similar method, we get Binet formula of Jacobsthal-Lucas generalized octonion $JLGO_n$. □

When using the Binet formulas to obtain identities for the Jacobsthal and Jacobsthal-Lucas generalized octonions, we require $\mu^{*2}, \upsilon^{*2}, \mu^* \upsilon^*$ and $\upsilon^* \mu^*$. These identities play important roles in this paper for calculations.

Lemma 2.2. *We have the following*

$$\begin{aligned} \mu^{*2} &= w_1 + JLGO_0 + 3\left(w_2 + JGO_0\right), & (2.5)\\ \upsilon^{*2} &= w_1 + JLGO_0 - 3\left(w_2 + JGO_0\right), & (2.6)\\ \mu^* \upsilon^* &= E + JLGO_0 + 6F, & (2.7)\\ \upsilon^* \mu^* &= E + JLGO_0 - 6F. & (2.8) \end{aligned}$$

where

$$\begin{aligned} w_1 &= -1-\frac{5}{2}a-\frac{17}{2}b-\frac{65}{2}ab-\frac{257}{2}c \\ &\quad -\frac{1025}{2}ac-\frac{4097}{2}bc-\frac{16385}{2}abc, \\ w_2 &= \frac{1}{2}a-\frac{5}{2}b-\frac{21}{2}ab-\frac{85}{2}c-\frac{341}{2}ac-\frac{1365}{2}bc-\frac{5461}{2}abc, \\ E &= 128abc+8ab+32ac-64bc+2a-4b-16c-1, \\ F &= 2\left(16bc-b-4c\right)e_1+\left(16ac-a-8c\right)e_2+40ce_3 \\ &\quad +5(-4ab-a+2b)e_4+34be_5+17ae_6. \end{aligned}$$

Using the multiplication table for the basis of $\mathbb{O}(a,b,c)$, *we have*

Proof. $\mu^{*2}=\left(\sum\limits_{s=0}^{7}\mu^s e_s\right)\left(\sum\limits_{s=0}^{7}\mu^s e_s\right)$

$$= -1-\tfrac{5}{2}a-\tfrac{17}{2}b-\tfrac{65}{2}ab-\tfrac{257}{2}c-\tfrac{1025}{2}ac-\tfrac{4097}{2}bc-\tfrac{16385}{2}abc+JLGO_0$$

$$+3\left(\tfrac{1}{2}a-\tfrac{5}{2}b-\tfrac{21}{2}ab-\tfrac{85}{2}c-\tfrac{341}{2}ac-\tfrac{1365}{2}bc-\tfrac{5461}{2}abc+JGO_0\right)$$

$$= w_1+JLGO_0+3\left(w_2+JGO_0\right)$$

The last equations is eq. (2.5). Similarly

$$\mu^*\upsilon^*=\left(\sum_{s=0}^{7}\mu^s e_s\right)\left(\sum_{s=0}^{7}\upsilon^s e_s\right)$$

$$= 128abc+8ab+32ac-64c+2a-4b-16c-1$$

$$+JLGO_0+6\left[2\left(16bc-b-4c\right)e_1+\left(16ac-a-8c\right)e_2+40ce_3\right.$$

$$\left.+5\left(-4ab-a+2b\right)e_4+34be_5+17ae_6\right]$$

$$= E+JLGO_0+6F.$$

The last equation is the eq. (2.7). The others can be proved similarly. □

Now we give the Catalan's identities involving the Jacobsthal and Jacobsthal-Lucas generalized octonions in the following theorem.

Theorem 2.3. *For any integers* n *and* r, *we have*

$$\begin{aligned} &JGO_{n+r}JGO_{n-r}-JGO_n^2 \\ &=(-2)^{n-r}\left[\frac{1}{9}\left(E+JLGO_0\right)\left(2\left(-2\right)^r-JL_{2r}\right)\right. \\ &\left.-2FJ_{2r}\right] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&JLGO_{n+r}JLGO_{n-r} - JLGO_n^2 \\
&= (-2)^{n-r}\left[(E + JLGO_0)\left(JL_{2r} - 2(-2)^r\right)\right. \\
&\left.+18FJ_{2r}\right].
\end{aligned} \tag{2.10}$$

Proof. $JGO_{n+r}JGO_{n-r} - JGO_n^2$

$$\begin{aligned}
&= \frac{1}{9}\left[(\mu^*\mu^{n+r} - \upsilon^*\upsilon^{n+r})(\mu^*\mu^{n-r} - \upsilon^*\upsilon^{n-r}) - (\mu^*\mu^n - \upsilon^*\upsilon^n)^2\right] \\
&= \frac{1}{9}\left(\mu^*\upsilon^*\mu^{n+r}\upsilon^{n-r} - \upsilon^*\mu^*\upsilon^{n+r}\mu^{n-r} + \mu^*\upsilon^*\mu^n\upsilon^n + \upsilon^*\mu^*\upsilon^n\mu^n\right) \\
&= \frac{1}{9}\left[-\mu^{n-r}\upsilon^{n-r}\left(\mu^*\upsilon^*\mu^{2r} + \upsilon^*\mu^*\upsilon^{2r}\right) + 2(-2)^n(E + JLGO_0)\right] \\
&= \frac{1}{9}\left[-(-2)^{n-r}\left((E + JGLO_0)\left(\mu^{2r} + \upsilon^{2r}\right) + 6F\left(\mu^{2r} - \upsilon^{2r}\right)\right)\right. \\
&\left.+2(-2)^n(E + JLGO_0)\right] \\
&= \frac{1}{9}\left[(-2)^{n-r}(E + JLGO_0)\left(2 - (-2)^{-r}JL_{2r}\right)\right. \\
&\left.+18(-2)^{n-r}FJ_{2r}\right] \\
&= (-2)^{n-r}\left[\frac{1}{9}(E + JLGO_0)\left(2(-2)^r - JL_{2r}\right) - 2FJ_{2r}\right].
\end{aligned}$$

The second identity in the theorem, i.e., Catalan's identity for the Jacobsthal-Lucas generalized octonion, can be proved similarly. □

For $r = 1$, Theorem 2.3 gives Cassini's identities for Jacobsthal and Jacobsthal-Lucas generalized octonions which are given in the following.

Corollary 2.4. *For any integer n, we have*

$$JGO_{n+1}JGO_{n-1} - JGO_n^2 = -(-2)^{n-1}\left[(E + JLGO_0) - 2F\right] \tag{2.11}$$

and

$$\begin{aligned}
&JLGO_{n+1}JLGO_{n-1} - JLGO_n^2 \\
= \ &(-2)^{n-1}\left[9(E + JLGO_0) + 18F\right].
\end{aligned} \tag{2.12}$$

D'Ocagne's identities for Jacobsthal and Jacobsthal-Lucas generalized octonions are given in the next theorem.

Theorem 2.5. *For any integers n and m, we have*

$$\begin{aligned}
&JGO_mJGO_{n+1} - JGO_{m+1}JGO_n \\
= \ &(-2)^n\left[(E + JLGO_0)J_{m-n} + 2FJL_{m-n}\right]
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned} & JLGO_m JLGO_{n+1} - JLGO_{m+1} JLGO_n \\ = & -(-2)^n \left[9\left(E + JLGO_0\right) J_{m-n} + 18FJL_{m-n}\right] \end{aligned} \tag{2.14}$$

Proof. Using the Binet formula for the Jacobsthal generelized octonions, we obtain

$$\begin{aligned} & JGO_m JGO_{n+1} - JGO_{m+1} JGO_n \\ & = \frac{1}{9}\left(\mu^* \mu^m - \upsilon^* \upsilon^m\right)\left(\mu^* \mu^{n+1} - \upsilon^* \upsilon^{n+1}\right) \\ & -\frac{1}{9}\left(\mu^* \mu^{m+1} - \upsilon^* \upsilon^{m+1}\right)\left(\mu^* \mu^n - \upsilon^* \upsilon^n\right) \\ & = \frac{1}{3}\left[(-2)^n \left(\mu^* \upsilon^* \mu^{m-n} - \upsilon^* \mu^* \upsilon^{m-n}\right)\right]. \end{aligned}$$

If we substitute equations (2.7) and (2.8) into the last equation, then we have

$$\begin{aligned} & JGO_m JGO_{n+1} - JGO_{m+1} JGO_n \\ & = \frac{1}{3}(-2)^n \left[3\left(E + JLGO_0\right) J_{m-n} + 6FJL_{m-n}\right] \\ & = (-2)^n \left[\left(E + JLGO_0\right) J_{m-n} + 2FJL_{m-n}\right]. \end{aligned}$$

The proof of the second identity can be done similarly by using the Binet formula in equation (2.2). □

3. Some Results for Jacobsthal and Jacobsthal-Lucas Generalized Octonions

In this section, after deriving famous three identities Catalan's, Cassini's and d'Ocagne's, we present some other identities for the Jacobsthal and Jacobsthal-Lucas generalized octonions.

Theorem 3.1. *Jacobsthal and Jacobsthal-Lucas generalized octonions satisfy the following identities*

$$\begin{aligned} & JGO_n^2 + JLGO_n^2 \\ = & \frac{10}{9}\left[\left(w_1 + JLGO_0\right) JL_{2n} + 9\left(w_2 + JGO_0\right) J_{2n}\right] \\ & +\frac{16}{9}(-2)^n \left(E + JLGO_0\right) \end{aligned} \tag{3.1}$$

$$\begin{aligned} & JGO_n^2 - JLGO_n^2 \\ = & -\frac{8}{9}\left[\left(w_1 + JLGO_0\right) JL_{2n} + 9\left(w_2 + JGO_0\right) J_{2n}\right] \\ & -\frac{20}{9}(-2)^n \left(E + JLGO_0\right) \end{aligned} \tag{3.2}$$

$$\begin{aligned} & JGO_{n+r}JLGO_{n+s} - JGO_{n+s}JLGO_{n+r} \\ = \; & -(-2)^{n+r+1}\left(E + JLGO_0\right) J_{s-r}, \end{aligned} \tag{3.3}$$

$$JLGO_{m+n} + (-1)^n JLGO_{m-n} = JL_n JLGO_n, \tag{3.4}$$

$$\begin{aligned} & JLGO_m JGO_n - JLGO_n JGO_m \\ = \; & 2\left(-2\right)^m \left(E + JLGO_0\right) J_{n-m}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} & JLGO_m JGO_n - JGO_m JLGO_n \\ = \; & -(-2)^{m+1}\left[\left(E + JLGO_0\right) J_{n-m} - 2FJL_{n-m}\right] \end{aligned} \tag{3.6}$$

Proof. We prove the second and fifth identities. We need the Binet formulas for the Jacobsthal and Jacobsthal-Lucas generelized octonion.

$$JGO_n^2 - JLGO_n^2 = \frac{1}{9}\left(\mu^*\mu^n - \upsilon^*\upsilon^n\right)\left(\mu^*\mu^n - \upsilon^*\upsilon^n\right)$$

$$-\left(\mu^*\mu^n + \upsilon^*\upsilon^n\right)\left(\mu^*\mu^n + \upsilon^*\upsilon^n\right)$$

$$= -\frac{8}{9}\left[\mu^{*2}\mu^{2n} + \upsilon^*\upsilon^{2n}\right] - \frac{10}{9}\left(-2\right)^n\left[\mu^*\upsilon^* + \upsilon^*\mu^*\right]$$

$$= -\tfrac{8}{9}\left[\left(w_1 + JLGO_0\right)\left(\mu^{2n} + \upsilon^{2n}\right) + 3\left(w_2 + JGO_0\right)\left(\mu^{2n} - \upsilon^{2n}\right)\right]$$

$$-\frac{10}{9}\left(-2\right)^n 2\left(E + JLGO_0\right)$$

$$= -\frac{8}{9}\left[\left(w_1 + JLGO_0\right) JL_{2n} + 9\left(w_2 + JGO_0\right) J_{2n}\right]$$

$$-\tfrac{20}{9}\left(-2\right)^n\left(E + JLGO_0\right)$$

Similarly, using Binet formulas again, we get

$$JLGO_m JGO_n - JLGO_n JGO_m = \tfrac{1}{3}\left(\mu^*\mu^m + \upsilon^*\upsilon^m\right)\left(\mu^*\mu^n - \upsilon^*\upsilon^n\right)$$

$$-\frac{1}{3}\left(\mu^*\mu^n + \upsilon^*\upsilon^n\right)\left(\mu^*\mu^n - \upsilon^*\upsilon^n\right)$$

$$= -\frac{1}{3}\left[\mu^m\upsilon^m\left(\mu^*\upsilon^* + \upsilon^*\mu^*\right) + \upsilon^m\mu^m\left(\upsilon^*\mu^* + \mu^*\upsilon^*\right)\right]$$

$$= \frac{2}{3}\left(-2\right)^m\left(E + JLGO_0\right)\left(\mu^{n-m} - \upsilon^{n-m}\right)$$

$$= 2\left(-2\right)^m\left(E + JLGO_0\right) J_{n-m}$$

The other identities can be proved similarly from the Binet formulas in equations (2.1) and (2.2). □

Since the algebra $\mathbb{O}(a, b, c)$ is non-commutative, then it can be seen what changes in the following theorem.

Theorem 3.2. *For any integers m and n, we have*

$$JGO_nJGO_m - JGO_mJGO_n = -4\,(-2)^m\,FJ_{n-m} \tag{3.7}$$

and

$$JLGO_nJLGO_m - JLGO_mJLGO_n = 36\,(-2)^m\,FJ_{n-m}. \tag{3.8}$$

Proof. From the Binet formula in equation (2.1) given

$$JGO_nJGO_m - JGO_mJGO_n = \frac{1}{9}\left(\mu^*\mu^n - \upsilon^*\upsilon^n\right)\left(\mu^*\mu^m - \upsilon^*\upsilon^m\right)$$

$$-\frac{1}{9}\left(\mu^*\mu^m - \upsilon^*\upsilon^m\right)\left(\mu^*\mu^n - \upsilon^*\upsilon^n\right)$$

$$= -\frac{1}{9}\left[-\mu^*\upsilon^*\mu^n\upsilon^m - \upsilon^*\mu^*\upsilon^n\mu^m + \mu^*\upsilon^*\mu^m\upsilon^n + \upsilon^*\mu^*\upsilon^m\mu^n\right]$$

$$= \frac{1}{9}\left[-\left(\mu^*\upsilon^* - \upsilon^*\mu^*\right)\left(\mu^n\upsilon^m - \upsilon^n\mu^m\right)\right]$$

$$= -\frac{4}{3}F\left[\mu^m\upsilon^m\left(\mu^{n-m} - \upsilon^{n-m}\right)\right]$$

$$= -4\,(-2)^m\,FJ_{n-m}$$

Eq. (3.8) can be proved similarly by using the Binet formula in equation (2.2). □

Corollary 3.3. *From Eq.*(3.7) *and Eq.*(3.8)*, it is clearly that*

$$JLGO_nJLGO_m - JLGO_mJLGO_n = -9\left(JGO_nJGO_m - JGO_mJGO_n\right).$$

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