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Shape Loop Space of Pro-discrete Spaces

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Abstract. In this paper, considering the kth shape loop space $\check{\Omega}_k^{\mathbf{p}}(X,x)$, for an HPol$_*$-expansion $\mathbf{p}:(X,x)\to((X_\lambda,x_\lambda),[p_{\lambda\lambda'}],\Lambda)$ of a pointed topological space (X,x), first we prove that $\check{\Omega}_k$ commutes with the product under some conditions and then we show that $\check{\Omega}_k^{\mathbf{p}}(X,x)\cong\varprojlim\check{\Omega}_k^{\mathbf{p}}(X_i,x_i)$, for a pro-discrete space $(X,x)=\varprojlim(X_i,x_i)$ of compact polyhedra. Finally, we conclude that these spaces are metric, second countable and separable.

Keywords: Shape theory, Inverse limit, Loop space
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1. Introduction

Cuchillo-Ibanez et al. [2] introduced a topology on the set of shape morphisms between arbitrary topological spaces X, Y, $Sh(X,Y)$. Moszyňska [6] showed that for a compact Hausdorff space (X,x), the kth shape group $\check{\pi}_k(X,x)$, $k\in\mathbb{N}$, is isomorphic to the set $Sh((S^k,*),(X,x))$ and Bilan [1] mentioned that the result can be extended for all topological spaces. Nasri et al. [11], considering the latter topology on the set of shape morphisms between pointed spaces, obtained a topology on the shape homotopy groups of arbitrary spaces, denoted by $\check{\pi}_k^{top}(X,x)$ and showed that with this topology, the kth shape group $\check{\pi}_k^{top}(X,x)$ is a Hausdorff topological group, for all $k\in\mathbb{N}$. Nasri et al. [11] introduced

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the kth shape loop space $\check{\Omega}_k^{\mathbf{p}}(X,x)$ as a subspace of $\prod_{\lambda\in\Lambda}\Omega^k(X_\lambda,x_\lambda)$, where $\Omega^k(X_\lambda,x_\lambda)$ is the set of all mappings $(S^k,*)\longrightarrow(X_\lambda,x_\lambda)$ endowed with compact-open topology. Also, they considered the quotient topology $q_{\mathbf{p}}:\check{\Omega}_k^{\mathbf{p}}(X,x)\to\check{\pi}_k(X,x)$ on the kth shape group. Then they showed that this quotient topology on the kth shape group coincides with the topology of $\check{\pi}_k^{top}$. In [9], Nasri and Mashayekhy showed that $\check{\Omega}_k^{\mathbf{p}}(X,x)$ is an H-group for every topological space (X,x) and every HPol$_*$-expansion $\mathbf{p}:(X,x)\to((X_\lambda,x_\lambda),[p_{\lambda\lambda'}],\Lambda)$. Also, they proved that $\check{\Omega}_k^{\mathbf{c}}:Top_*\longrightarrow Top_*$ is a functor, for all $k\in\mathbb{N}\cup\{0\}$, where $\mathbf{c}$ is the Čech HPol$_*$-expansion of spaces and then they showed that this functor preserves the homotopy on compact Hausdorff spaces. It is well-known that if the cartesian product of two spaces X and Y admits an Hpol-expansion, which is the cartesian product of Hpol-expansions of these space, then $X\times Y$ is a product in the shape category [7]. In this case, Nasri et al. [10] proved that the shape groups and the coarse shape groups commute with the product. Also, Nasri et al. [11] showed that topological kth shape group $\check{\pi}_k^{top}$ of compact Hausdorff spaces commutes with finite products, for all $k\in\mathbb{N}$. In this paper, we prove that if the Cartesian product of two pointed topological spaces X and Y admits an HPol$_*$-expansion, which is the Cartesian product of HPol$_*$-expansions of these spaces, then $\check{\Omega}_k$ commutes with the product. As a consequence we show that if (X,x) and (Y,y) are two pointed compact Hausdorff spaces with HPol$_*$-expansions $\mathbf{p}:(X,x)\to((X_\lambda,x_\lambda),[p_{\lambda\lambda'}],\Lambda)$ and $\mathbf{q}:(Y,y)\to((Y_\lambda,y_\lambda),[q_{\lambda\lambda'}],\Lambda)$, then

$$\check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X\times Y,(x,y))\cong\check{\Omega}_k^{\mathbf{p}}(X,x)\times\check{\Omega}_k^{\mathbf{q}}(Y,y).$$

Also, we prove that $\check{\Omega}_k^{\mathbf{p}}(X,x)\cong\varprojlim\check{\Omega}_k^{\mathbf{p}}(X_i,x_i)$, for a pro-discrete $(X,x)=\varprojlim(X_i,x_i)$ of compact polyhedra. Then we show that these spaces are metric, second countable and separable.

2. Preliminaries

In this section, we recall some of the main notions concerning the shape category and pro-HTop (see [8]).

Let $\mathbf{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)$ and $\mathbf{Y}=(Y_\mu,q_{\mu\mu'},M)$ be two inverse systems in HTop. A *pro-morphism* of inverse systems, $(f,f_\mu):\mathbf{X}\to\mathbf{Y}$, consists of an index function $f:M\to\Lambda$ and of mappings $f_\mu:X_{f(\mu)}\to Y_\mu$, $\mu\in M$, such that for every related pair $\mu\le\mu'$ in M, there exists a $\lambda\in\Lambda$, $\lambda\ge f(\mu),f(\mu')$ so that,

$$q_{\mu\mu'}f_{\mu'}p_{f(\mu')\lambda}\simeq f_\mu p_{f(\mu)\lambda}.$$

The *composition* of two pro-morphisms $(f, f_\mu) : \mathbf{X} \to \mathbf{Y}$ and $(g, g_\nu) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ is also a pro-morphism $(h, h_\nu) = (g, g_\nu)(f, f_\mu) : \mathbf{X} \to \mathbf{Z}$, where $h = fg$ and $h_\nu = g_\nu f_{g(\nu)}$. The *identity pro-morphism* on $\mathbf{X}$ is the pro-morphism $(1_\Lambda, 1_{X_\lambda}) : \mathbf{X} \to \mathbf{X}$, where 1_Λ is the identity function. A pro-morphism $(f, f_\mu) : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to a pro-morphism $(f', f'_\mu) : \mathbf{X} \to \mathbf{Y}$, denoted by $(f, f_\mu) \sim (f', f'_\mu)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$ such that $\lambda \geq f(\mu), f'(\mu)$ and

$$f_\mu p_{f(\mu)\lambda} \simeq f'_\mu p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation. The *category* pro-HTop has as objects all inverse systems $\mathbf{X}$ in HTop and as morphisms all equivalence classes $\mathbf{f} = [(f, f_\mu)]$. The composition of $\mathbf{f} = [(f, f_\mu)]$ and $\mathbf{g} = [(g, g_\nu)]$ in pro-HTop is well defined by putting

$$\mathbf{gf} = \mathbf{h} = [(h, h_\nu)].$$

An HPol-expansion of a topological space X is a morphism $\mathbf{p} : X \to \mathbf{X}$ of pro-HTop, where $\mathbf{X}$ belongs to pro-HPol characterized by the following two properties:
(E1) For every $P \in HPol$ and every map $h : X \to P$ in Top, there is a $\lambda \in \Lambda$ and a map $f : X_\lambda \to P$ such that $fp_\lambda \simeq h$.
(E2) If $f_0, f_1 : X_\lambda \to P$ satisfy $f_0 p_\lambda \simeq f_1 p_\lambda$, then there exists a $\lambda' \geq \lambda$ such that $f_0 p_{\lambda\lambda'} \simeq f_1 p_{\lambda\lambda'}$.

Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{p'} : X \to \mathbf{X'}$ be two HPol-expansions of the same space X in HTop, and let $\mathbf{q} : Y \to \mathbf{Y}$ and $\mathbf{q'} : Y \to \mathbf{Y'}$ be two HPol-expansions of the same space Y in HTop. Then there exist two natural isomorphisms $\mathbf{i} : \mathbf{X} \to \mathbf{X'}$ and $\mathbf{j} : \mathbf{Y} \to \mathbf{Y'}$ in pro-HTop. A morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is said to be *equivalent* to a morphism $\mathbf{f'} : \mathbf{X'} \to \mathbf{Y'}$, denoted by $\mathbf{f} \sim \mathbf{f'}$, provided the following diagram in pro-HTop commutes:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{i}} & \mathbf{X'} \\ \Big\downarrow \mathbf{f} & & \mathbf{f'} \Big\downarrow \\ \mathbf{Y} & \xrightarrow{\mathbf{j}} & \mathbf{Y'}. \end{array} \tag{2.1}$$

Now, the *shape category* Sh is defined as follows: The objects of Sh are topological spaces. A morphism $F : X \to Y$ is the equivalence class $< \mathbf{f} >$ of a mapping $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-HTop.

The *composition* of $F =< \mathbf{f} >: X \to Y$ and $G =< \mathbf{g} >: Y \to Z$ is defined by representatives, i.e., $GF =< \mathbf{gf} >: X \to Z$. The *identity shape morphism* on a space X, $1_X : X \to X$, is the equivalence class $< 1_{\mathbf{X}} >$ of the identity morphism $1_{\mathbf{X}}$ in pro-HTop.

Let $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ be HPol-expansions of X and Y, respectively. Then for every morphism $f : X \to Y$ in HTop, there is a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-HTop such that the following diagram commutes in pro-HTop.

$$\begin{array}{ccc} \mathbf{X} & \xleftarrow[\mathbf{p}]{} & X \\ \Big\downarrow \mathbf{f} & & f \Big\downarrow \\ \mathbf{Y} & \xleftarrow[\mathbf{q}]{} & Y. \end{array} \tag{2.2}$$

If we take other HPol-expansions $\mathbf{p}' : X \to \mathbf{X}'$ and $\mathbf{q}' : Y \to \mathbf{Y}'$, we obtain another morphism $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ in pro-HTop such that $\mathbf{f}'\mathbf{p}' = \mathbf{q}'f$ and so we have $\mathbf{f} \sim \mathbf{f}'$. Hence every morphism $f \in HTop(X, Y)$ yields an equivalence class $< [\mathbf{f}] >$, i.e., a shape morphism $F : X \to Y$ which is denoted by $\mathcal{S}(f)$. If we put $\mathcal{S}(X) = X$ for every topological space X, then we obtain a functor $\mathcal{S} : HTop \to Sh$, called the *shape functor*.

Similarly, we can define the categories pro-HTop$_*$ and Sh$_*$ on pointed topological spaces (see [8]).

3. Shape loop space of pro-discrete

Recall that for an HPol$_*$-expansion $\mathbf{p} : (X, x) \to ((X_\lambda, x_\lambda), [p_{\lambda\lambda'}], \Lambda)$ of a pointed topological space (X, x), a *k-shape loop* in X is defined as an element $(a_\lambda) \in \prod_{\lambda\in\Lambda} \Omega^k(X_\lambda, x_\lambda)$ such that $p_{\lambda\lambda'}(a_{\lambda'}) \simeq a_\lambda$, for all $\lambda' \geq \lambda$. The set of all k-shape loops in X is called the *kth shape loop space* and it is denoted by $\check{\Omega}_k^{\mathbf{p}}(X, x)$. Then $\check{\Omega}_k^{\mathbf{p}}(X, x)$ is a topological space as a subspace of $\prod_{\lambda\in\Lambda} \Omega^k(X_\lambda, x_\lambda)$ [11]. In [9] Nasri and Mashayekhy proved that $\check{\Omega}_k^{\mathbf{c}} : Top_* \longrightarrow Top_*$ is a functor, for all $k \in \mathbb{N}_0$, where $\mathbf{c}$ is the Čech HPol$_*$-expansion of spaces. Now, we intend to prove that $\check{\Omega}_k^{\mathbf{c}}$ commutes with the product under some conditions. First, we need the following results.

Lemma 3.1. *Let (X, x) and (Y, y) admit HPol$_*$-expansions $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) - ((X_\lambda, x_\lambda), [p_{\lambda\lambda'}], \Lambda)$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y}) = ((Y_\lambda, y_\lambda), [q_{\lambda\lambda'}], \Lambda)$, respectively and let $f : (X, x) \longrightarrow (Y, y)$ be a continuous map represented by $[(f_\lambda)] : (\mathbf{X}, \mathbf{x}) \longrightarrow (\mathbf{Y}, \mathbf{y})$. Then the map $\check{\Omega}_k(f) : \check{\Omega}_k^{\mathbf{p}}(X, x) \longrightarrow \check{\Omega}_k^{\mathbf{q}}(Y, y)$ given by $\check{\Omega}_k(f)(a_\lambda) = (f_\lambda \circ a_\lambda)$ is continuous.*

Proof. Since $f : (X, x) \longrightarrow (Y, y)$ is represented by $[(f_\lambda)] : (\mathbf{X}, \mathbf{x}) \longrightarrow (\mathbf{Y}, \mathbf{y})$, for all $\lambda' \geq \lambda$, $q_{\lambda\lambda'} \circ f_{\lambda'} \simeq f_\lambda \circ p_{\lambda\lambda'}$. The map $\check{\Omega}_k(f) : \check{\Omega}_k^{\mathbf{p}}(X, x) \longrightarrow \check{\Omega}_k^{\mathbf{q}}(Y, y)$ given by $\check{\Omega}_k(f)(a_\lambda) = (f_\lambda \circ a_\lambda)$ is well defined; because for all $\lambda' \geq \lambda$,

$$q_{\lambda\lambda'}(f_{\lambda'} \circ a_{\lambda'}) \simeq f_\lambda \circ p_{\lambda\lambda'} \circ a_{\lambda'} \simeq f_\lambda \circ a_\lambda.$$

$\check{\Omega}_k^{\mathbf{q}}(Y, y)$ is a subspace of $\prod_{\lambda\in\Lambda} \Omega^k(Y_\lambda, y_\lambda)$ and $f_\lambda \circ a_\lambda$ is continuous, for all $\lambda \in \Lambda$. Thus the map $\check{\Omega}_k(f)$ from product topology is continuous. □

In the following remark, we show that $\check{\Omega}_k^{\mathbf{p}}(f)$ is continuous, for some spectial maps.

Remark 3.2. (i) Let $f : (P_1, p_1) \longrightarrow (P_2, p_2)$ be a continuous map between two pointed polyhedra. Because the HPol$_*$-expansions of polyhedra are trivial, then $\check{\Omega}_k(P_i, p_i) = \Omega_k(P_i, p_i)$, for $i = 1, 2$. Thus we can define $\check{\Omega}_k(f) : \check{\Omega}_k(P_1, p_1) \longrightarrow \check{\Omega}_k(P_2, p_2)$ by $\check{\Omega}_k(f)(\alpha) = f \circ \alpha$. It is easy to see that $\check{\Omega}_k(f)$ is continuous.
(ii) Let $\mathbf{p} : (X, x) \to ((X_\lambda, x_\lambda), [p_{\lambda\lambda'}], \Lambda)$ be an HPol$_*$-expansion of pointed topological space (X, x). Let (P, p) be a pointed polyhedron and $f : (X, x) \longrightarrow (P, p)$, $g : (P, p) \longrightarrow (X, x)$ be continuous maps represented by $[(f_\lambda)] : (\mathbf{X}, \mathbf{x}) \longrightarrow (P, p)$ and $[(g_\lambda)] : (P, p) \longrightarrow (\mathbf{Y}, \mathbf{y})$, respectively. Because for all $\lambda' \geq \lambda$, $f_{\lambda'} \circ a_{\lambda'} \simeq f_\lambda \circ a_\lambda$, then we can define $\check{\Omega}_k^{\mathbf{p}}(f) : \check{\Omega}_k^{\mathbf{p}}(X, x) \longrightarrow \check{\Omega}_k(P, p)$ by $\check{\Omega}_k^{\mathbf{p}}(f)(a_\lambda) = f_\lambda \circ a_\lambda$, for any $\lambda \in \Lambda$. Also we can define $\check{\Omega}_k^{\mathbf{p}}(g) : \check{\Omega}_k(P, p) \longrightarrow \check{\Omega}_k^{\mathbf{p}}(X, x)$ by $\check{\Omega}_k^{\mathbf{p}}(g)(a) = (g_\lambda \circ a)$, for all $\lambda \in \Lambda$.

The following theorem is one of the main result of this paper.

Theorem 3.3. *Let (X, x) and (Y, y) admit HPol$_*$-expansions $\mathbf{p} : (X, x) \to (\mathbf{X}, \mathbf{x}) - ((X_\lambda, x_\lambda), [p_{\lambda\lambda'}], \Lambda)$ and $\mathbf{q} : (Y, y) \to (\mathbf{Y}, \mathbf{y}) = ((Y_\lambda, y_\lambda), [q_{\lambda\lambda'}], \Lambda)$, respectively such that $\mathbf{p} \times \mathbf{q} : (X \times Y, (x, y)) \to (\mathbf{X} \times \mathbf{Y}, (\mathbf{x}, \mathbf{y}))$ is an HPol$_*$-expansion. Then*

$$\check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X \times Y, (x, y)) \cong \check{\Omega}_k^{\mathbf{p}}(X, x) \times \check{\Omega}_k^{\mathbf{q}}(Y, y).$$

Proof. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the canonical projections and assume that $\check{\Omega}_k(\pi_X) : \check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X \times Y, (x, y)) \longrightarrow \check{\Omega}_k^{\mathbf{p}}(X, x)$ and $\check{\Omega}_k(\pi_Y) : \check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X \times Y, (x, y)) \longrightarrow \check{\Omega}_k^{\mathbf{q}}(Y, y)$ be the induced morphisms of canonical projections which are continuous, by Lemma 3.1. Then there is a continuous map $\alpha : \check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X \times Y, (x, y)) \to \check{\Omega}_k^{\mathbf{p}}(X, x) \times \check{\Omega}_k^{\mathbf{q}}(Y, y)$. We define a map $\beta : \check{\Omega}_k^{\mathbf{p}}(X, x) \times \check{\Omega}_k^{\mathbf{q}}(Y, y) \to \check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X \times Y, (x, y))$ by $\beta((a_\lambda), (b_\lambda)) = ((a_\lambda, b_\lambda))$, where $(a_\lambda, b_\lambda) : S^k \longrightarrow X_\lambda \times Y_\lambda$ is given by $(a_\lambda, b_\lambda)(s) = (a_\lambda(s), b_\lambda(s))$, for all $\lambda \in \Lambda$. The map β is well defined; because for all $\lambda' \geq \lambda$,

$$(p_{\lambda\lambda'} \times q_{\lambda\lambda'})(a_{\lambda'}, b_{\lambda'}) = (p_{\lambda\lambda'}(a_{\lambda'}) \times q_{\lambda\lambda'}(b_{\lambda'})) \simeq (a_\lambda, b_\lambda).$$

It is routine to check that $\alpha \circ \beta = id$ and $\beta \circ \alpha = id$. □

Mardešić showed that if $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$ are HPol-expansions of compact Hausdorff spaces X and Y, respectively, then $\mathbf{p} \times \mathbf{q} : X \times Y \to \mathbf{X} \times \mathbf{Y}$ is also an HPol-expansion of $X \times Y$ [8, Lemma 2 and Theorem 4]. Therefore we have the following result from Theorem 3.3.

Corollary 3.4. *If (X,x) and (Y,y) are two pointed compact Hausdorff spaces with HPol$_*$-expansions $\mathbf{p} : (X,x) \to ((X_\lambda, x_\lambda), [p_{\lambda\lambda'}], \Lambda)$ and $\mathbf{q} : (Y,y) \to ((Y_\lambda, y_\lambda), [q_{\lambda\lambda'}], \Lambda)$, then*

$$\check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X \times Y, (x,y)) \cong \check{\Omega}_k^{\mathbf{p}}(X,x) \times \check{\Omega}_k^{\mathbf{q}}(Y,y).$$

For every topological space one can associate an inverse system $\mathbf{C}(\mathbf{X}) = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in the category HPol which is called the Čech system of X [8]. The set Λ is the set of all normal coverings λ of X ordered by the relation of being a finer covering. $X_\lambda = |N(\lambda)|$ is the nerve of λ and $[p_{\lambda\lambda'}]$, $\lambda \le \lambda'$, is the unique homotopy class to which belong the projections $p_{\lambda\lambda'} : |N(\lambda')| \longrightarrow |N(\lambda)|$. For $\lambda \in \Lambda$ let $[p_\lambda] : X \longrightarrow X_\lambda$ be the unique homotopy class of the canonical mappings $p_\lambda : X \longrightarrow X_\lambda = |N(\lambda)|$. For every topological space X the morphism $\mathbf{p} = (p_\lambda) : X \longrightarrow \mathbf{C}(\mathbf{X})$ of pro-HTop is an HPol-expansion [8, Appendix 1,Theorem 3.8] which is called it Čech HPol-expansion. In [9] Nasri and Mashayekhy proved that $\check{\Omega}_k^{\mathbf{c}} : Top_* \longrightarrow Top_*$ is a functor, for all $k \in \mathbb{N}_0$. By Theorem 3.3 we obtain the following result.

Corollary 3.5. *Let $\mathbf{p} : (X,x) \to (\mathbf{C}(\mathbf{X}), \mathbf{x}) = ((X_\lambda, x_\lambda), [p_{\lambda\lambda'}], \Lambda)$ and $\mathbf{q} : (Y,y) \to (\mathbf{C}(\mathbf{Y}), \mathbf{y}) = ((Y_\lambda, y_\lambda), [q_{\lambda\lambda'}], \Lambda)$ be the Čech HPol$_*$-expansions of the pointed topological spaces (X,x) and (Y,y), respectively such that $\mathbf{p} \times \mathbf{q} : (X \times Y, (x,y)) \to (\mathbf{C}(\mathbf{X}) \times \mathbf{C}(\mathbf{Y}), (\mathbf{x}, \mathbf{y}))$ is an HPol$_*$-expansion. Then*

$$\check{\Omega}_k^{\mathbf{p}\times\mathbf{q}}(X \times Y, (x,y)) \cong \check{\Omega}_k^{\mathbf{p}}(X,x) \times \check{\Omega}_k^{\mathbf{q}}(Y,y).$$

Now, we intend to prove that $\check{\Omega}_k^{\mathbf{p}}$ commutes with the inverse limit under some conditions.

Remark 3.6. Let (X,x) be a pointed topological space with HPol$_*$-expansion $\mathbf{p} : (X,x) \to (\mathbf{X}, \mathbf{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$. If X_λ's are discrete, then the homotopies in the definition of $\check{\Omega}_k^{\mathbf{p}}(X,x) = \{(a_\lambda) \in \prod_{\lambda\in\Lambda} \Omega^k(X_\lambda, x_\lambda) |\ p_{\lambda\lambda'}(a_{\lambda'}) \simeq a_\lambda,\ \textit{for all}\ \lambda' \ge \lambda\}$ will become equality and therefore

$$\check{\Omega}_k^{\mathbf{p}}(X,x) = \lim_{\leftarrow} \Omega^k(X_\lambda, x_\lambda).$$

The following theorem is the second main result of this paper.

Theorem 3.7. *Let $(X,x) = \lim\limits_{\leftarrow}(X_i, x_i)$ be an inverse limit space of an inverse system $\{(X_i, x_i), p_{ij}\}_I$, where X_i's are discrete compact polyhedra. Then for all $k \in \mathbb{N}_0$,*

$$\check{\Omega}_k^{\mathbf{p}}(X,x) = \lim_{\leftarrow} \check{\Omega}_k^{\mathbf{p}}(X_i, x_i).$$

Proof. Since X_i's are compact, $\prod_{i\in\mathbb{N}} X_i$ is compact by [4, Theorem 3.2.4] and since X_i's are Hausdorff, $X = \lim\limits_{\leftarrow} X_i$ is a closed subspace of $\prod_{i\in\mathbb{N}} X_i$

by [4, Proposition 2.5.1]. Hence $\lim\limits_{\leftarrow} X_i$ is compact by [4, Theorem 3.1.2]. Therefore, the limit $\mathbf{p} : X \to (X_i, p_{ii+1}, \mathbb{N})$ is an HPol-expansion of X by [5, Remark 1]. Since X_i's are discrete, $\check{\Omega}_k^{\mathbf{P}}(X,x) = \lim\limits_{\leftarrow} \Omega^k(X_i, x_i)$, by Remark 3.6. Because the HPol$_*$-expansions of polyhedra are trivial, then $\check{\Omega}_k^{\mathbf{P}}(X_i,x_i) = \Omega^k(X_i, x_i)$ which completes the proof. □

Corollary 3.8. *Let* $(X,x) = \lim\limits_{\leftarrow}(X_i,x_i)$ *be an inverse limit space of an inverse system* $\{(X_i,x_i), p_{ij}\}_I$*, where* X_i*'s are discrete compact polyhedra. If* I *is countable, then for all* $k \in \mathbb{N}_0$*, The space* $\check{\Omega}_k^{\mathbf{P}}(X,x)$ *is a*
(i) *metric space.*
(ii) *second countable space.*
(iii) *separable space.*

Proof. (i) Since X_i's are metric, the loop spaces $\Omega^k(X_i,x_i)$'s are metric by [3, Theorem 12.8.2], for all $k \in \mathbb{N}_0$. Since $\Omega^k(X_i,x_i)$'s are metric and I is countable, the limit $\lim\limits_{\leftarrow} \Omega^k(X_i,x_i)$ is metric, by [4, Corollary 4.2.5]. Since X_i's are discrete compact polyhedra, $\check{\Omega}_k^{\mathbf{P}}(X,x) = \lim\limits_{\leftarrow} \check{\Omega}_k^{\mathbf{P}}(X_i,x_i) = \lim\limits_{\leftarrow} \Omega^k(X_i,x_i)$, by Theorem 3.7. Therefore $\check{\Omega}_k^{\mathbf{P}}(X,x)$ is a metric space.
(ii) The argument is similar to part (i). Since X_i's are compact polyhedra, they are second countable, by [4, Theorem 4.2.8]. Thus $\Omega^k(X_i,x_i)$'s are second countable by [3, Theorem 12.5.2]. Since $\Omega^k(X_i,x_i)$'s are second countable and I is countable, the limit $\lim\limits_{\leftarrow} \Omega^k(X_i,x_i)$ is second countable by [3, Theorem 8.6.2]. Since X_i's are discrete compact polyhedra, $\check{\Omega}_k^{\mathbf{P}}(X,x) = \lim\limits_{\leftarrow} \check{\Omega}_k^{\mathbf{P}}(X_i,x_i) = \lim\limits_{\leftarrow} \Omega^k(X_i,x_i)$, by Theorem 3.7. Therefore $\check{\Omega}_k^{\mathbf{P}}(X,x)$ is a second countable space.
(iii) It follows from part (ii) and [3, Theorem 8.7.3]. □

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