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Ritz approximation for the fractional optimal control problems

Mohammad Arab Firoozjaee [1]
Department of Mathematics, University of Science and Technology of Mazandaran, Behshahr, Iran

Abstract. The manuscript deals of the fractional optimal control problems (FOCPs) based on the Caputo fractional derivative by the ritz method. To use this method, we transform the FOCPs into an optimization problem and obtain the system of nonlinear algebraic equations. By polynomial basis functions, we approximate solutions. Then we have the coefficients of polynomial expansions by solving the system of nonlinear equations. numerical examples are presented which illustrate the performance of the method.

Keywords: Fractional Optimal Control Problems, Caputo fractional derivative, Optimal Control Problems, Polynomial basis functions.

2000 Mathematics subject classification: 35R10; Secondary 49L20.

1. Introduction

In dynamical systems, Optimal control theory allows us to choose control functions to attain a certain result. The theory generalizes that of the classical calculus of variations and has found many applications in engineering, physics, economics, and life sciences [10, 24]. In 1990 decade , motivated by nonconservative physical processes in mechanics, the subject was extended to the case in which derivatives and integrals are understood as fractional operators of arbitrary order[5]. Fractional

[1]Corresponding author: m_firoozjaee@mazust.ac.ir
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optimal control is nowadays an important research area, allowing to apply the power of variational methods to real systems [7]. The modeling of many phenomenons leads to a set of fractional differential equations. Also, fractional calculs appear in several problems in engineering and science such as, bioengineering [25], viscoelasticity [8, 9], dynamics of interfaces among nanoparticles and substrates [13], etc. Interested researcher in fractional differential equations and fractional calculus can study [14, 21, 19]. When the fractional differential equations are used in conjunction with the performance index and a set of initial conditions, they direct to FOCPs [1]. Nowadays FOCPs is an important research area, allowing to use the power of variational methods to real systems [6, 7]. Riemann—Liouville and Caputo fractional derivatives are the most important types of fractional derivatives but a FOCP can be defined with respect to different definitions of fractional derivative. For FOCPs, general necessary conditions of optimality have been developed. In [2], Hamiltonian formulas for FOCPs with the Riemann–Liouville fractional derivative considered. In [4], such formulas are achieved for FOCPs with the Caputo fractional derivative. the Hamiltonian system of equations provides necessary conditions of optimization. For find the optimal solution of a FOCP, the Hamiltonian system can be solved because an optimal solution of the FOCP should satisfy the system [2, 4]. In the Hamiltonian system of equations there exist both right and left fractional differential operators which makes it so hard to find exact solution for the system. In [11, 27, 28], there exist some numerical simulations for FOCPs with Riemann-–Liouville fractional derivatives. Also, there exist numerical simulations for FOCPs with the Caputo fractional derivative such as [4, 3], where the researcher by solving the Hamiltonian equations approximately has solved the problem . A linear quadratic FOCP is solved directly without using Hamiltonian formulas [23]. In this article, we consider a class of optimal control problems with the Caputo fractional derivative in a dynamical system. The numerical approach developed and applied to solve different classes of FOCPs are generally based on two major approaches: an approximate solution of the Pontryagin's system of equations or a direct approximation of the FOCP without need to necessary conditions of optimality. The approachs based on the first solution are called indirect methods, whereas the methods of second approach are categorized as direct methods. Examples of indirect methods can be found in [1, 3]. The Ritz approximation is one of the most widely used direct methods for solving variational problems approximately. The idea consists of reducing the problem of searching for the extremum of a functional in the function space to the problem of optimizing a real–valued multivariate function [15]. The Ritz direct

method is based on the approximation of unknown functions by a linear combination of iofinite basis functns. In the Ritz method, various types of basis functions can be used. In [16, 18], respectively, Walsh and Haar wavelets are used as basis functions. Despite the wide range of applicability of the Ritz method, its practical application is often limited by the so–called "curse of dimensionality"[20]. Accordingly, the number of basis functions should be huge to give an acceptable numerical and approximate solutions for the problem.
The manuscript is arranged as follows: section 2, defines ritz approximation for the FOCPs. In section 3, the proposed technique is utilized for some examples. Finally, section 4 concludes the paper.

2. Solution of The Fractional Optimal Control Problem

In this article, we consider the fractional optimal control problems in the following. Let $0 < \alpha < n$ and let H, g: $[b, +\infty[\, \times R^2 \to R$ be two differentiable functions. Consider the following FOCP:

$$minimize I(y,u,T) = \int_0^1 H(t, y(t), u(t))dt, \tag{2.1}$$

subject to the dynamic system

$$K D_t^{\alpha} y(t) = g(t, y(t), u(t)), \tag{2.2}$$

where the initial conditions are as follows:

$$y(0) = y_0,\ y'(0) = y_1, \tag{2.3}$$

where K, y_a are fixed real numbers. fractional derivatives are taken in the Caputo sense that defind[12]

$$D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(\tau)}{(t-\tau)^{-n+\alpha+1}} d\tau$$

The method consists of conversion fractional optimal control problem to optimization problem and expanding the solution by polynomial basis functions with unknown coefficients.

We approximate $y(t)$ as

$$y(t) \cong y_m(t) = \sum_{i=0}^{l} a_i t^2 \psi_i(t) + w(t), \tag{2.4}$$

and $u(t)$ obtain of Eq. (2.2). where $\psi_i(t)$ are polynomial basis functions and a_i are unknown coefficients. In following, we determine $w(t)$ as $w(0) = y_0$ and $w'(0) = y_1$.

Now we have the following optimal problem

$$I[a_0, a_1, ..., a_l] = \int_0^1 H(t, \tilde{y}(t), \tilde{u}(t))dt, \tag{2.5}$$

If $a_i$are decided by the optimizing function I, then by (2.4), we obtain functions which approximate the optimum value of I in (2.5). To find unknowns a_i, $i = 0, 1, ..., l$ in $\widetilde{y}(t)$, according to the necessary conditions of optimization for (2.5), we have

$$\frac{\partial I}{\partial a_i} = 0, \;\; i = 0, ..., l. \tag{2.6}$$

Then by solving the above system of l algebraic equations (2.6), we obtain a_i, $i = 0, 1, ..., l$. The approached demonstrated here relies on the Ritz method. Then with solving this problem by mathematica software, we obtain a_i. The method presented here is based on the Ritz method. For more study, we refer the interested researcher to [14].

3. Illustrative examples

To demonstrate the effectiveness of the method, here we consider fractional optimal control problems. The following examples demonstrate that the desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative example show that this method in comparison to other methods has high accuracy and is easily implemented.

Example 3.1. Consider the nonlinear FOCP[26]

$$\begin{aligned} \min I(y,u) = \textstyle\int_0^1 \Big(& e^t \left(-t^4 + y(t) + t - 1\right)^2 + \\ & \left(t^2+1\right)\left(-10.9209t^{21/10} + t^4 + u(t) - t + 1\right)^2 \Big)\, dt, \end{aligned} \tag{3.1}$$

subject to the dynamical system

$$D^{1.9}y(t) = u(t) + y(t), \tag{3.2}$$

with initial conditions: $y(0) = 1, y'(0) = -1$.
The state and control functions that minimize the performance index I are given by

$$\begin{aligned} y(t) &= 1 - t + t^4, \\ u(t) &= -1 + t - t^4 + \tfrac{8000}{77\Gamma(\frac{1}{10})} t^{\frac{21}{10}}, \end{aligned} \tag{3.3}$$

that are exact solutions and optimal value is $I_{opt} = 0$. We used the Ritz approximation that in section 2 explained. In first, we determine

$$y_l(t) = \sum_{i=0}^{l} a_i t^{i+2} + 1 - t + t^2. \quad (3.4)$$

By solved system (2.6) with different value of l, we obtain

$$a_0 = -1, a_1 = -2.62 \times 10^{-6}, a_2 = 1, a_3 = 0, a_4 = 9.82 \times 10^{-6}, a_5 = 2.75 \times 10^{-6}.$$

To replace this coefficients in (3.4), we obtain

$$\begin{aligned} y_5(t) &= -2.753 \times 10^{-6}t^7 + 9.825 \times 10^{-6}t^6 - 0.00001t^5 + 1.00001t^4 - 2.629 \times 10^{-6}t^3 \\ &+ 2.897 \times 10^{-7}t^2 - t + 1 + ... \cong t^4 - t + 1, \end{aligned}$$

Which obviously converges to the the exact solution.
By applying our method with different values of l, we obtain the numerical results. Fig. 1 shows the absolute error of this problem obtained by the present method with $l = 5$. From Fig. 1, we can see that the present method provides accurate results.

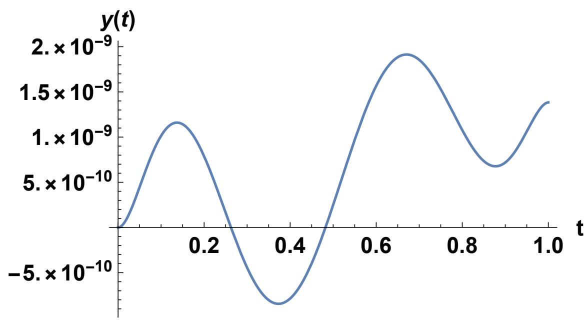

Fig.1. The absolute error between exact and numerical solution for $l = 5$.

The following table shows the values of minimum I for different values of approximations.

	$l=1$	$l=3$	$l=5$
I	0.720698	1.92134×10^{-12}	1.89784×10^{-12}

Example 3.2. Consider the nonlinear FOCP[17]

$$\min I(x,u) = \int_0^1 \left(y(t) - t^2\right)^2 + \left(u(t) + t^4 - \frac{2t^{\frac{9}{10}}}{\Gamma(3-\alpha)}\right)^2 dt, \quad (3.5)$$

subject to the dynamical system

$$D^\alpha y(t) = u(t) + t^2 y(t), \quad (3.6)$$

with initial conditions: $y(0) = 0, y'(0) = 0$.
In this example $0 < \alpha < 2$ and the minimizing solutions for the state and control variables are the functions: $y(t) = t^2$ and $u(t) = \frac{2t^{\frac{9}{10}}}{\Gamma(3-\alpha)} - t^4$ respectively.The performance index I has the minimum value of 0. We used the Ritz approximation that in section 2 explained. In first, we determine

$$y_l(t) = \sum_{i=0}^{l} a_i t^{i+2}, \tag{3.7}$$

By solved system (2.6) with value of $l = 7$ and $\alpha = 1.1$, we obtain

$$\begin{aligned} a_0 &= 1, a_1 = -1.30508 \times 10^{-8}, a_2 = 7.8926 \times 10^{-8}, \\ a_3 &= -2.47662 \times 10^{-8}, a_4 = 4.39582 \times 10^{-7}, a_5 = -4.45345 \times 10^{-7}, \\ a_6 &= 2.40267 \times 10^{-7}, a_7 = -5.35796 \times 10^{-8}. \end{aligned}$$

To replace this coefficients in (3.15), we obtain

$$\begin{aligned} y_7(t) &= t^2 - 4.2443 \times 10^{-12} t^3 + 1.5333 \times 10^{-11} t^4 - 2.565 \times 10^{-11} t^5 \\ &+ 2.0201 \times 10^{-11} t^6 - 6.0668 \times 10^{-12} t^7, \end{aligned}$$

Which obviously converges to the the exact solution. If we choose other α, we will achieve the same result.

Example 3.3. Consider the following nonlinear problem[22]

$$\min I(y,u) = \int_0^1 \left[(y(t) - t^{\frac{5}{2}})^4 + (1+t^2)(u(t) + t^6 - \frac{15\sqrt{\pi}t}{8})^2 \right] dt, \tag{3.8}$$

subject to the dynamical system

$$D^{1.5} y(t) = u(t) + t y^2(t), \tag{3.9}$$

with initial conditions: $y(0) = 0, y'(0) = 0$.
The state function that minimize the performance index I are given by

$$y(t) = t^{\frac{5}{2}}, \tag{3.10}$$

that are exact solutions and optimal value is $I_{opt} = 0$. We used the Ritz approximation that in section 2 explained. In first, we determine

$$y_l(t) = \sum_{i=0}^{l} a_i t^{i+2}, \tag{3.11}$$

By solved system (2.6) with different value of l, we obtain a_i. For example, with $l = 5$, we have

$$\begin{aligned} a_0 &= 0.175537, a_1 = 1.70658, a_2 = -2.08146, a_3 = 2.14001, \\ a_4 &= -1.23854, a_5 = 0.298111. \end{aligned}$$

To replace this coefficients in (3.15), we obtain

$$\begin{aligned} y_5(t) &= 0.175537t^2 + 1.70658t^3 - 2.08146t^4 + 2.14001t^5 \\ &- 1.23854t^6 + 0.298111t^7, \end{aligned}$$

By applying our method with different values of l, we obtain the numerical results. Fig. 2 shows the absolute error of this problem obtained by the present method with $l = 5$. From Fig. 2, we can see that the present method provides accurate results.

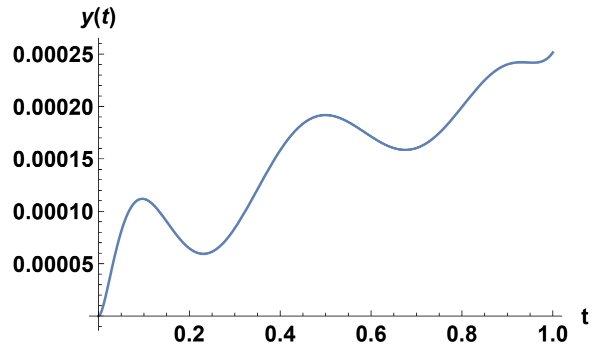

Fig.2. The absolute error between exact and numerical solution for $l = 5$.

The following table shows the values of minimum I for different values of approximations.

	$l = 1$	$l = 3$	$l = 5$
I	0.00326428	0.0000777	7.85652×10^{-6}

Example 3.4. Consider the nonlinear FOCP[17]

$$\min I(x, u) = \frac{1}{2} \int_0^1 \left(3y^2(t) - u^2(t)\right) dt, \tag{3.12}$$

subject to the dynamical system

$$\frac{1}{4} y'(t) + \frac{3}{4} D^\alpha y(t) = y(t) - u(t), \tag{3.13}$$

with initial conditions: $y(0) = 1, y'(0) = \frac{4e^2}{3+e^4}$.
For $\alpha = 1$, state functions that minimize the performance index I are given by

$$\begin{aligned} y(t) &= \tfrac{3}{3+e^4} e^{2t} + \tfrac{e^4}{3+e^4} e^{-2t}, \\ u(t) &= \tfrac{3e^4}{3+e^4} e^{-2t} - \tfrac{3e^4}{3+e^4} e^{2t}, \end{aligned} \tag{3.14}$$

that are exact solutions and optimal value is $I_{opt} = 1.39583$. We used the Ritz approximation that in section 2 explained. In first, we determine

$$y_l(t) = \sum_{i=0}^{l} a_i \psi_i(t) t^2 + (1 - t^2) + \frac{4e^2}{3 + e^4} t, \tag{3.15}$$

that $\psi_i(t)$ are legendre polynomials. By solved system (2.6) with different value of l, we obtain a_i. To replace coefficients in (3.15), we obtain approximation solution. By applying our method with different values of l, we obtain the numerical results. Fig. 3 shows the absolute error of this problem obtained by the present method with $l = 13$ and $\alpha = 1$. From Fig. 3, we can see that the present method provides accurate results. In Fig. (4), we represent the numerical solutions of $y(t)$ for $\alpha = 0.5, 0.7, 0.9, 1$ with $l = 3$ in comparison with the exact solutions.

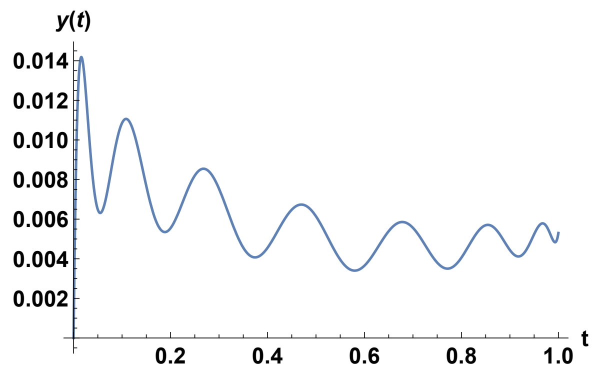

Fig.3. The absolute error between exact and numerical solution for $l = 13$.

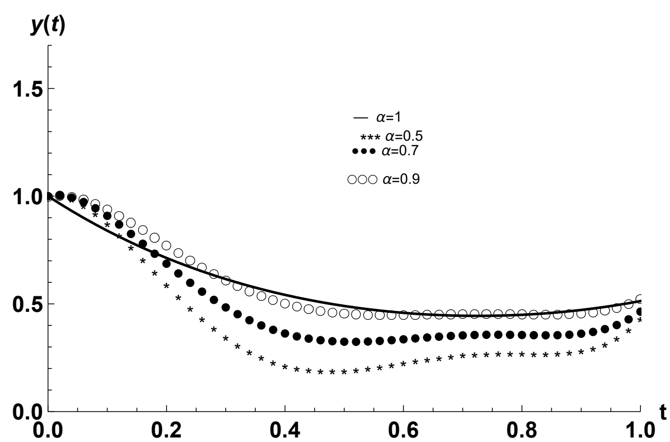

Fig.4. Approximate solution for $\alpha = 0.5, 0.7, 0.9$ with $l = 3$ and Exact solution(—).

The following table shows the values of minimum I for different values of approximations. This table shows that the values obtained by increasing l to the optimal value is convergent.

	$l=7$	$l=10$	$l=13$
I	1.42941	1.41452	1.40773

4. Conclusion

In this manuscript, we apply an effective and simple technique for solving a wide class of the FOCPs. The present approximate solutions can be determined by using solving the resulting system of equations, which can be effectively computed using symbolic computing codes. This method can also be used to other FOCPs. To represent that this technique has high performance and is easily implemented, we have solved some illustrative examples.

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