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Multi-step conformable fractional differential transform method for solving and stability of the conformable fractional differential systems

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Abstract. In this article, the multi-step conformable fractional differential transform method (MSCDTM) is applied to give approximate solutions of the conformable fractional-order differential systems. Moreover, we check the stability of conformable fractional-order Lü system with the MSCDTM to demonstrate the efficiency and effectiveness of the proposed procedure.

Keywords: Multi-step, Differential transform method, Conformable fractional, Stability.

2010 Mathematics subject classification: 34D20, 26A33, 34A34.

1. Introduction

Fractional differential equations rather than ordinary and partial ones more accurately describe physical phenomena having memory and hereditary characteristics thanks to memory effects of fractional derivatives. Because of these important characteristics, fractional differential equations have become more important in many fields of science in recent

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years. There are many definitions for fractional differential equations, such as Riemann-Liouville and Caputo's fractional derivatives [1].

(i) The Riemann-Liouville fractional derivative of a function says f is defined as

$$D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(x)}{(t-x)^\alpha} dx, \qquad 0 < \alpha < 1. \tag{1.1}$$

(ii) The Caputo fractional derivative of a differentiable function says f is defined as

$$^C D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(x)}{(t-x)^\alpha} dx, \qquad 0 < \alpha < 1. \tag{1.2}$$

Recently, Khalil et al. [2] introduced a new simple well-behaved definition of the fractional derivative called conformable fractional derivative. This fractional derivative is theoretically very easier to handle and also obeys some conventional properties that cannot be satisfied by the existing fractional derivatives, for instance, the chain rule [3]. In short time, many studies related to this new fractional derivative definition was done [4,5]. The fractional derivative of f in the conformable sense is defined as follows

Definition 1.1. [2] Let $f : (0, \infty) \to \mathbb{R}$, then, the conformable fractional derivative of f of order α is defined as

$$_tT_\alpha^{t_0}\left(f\right)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-t_0)^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1.3}$$

for all $t > 0, \alpha \in (0, 1]$. Also, if the conformable fractional derivative of f of order α exists, then we simply say is α-differentiable.

The new definition satisfies the properties which given in the following theorem:

Theorem 1.2. *[2] Let $\alpha \in (0, 1]$, and f, g be α-differentiable at a point t, then*

(i) $_tT_\alpha^{t_0}(af + bg) = a_tT_\alpha^{t_0}(f) + b_tT_\alpha^{t_0}(g)$, *for all $a, b \in \mathbb{R}$.*

(ii) $_tT_\alpha^{t_0}(t^\mu) = \mu t^{\mu-\alpha}$, *for all $\mu \in \mathbb{R}$.*

(iii) $_tT_\alpha^{t_0}(fg) = f_tT_\alpha^{t_0}(g) + g_tT_\alpha^{t_0}(f)$.

(iv) $_tT_\alpha^{t_0}(\frac{f}{g}) = \frac{g_tT_\alpha^{t_0}(f) - f_tT_\alpha^{t_0}(g)}{g^2}$.

If, in addition, f is differentiable, then $_tT_\alpha^{t_0}(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

For simplicity and without loss of generality, we assume that ${}_tT_\alpha = {}_tT_\alpha^0$. In [3] T. Abdeljawad established the chain rule for conformable fractional derivatives as following theorem.

Theorem 1.3. *Let $f : (0,\infty) \to \mathbb{R}$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable; then, one has the following rule*

$$
{}_tT_\alpha(fog)(t) = \left({}_tT_\alpha f\right)(g(t))\left({}_tT_\alpha g\right)(t)\; g(t)^{\alpha-1}. \tag{1.4}
$$

If $t = 0$, then

$$
{}_tT_\alpha(fog)(0) = \lim_{t\to 0^+}\left({}_tT_\alpha f\right)(g(t))\left({}_tT_\alpha g\right)(t)\; g(t)^{\alpha-1}.
$$

The fractional exponential function plays a very important role in the conformable fractional differential equations. The fractional exponential function $e^{\frac{1}{\alpha}t^\alpha}$, where $0 < \alpha \leq 1$, is defined by the following series representation,

$$
e^{\frac{1}{\alpha}t^\alpha} = \sum_{k=0}^{\infty}\frac{t^{\alpha k}}{\alpha^k k!}.
$$

Now, we list here the fractional derivatives of certain functions [2]

(i) ${}_tT_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$,

(ii) ${}_tT_\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos\frac{1}{\alpha}t^\alpha$,

(iii) ${}_tT_\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin\frac{1}{\alpha}t^\alpha$,

(iv) ${}_tT_\alpha(\frac{1}{\alpha}t^\alpha) = 1$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

Theorem 1.4. *[3] Assume f is infinitely α-differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of a point t_0. Then f has the fractional power series expansion*

$$
f(t) = \sum_{k=0}^{\infty}\frac{\left(T_\alpha^{t_0}f\right)^{(k)}(t_0)\,(t-t_0)^{\alpha k}}{\alpha^k k!}, \qquad t_0 < t < t_0 + R^{1/\alpha}, \quad R > 0. \tag{1.5}
$$

Here $\left(T_\alpha^{t_0}f\right)^{(k)}(t_0)$ denotes the application of the fractional derivative for k times.

Definition 1.5. (fractional Laplace transform) [3] Let $0 < \alpha \leq 1$ and $f : [0,\infty) \to \mathbb{R}$ be real valued function. Then the fractional Laplace transform of order α starting from a of f is defined by,

$$L_\alpha\{f(t)\} = F_\alpha(s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\, d\alpha(t), \tag{1.6}$$

where $d\alpha(t) = t^{\alpha-1}dt$.

Furthermore, using the properties of the fractional exponential function and integration by parts, we have

$$L_\alpha\{{}_tT_\alpha\,(f)\,(t)\} = s\; F_\alpha(s) - f(0). \tag{1.7}$$

In this paper, at first we produce sufficient conditions for asymptotical stability of linear conformable fractional differential system [6]. Then, we present the MSCDTM for obtain approximate analytical solution and stability of the conformable fractional-order Lü system to illustrate the validity of the results.

2. Stability analysis

In this section, we consider the stability of the following linear conformable fractional differential system

$$\begin{cases} {}_tT_{\alpha_1}\, x_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t), \\ {}_tT_{\alpha_2}\, x_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t), \\ \qquad\vdots \qquad\qquad\qquad\qquad \vdots \\ {}_tT_{\alpha_n}\, x_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t), \end{cases} \tag{2.1}$$

where $x_i(0) = x_{i0}$ and $0 < \alpha_i \leq 1$ for $i = 1, 2, \ldots, n$.

Definition 2.1. The zero solution of linear conformable fractional differential system (2.1) is said to be stable if, for any initial value x_0, there exists an $\varepsilon > 0$ such that $\|x(t)\| \leq \varepsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if, in addition to being stable, $\|x(t)\| \to 0$ as $t \to \infty$.

We study the stability of system (2.1) by applying the fractional Laplace transforms on both sides of this system, we have

$$sX_{\alpha_i}(s) - x_{i0} = \sum_{j=1}^{n} a_{ij}X_{\alpha_i}(s), \tag{2.2}$$

for $i = 1, \ldots, n$, where $X_{\alpha_i}(s)$ is the fractional Laplace transform of $x_i(t)$.

We can rewrite (2.2) as follows

$$\Delta(s). \begin{pmatrix} X_{\alpha_1}(s) \\ X_{\alpha_2}(s) \\ \vdots \\ X_{\alpha_n}(s) \end{pmatrix} = x_0. \tag{2.3}$$

in which

$$\Delta(s) = \begin{pmatrix} \Delta_{11}(s) & \Delta_{12}(s) & \dots & \Delta_{1n}(s) \\ \Delta_{21}(s) & \Delta_{22}(s) & \dots & \Delta_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1}(s) & \Delta_{n2}(s) & \cdots & \Delta_{nn}(s) \end{pmatrix},$$

where

$$\Delta_{ij}(s) = \begin{cases} s - a_{ii} & \text{if } i = j, \\ -a_{ij} & \textit{otherwise}. \end{cases}$$

and $x_0 = (x_{10}, x_{20}, \dots, x_{n0})^T$. For simplicity, we call $\Delta(s)$ a characteristic matrix of (2.1), moreover $\det(\Delta(s)) = 0$ is the characteristic equation of system (2.1). Now, we express the main theorem for checking the stability of system (2.1).

Theorem 2.2. *[6] If all roots of* $\det(\Delta(s)) = 0$ *have negative real parts, then zero solution system of* (2.1) *is asymptotically stable.*

3. Description of the method

In this section, the MSCDTM is applied to solve the conformable fractional non-linear systems of differential equations.

Definition 3.1. [7] Assume $f(t)$ is infinitely α-differentiable function for some $\alpha \in (0, 1]$. Conformable fractional differential transform of $f(t)$ is defined as

$$F_\alpha(k) = \frac{1}{\alpha^k k!} \left[\left(T_\alpha^{t_0} f \right)^{(k)}(t) \right]_{t=t_0}. \tag{3.1}$$

Definition 3.2. [7] Let $F_\alpha(k)$ be the conformable fractional differential transform of $f(t)$. The inverse of the conformable fractional differential transform of a sequence $\{F_\alpha(k)\}_{k=0}^\infty$ is defined as follow

$$f(t) = \sum_{k=0}^{\infty} F_\alpha(k) \, (t - t_0)^{\alpha k}.$$

From definitions (1.5) and (3.1), we obtain

$$f(t) = \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!} \left[\left(T_\alpha^{t_0} f \right)^{(k)}(t) \right]_{t=t_0} (t - t_0)^{\alpha k}. \tag{3.2}$$

Equation (3.2) implies that the concept of conformable fractional differential transformation (CDTM) is derived from the fractional power series expansion. Some basic properties of the CDTM obtained from definitions (1.5) and (3.1) are summarized in table 1 [7].

TABLE 1. Operations of CDTM.

Original function	Transformation function
$f(t) = u(t) \pm v(t)$	$F_\alpha(k) = U_\alpha(k) \pm V_\alpha(k)$
$f(t) = cu(t)$	$F_\alpha(k) = cU_\alpha(k)$
$f(t) = T_\alpha^{t_0}(u(t))$	$F_\alpha(k) = \alpha(k+1)U_\alpha(k+1)$
$f(t) = T_\beta^{t_0}(u(t)) \quad m < \beta \leq m+1$	$F_\alpha(k) = \frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+\beta-m)}U_\alpha(k+\beta/\alpha)$
$f(t) = u(t)v(t)$	$F_\alpha(k) = \sum_{l=0}^{k} U_\alpha(l)V_\alpha(k-l)$

To demonstrate the effectiveness of the CDTM, we present the following initial value problem for systems of conformable fractional differential equations

$$\begin{cases} T_{\alpha_1}^{t_0} y_1(t) = f_1(t, y_1(t), y_2(t), \dots, y_n(t)), \\ T_{\alpha_2}^{t_0} y_2(t) = f_2(t, y_1(t), y_2(t), \dots, y_n(t)), \\ \qquad \vdots \\ T_{\alpha_n}^{t_0} y_n(t) = f_n(t, y_1(t), y_2(t), \dots, y_n(t)), \end{cases} \tag{3.3}$$

subject to the initial conditions

$$y_{i0}(t_0) = c_i, \qquad i = 1, 2 \dots, n. \tag{3.4}$$

Let $[t_0, L]$ be the interval over which we want to find the solution of the initial value problem (3.3)-(3.4). In actual applications of the CDTM, the Nth-order approximate solution of the initial value problem (3.3)-(3.4) can be expressed by the finite series

$$y_i(t) = \sum_{k=0}^{\infty} Y_{\alpha_i}(k)\,(t-t_0)^{\alpha_i k}, \qquad t \in [t_0, L], \qquad i = 0, 1, \dots, n, \tag{3.5}$$

where $Y_{\alpha_i}(k)$ is the differential transform for $y_i(t)$ and satisfies the recurrence relation

$$Y_{\alpha_i}(k+1) = \frac{1}{\alpha_i(k+1)} F_{\alpha_i}(k, Y_{\alpha_1}(k), Y_{\alpha_2}(k), \dots, Y_{\alpha_n}(k)), \tag{3.6}$$

$Y_{\alpha_i}(0) = c_i$, and $F_{\alpha_i}(k, Y_{\alpha_1}(k), \dots, Y_{\alpha_n}(k))$ is the differential transform of function $f_i(t, y_1(t),\, y_2(t), \dots, y_n(t))$ for $i = 0, 1, \dots, n$.
Assume that the interval $[t_0, L]$ is divided into M sub-intervals $[t_{m-1}, t_m]$, $m = 1, \dots, M$, of equal step size $h = (L - t_0)/M$ by using the nodes $t_m = t_0 + mh$. The main ideas of the MSCDTM are as follows:
First, we apply the CDTM to the initial value problem (3.3)-(3.4) over the interval $[t_0, t_1]$, we will obtain the approximate solution $y_{i,1}(t)$ using the initial condition $y_i(t_0) = c_i$, for $i = 0, 1, \dots, n$. For $m \geq 2$ and at each subinterval $[t_{m-1}, t_m]$ we will use the initial condition $y_{i,m}(t_{m-1}) =$

$y_{i,m-1}(t_{m-1})$ and apply the CDTM to the initial value problem (3.3)-(3.4) over the interval $[t_{m-1}, t_m]$. The process is repeated and generates a sequence of approximate solutions $y_{i,m}(t)$, $m = 1, \ldots, M$, for $i = 1, \ldots, n,$. Finally the MSCDTM assumes the following solution

$$y_i(t) = \begin{cases} y_{i,1}(t) & t \in [t_0, t_1]\,, \\ y_{i,2}(t) & t \in [t_1, t_2]\,, \\ & \vdots \\ y_{i,M}(t) & t \in [t_{M-1}, t_M]\,. \end{cases}$$

The algorithm, MSCDTM, is simple for computational performance for all values of h. As we will see in the next section, the main advantage of the algorithm is that the solution obtained converges for wide time regions.

4. Example

The so-called Lü's system is known as a bridge between the Lorenz system and Chen's system [8]. Its conformable fractional version is described as follows

$$\begin{aligned} T_{\alpha_1} x(t) &= a(y(t) - x(t)), \\ T_{\alpha_2} y(t) &= -x(t)z(t) + cy(t), \\ T_{\alpha_3} z(t) &= x(t)y(t) - bz(t), \end{aligned} \tag{4.1}$$

where $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$, are derivatives orders, and a, b, c are system parameters. The system (4.1) has three equilibrium points $E_1 = (0, 0, 0)$, $E_2 = (\sqrt{bc}, \sqrt{bc}, c)$ and $E_3 = (-\sqrt{bc}, -\sqrt{bc}, c)$. The Jacobian matrix for equilibria $E^* = (x^*, y^*, z^*)$ is defined as

$$J = \begin{bmatrix} -a & a & 0 \\ -z^* & c & -x^* \\ y^* & x^* & -b \end{bmatrix}. \tag{4.2}$$

Let us consider the following parameters $a = 25, b = 1, c = 40$ of the system (4.1). For equilibrium points $E_1 = (0, 0, 0)$ we obtain the following eigenvalus of the Jacobian matrix (4.2): $\lambda_1 = -1, \lambda_2 = 40$ and $\lambda_3 = -25$. For the equilibrium $E_2 = (6.3245, 6.3245, 40)$ we get the eigenvalues $\lambda_1 \approx -8.8284$ and $\lambda_2, \lambda_3 \approx 11.4142 \pm 9.8109j$. The equilibrium point $E_3 = (-6.3245, -6.3245, 40)$ has the same eigenvalues as the equilibrium E_2. All these eigenvalues satisfy the condition for the system to be unstable. Figure 1 shows that the system (4.1) with order $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.95, 0.9)$ is unstable.

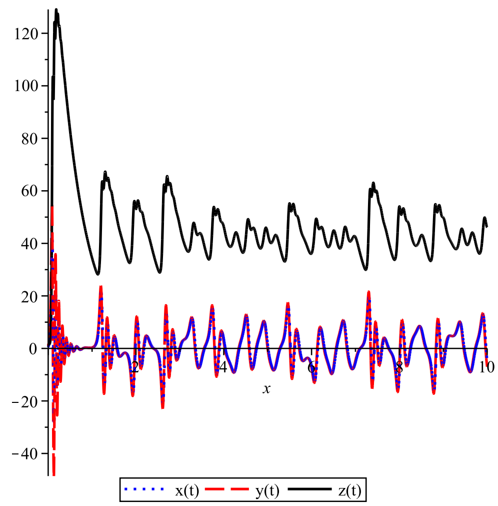

FIGURE 1. The system (4.1) with $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.95, 0.9)$ is unstable.

Let us consider the following parameters $a = 1, b = 40, c = 10$ of the system (4.1). For equilibrium points $E_1 = (0, 0, 0)$ we obtain the following eigenvalus of the Jacobian matrix (4.2): $\lambda_1 = -1, \lambda_2 = 10$ and $\lambda_3 = -40$. For the equilibrium $E_2 = (20, 20, 10)$ we get the eigenvalues $\lambda_1 \approx -30.5478$ and $\lambda_2, \lambda_3 \approx -0.2260 \pm 5.1124j$. The equilibrium point $E_3 = (-20, -20, 10)$ has the same eigenvalues as the equilibrium E_2, thus the equilibrium point E_2 is asymptotically stable. Figure 2 shows that the system (4.1) with order $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.95, 0.9)$ isasymptotically stable.
All the results are calculated by using the computer algebra package Maple. The term-number of MSCDTM series solutions is fixed $N = 10$ and the time step size $h = 0.01$, with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 1.5)$.

5. Conclusion

In this paper, the multi-step conformable fractional differential transform method has been successfully applied to find the numerical solutions of the nonlinear system of conformable fractional order. The numerical simulations and stability of conformable fractional Lü system are used to illustrate our main result. The approximate solutions obtained by MSCDTM are highly accurate and valid for a long time. The

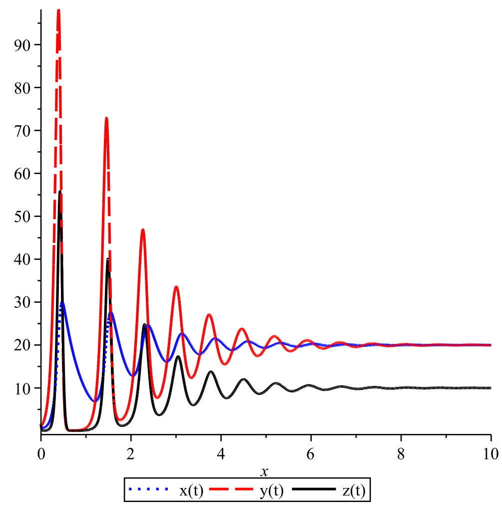

FIGURE 2. The equilibrium point E_2 of the system (4.1) with $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.95, 0.9)$ is asymptotically stable.

results presented in this paper suggest that this algorithm is also readily applicable to more general classes of linear and nonlinear differential equations of conformable fractional order.

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