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Realizable List by Circulant and Skew-Circulant Matrices

A.M. Nazari [1] and R. Mohammadi [2]
[1] Department of Mathematics, Faculty of science, Arak University, Arak 38156-8-8349, Arak, Iran.
[2] Department of Mathematics, Faculty of science, Arak University, Arak 38156-8-8349, Arak, Iran.

Abstract. In this paper for two given sets of eigenvalues, which one of them is the eigenvalues of circulant matrix and the other is the eigenvalues of skew-circulant matrix, we find a nonnegative matrix, such that the union of two sets be the spectrum of nonnegative matrices.

Keywords: Nonnegative matrices, Circulant and Skew-Circulant matrices, Inverse eigenvalue problem.

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1. Introduction

The class of circulant and skew-circulant matrices and their properties are introduced in [**?**]. A circulant matrix is a special kind of Toeplitz matrix where each row vector is rotated one element to the right or to the left relative to the preceding row vector. Circulant matrices are very useful in digital image processing [**?**]. In numerical analysis, circulant matrices are important because they are diagonalized by a discrete Fourier transform, and hence linear equations that contain them may be quickly solved by using a fast Fourier transform (FFT).

[1]Corresponding author: a-nazari@araku.ac.ir
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The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of complex numbers in order that it be the spectrum of a nonnegative matrix. In this case, one says that σ is realizable and a nonnegative matrix A with spectrum σ is said to realize σ and it is referred to as a realizing matrix. A number of necessary conditions for realizability are known, as well as a number of sufficient conditions. In many cases, sufficiency is established by direct construction of a realizing matrix [1-6]. So far, in general, this problem has been solved for state $n = 5$. Of course, for the symmetric case(SNIEP), many solutions have been presented so far. Nazari and Sherefat solved NIEP for $n = 5$ solved the problem in 17 different cases[**?**].

In this paper we solve the special case of nonnegative inverse eigenvalue problem(NIEP). For given set $\sigma(C)$ that is the spectrum of neonnegative circulant matrix C and the other set $\sigma(S)$ that is the spectrum of skew-circulat matrix we find a nonnegative matrix that its spectrum is the $\sigma(C) \cup \sigma(S)$.

2. Properties of circulant and skew circulant matrix

Let $s = (s_0, s_1, ..., s_{n-1})^T, c = (c_0, c_1, ..., c_{n1})^T \in \mathbb{R}^n$ be given.

Definition 2.1. [**?**, **?**] An $n \times n$ real right (left) circulant matrix is a matrix of form:

$$C_R(c) = \begin{bmatrix} c_0 & \multicolumn{3}{c}{\dots\dots\dots\dots\dots} & c_{n-1} \\ c_{n-1} & c_0 & \multicolumn{2}{c}{\dots\dots\dots} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & .. & c_{n-1} & c_0 \end{bmatrix}$$

where each row is a cyclic shift of the row above to the right (left).

Definition 2.2. [**?**] An $n \times n$ real skew right (left) circulant matrix is a matrix of the form:

$$S_R(s) = \begin{bmatrix} s_0 & \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots} & s_{n-1} \\ -s_{n-1} & s_0 & \multicolumn{2}{c}{\dots\dots\dots\dots} & s_{n-2} \\ -s_{n-2} & -s_{n-1} & s_0 & \dots\dots & s_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_1 & -s_2 & .. & -s_{n-1} & s_0 \end{bmatrix}.$$

The next concepts can be seen in [**?**]. Define the orthogonal (anti-diagonal unit) matrix $J_m \in \mathbb{R}^{m\times m}$as

$$J_m := \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

The matrix

$$\Gamma_n := \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & J_{n-1} \end{array}\right]$$

is an orthogonal cyclic shift matrix (and a left circulant matrix). It follows that,

$$\Gamma_n = F \cdot F^T = F^2,$$

where $F = (f_{pq})$ are given by

$$f_{pq} := \frac{1}{\sqrt{n}}\omega^{pq}, \quad p = 0, 1, \dots, n-1, \quad q = 0, 1, \dots, n-1$$

where

$$\omega = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n} = \exp\frac{2\pi i}{n}.$$

For the orthogonal matrix

$$\Xi_n = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & -J_{n-1} \end{array}\right],$$

it is straightforward to verify that

$$\Xi_n = G \cdot G^T,$$

where $G = (g_{pq})$ with

$$g_{pq} = \frac{1}{\sqrt{n}}\omega^{p(q+\frac{1}{2})}, \quad p = 0, 1, \dots, n-1, \quad , q = 0, 1, \dots, n-1.$$

is strongly related to the DFT(Discrete Fourier Transform) matrix, i.e.,

$$G := \mathrm{diag}(1, \tau, ..., \tau^{n-1})F$$

with

$$\tau = \omega^{\frac{1}{2}}$$

Therefore, G is also unitary. Let M and N be two circulant (skew circulant)matrices then (see, for instance [**?**]):
1. $M + M$ and MN are circulant(skew circulant) matrices;
2. M^T is a circulant (skew circulant) matrix;
3. $M.N$ is a circulant (skew circulant) matrix;
4. $\sum_{l=1}^{k} \alpha_l M^l$ is a circulant matrix.
5. The rank of a circulant matrix C is equal to $n - d$, where d is the degree of $\gcd(f(x), x^n - 1)$.

6.The polynomial $f(x) = c_0+c_1x+\cdots+c_{n-1}x^{n-1}$ is called the associated polynomial of matrix C (see in[**?**]).
7. One amazing property of circulant matrices is that the eigenvectors are always the same.

Theorem 2.3. [?] *$C(c) = F^* \cdot \Lambda \cdot F$ where*

$$\Lambda(s) = \operatorname{diag}(\lambda_0(c), \lambda_1(c), \ldots, \lambda_{n-1}(c))$$

and

$$\lambda_k(c) = \sum_{j=0}^{n-1} c_j \omega^{kj}, \qquad k = 0, 1, \ldots, n-1$$

Theorem 2.4. [?] *$S(s) = G \cdot M \cdot G^*$ where*

$$M(s) = \operatorname{diag}(\mu_0, \mu_1, \ldots, \mu_{n-1})$$

and

$$\mu_k = \sum_{j=0}^{n-1} s_j \omega^{(k+\frac{1}{2})j}, \qquad k = 0, 1, \ldots, n-1$$

Theorem 2.5. [?]

1. $\lambda_{n-k}(c) = \overline{\lambda_k(c)}$, *for* $k = 1, 2, ..., n-1$ *and* $\lambda_0(c) = \sum_{j=1}^{n-1} c_j$
2. $\mu_{n-1-k}(s) = \overline{\mu_k(s)}$, *for* $k = 0, 1, ..., n-1$

3. Main result

In this section we will generalize the method that introduced in [**?**]. Let σ be a spectrum of order $2n+k$, we will solve the NIEP by circulant matrices and skew-circulant matrices. At first we express an important theorem.

Theorem 3.1. [?] *Let $C = (c_{ij})$ be a nonnegative matrix of order $n+1$and consider the $S = skew_circ(s_0, s_1, ..., s_{n-1}) := (s_{ij})$ whose spectra (counted with their multiplicities) are $(\lambda_0, \lambda_1, ..., \lambda_n)and(\mu_0, \mu_1, ..., \mu_n)$, respectively. Moreover, suppose that $|s_{ij}| \leq c_{ij}$, $1 \leq i, j \leq n$ and $0 \leq \gamma \leq 1$.Then the nonnegative matrix*

$$M_{\pm\gamma} = \begin{bmatrix} \frac{c_{11}\pm\gamma s_{11}}{2} & \frac{c_{11}\mp\gamma s_{11}}{2} & \cdots & \cdots & \frac{c_{1n}\pm\gamma s_{1n}}{2} & \frac{c_{1n}\mp\gamma s_{1n}}{2} & c_{1n+1} \\ \frac{c_{11}\mp\gamma s_{11}}{2} & \frac{c_{11}\pm\gamma s_{11}}{2} & \cdots & \cdots & \frac{c_{1n}\mp\gamma s_{1n}}{2} & \frac{c_{1n}\pm\gamma s_{1n}}{2} & c_{1n+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \frac{c_{n1}\pm\gamma s_{n1}}{2} & \frac{c_{n1}\mp\gamma s_{n1}}{2} & \cdots & \cdots & \frac{c_{nn}\pm\gamma s_{nn}}{2} & \frac{c_{nn}\mp\gamma s_{nn}}{2} & c_{nn+1} \\ \frac{c_{n1}\mp\gamma s_{n1}}{2} & \frac{c_{n1}\pm\gamma s_{n1}}{2} & \cdots & \cdots & \frac{c_{nn}\mp\gamma s_{nn}}{2} & \frac{c_{nn}\pm\gamma s_{nn}}{2} & c_{nn+1} \\ \varphi^{(1)}_{n+1,1} & \varphi^{(2)}_{n+1,1} & \cdots & \cdots & \varphi^{(1)}_{n+1,n} & \varphi^{(2)}_{n+1,n} & c_{n+1,n+1} \end{bmatrix},$$

where

$$\phi^{(1)}_{n+1,i} + \phi^{(2)}_{n+1,i} = c_{n+1,i}, \quad 1 \leq i \leq n$$

realizes the list

$$\{\lambda_0, \lambda_1, \ldots, \lambda_n, \pm\gamma\mu_0, \pm\gamma\mu_1, \ldots, \pm\gamma\mu_{n-1}\}.$$

Now we present the extension of above theorem.

Theorem 3.2. *Suppose that $A = (A_{ij})$ is an into block square matrix of order $2n + k$ where*

$$A_{ij} = \begin{cases} \begin{bmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{bmatrix} & 1 \leq i, j \leq n \\ \begin{bmatrix} a_{ij} \\ a_{ij} \end{bmatrix} & 1 \leq i \leq n, \quad n+1 \leq j \leq n+k \\ \begin{bmatrix} a_{ij} & b_{ij} \end{bmatrix} & 1 \leq j \leq n, \quad n+1 \leq i \leq n+k \\ a_{ij} & n+1 \leq i, j \leq n+k \end{cases}$$

If

$$c_{ij} = \begin{cases} a_{ij} + b_{ij} & 1 \leq i, j \leq n \\ a_{ij} & n+1 \leq j \leq n+k, \quad 1 \leq i \leq n \\ a_{ij} + b_{ij} & n+1 \leq i \leq n+k, \quad 1 \leq j \leq n \\ a_{ij} & n+1 \leq i, j \leq n+k \end{cases}$$

and

$$s_{ij} = a_{ij} - b_{ij}, \quad 1 \leq i, j \leq n$$

Then

$$\sigma(A) = \sigma(S) \cup \sigma(C)$$

where

$$S = (s_{ij}) \quad \textit{and} \quad C = (c_{ij})$$

Proof. Let (λ, v) be an eigenpair of C, with $v := (v_1, v_2, ..., v_{n+k})$, and consider the $(2n+k)$-by-1 block vector $\begin{bmatrix} (\omega_j) \\ \omega_{n+1} \\ \omega_{n+2} \\ \vdots \\ \omega_{n+k} \end{bmatrix}$, where

$$w_j = \begin{cases} v_j \cdot e & 1 \leq j \leq n \\ v_j & n+1 \leq j \leq n+k \end{cases}$$

and $e = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

Since

$$A_{ij} \cdot w_j = \begin{bmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{bmatrix} \cdot v_j.e = \begin{bmatrix} a_{ij} + b_{ij} \\ b_{ij} + a_{ij} \end{bmatrix} .v_j = c_{ij}.v_j \cdot e$$
$$i, j = 1, 2, \ldots, n$$

$$A_{ij} \cdot w_j = \begin{bmatrix} a_{ij} \\ a_{ij} \end{bmatrix} \cdot w_j = c_{ij}.v_j.e$$
$$1 \leq i \leq n, \;\; n+1 \leq j \leq n+k$$

$$A_{ij} \cdot w_j = a_{ij}.v_j = c_{ij}.v_j \qquad\qquad n+1 \leq i, j \leq n+k$$

$$A_{ij} \cdot w_j = \begin{bmatrix} a_{ij} & b_{ij} \end{bmatrix} \cdot w_j = \begin{bmatrix} a_{ij} & b_{ij} \end{bmatrix} \cdot v_j.e = (a_{ij} + b_{ij}).v_j = c_{ij}.v_j$$
$$n+1 \leq i \leq n+k, \;\; 1 \leq j \leq n$$

Notice that, for every $i \in \{1, 2, ..., n\}$

$$\sum_{j=1}^{n+k} A_{ij}.w_j = \sum_{j=1}^{n} A_{ij}.w_j + \sum_{j=n+1}^{n+k} A_{ij}.w_j = \sum_{j=1}^{n} c_{ij} v_j e + \sum_{j=n+1}^{n+k} c_{ij} v_j e = \lambda v_i e = \lambda w_i.$$

and for every $i \in \{n+1, ..., n+k\}$

$$\sum_{j=1}^{n+k} A_{ij}.w_j = \sum_{j=1}^{n} A_{ij} w_j + \sum_{j=n+1}^{n+k} A_{ij} w_j = \sum_{j=1}^{n} c_{ij} v_j + \sum_{j=n+1}^{n+k} c_{ij} v_j = \lambda v_i = \lambda w_i.$$

i.e (λ, ω) is an eigenpair of A. Thus $\sigma(C) \subseteq \sigma(A)$.
Similarly, let (μ, x) be an eigenpair of S, with $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ and consider the $(2n+k)$-by-1 block vector $Y := \begin{bmatrix} y_j \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, where $y_j := x_j f$ and $f = (1, -1)^T$. Since

$$A_{ij} y_j = \begin{bmatrix} a_{ij} & b_{ij} \\ b_{ij} & a_{ij} \end{bmatrix} \cdot x_j.f = \begin{bmatrix} a_{ij} - b_{ij} \\ b_{ij} - a_{ij} \end{bmatrix} x_j = s_{ij} x_j.f$$

notice that, for every $i = 1, 2, ..., n$

$$\sum_{j=1}^{n+k} A_{ij} y_j = \sum_{j=1}^{n} A_{ij} y_j + \sum_{j=n+1}^{n+k} A_{ij} y_j = \sum_{j=1}^{n} s_{ij} y_j = (\mu x_i) f = \mu y_i$$

i.e (μ, y) be an eigenpair of A. Thus $\sigma(S) \subseteq \sigma(A)$. Suppose that

$$\Theta_c = \{(x_{1i}, x_{2i}, \ldots, x_{ni}, x_{n+1i}, \ldots x_{n+ki})^T : i = 1, 2, \ldots, n+k\}$$

$$\Theta_s = \{(y_{1i}, y_{2i}, \ldots, y_{ni})^T : i = 1, 2, \ldots, n\}$$

are bases of eigenvectors of C and S, respectively. The result will follow after proving the linear independence of the following set $\Upsilon = \Upsilon_1 \cup \Upsilon_2$, where

$$\Upsilon_1 = \{(x_1 e^T, x_2 e^T, \ldots, x_{n+k} e^T)^T : (x_1, x_2, \ldots, x_{n+k})^T \in \Theta_c\}$$

$$\Upsilon_2 = \{(y_1 f^T, y_2 f^T, \ldots, y_n f^T, 0, \ldots, 0)^T : (y_1, y_2, \ldots, y_n)^T \in \Theta_s\}$$

To this aim, we study the next determinant:

$$d = \begin{vmatrix} y_{11} & \cdots & y_{1n} & x_{11} & \cdots & x_{1n} & x_{1,n+1} & \cdots & x_{1,n+k} \\ -y_{11} & \cdots & -y_{1n} & x_{11} & \cdots & x_{1n} & x_{1,n+1} & \cdots & x_{1,n+k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} & x_{n1} & \cdots & x_{nn} & x_{n,n+1} & \cdots & x_{n,n+k} \\ -y_{n1} & \cdots & -ynn & x_{n1} & \cdots & x_{nn} & x_{n,n+1} & \cdots & x_{n,n+k} \\ 0 & \cdots & 0 & x_{n+1,1} & \cdots & x_{n+1,n} & x_{n+1,n+1} & \cdots & x_{n+1,n+k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{n+k,1} & \cdots & x_{n+k,n} & x_{n+k,n+1} & \cdots & x_{n+k,n+k} \end{vmatrix}.$$

Note that d stands for the determinant of a $(2n+k)$-by-$(2n+k)$ matrix obtained from the coordinates of the vectors in Υ. As before, adding rows and making suitable row permutations we conclude that the absolute value of d coincides with the absolute value of the following determinant

$$d = \begin{vmatrix} y_{11} & \cdots & y_{1n} & x_{11} & \cdots & x_{1n} & x_{1,n+1} & \cdots & x_{1,n+k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} & x_{n1} & \cdots & x_{nn} & x_{n,n+1} & \cdots & x_{n,n+k} \\ 0 & \cdots & 0 & 2x_{11} & \cdots & 2x_{1n} & 2x_{1,n+1} & \cdots & 2x_{1,n+k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 2x_{n,1} & \cdots & 2x_{n,n} & 2x_{n,n+1} & \cdots & 2x_{n,n+k} \\ 0 & \cdots & 0 & x_{n+1,1} & \cdots & x_{n+1,n} & x_{n+1,n+1} & \cdots & x_{n+1,n+k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & x_{n+k,1} & \cdots & x_{n+k,n} & x_{n+k,n+1} & \cdots & x_{n+k,n+k} \end{vmatrix}$$

which is nonzero by the linear independence of the set Θ_c and Θ_s. Thus the statement follows.

□

Example 3.3. Let $\sigma(S) = \{\frac{1+i\sqrt{3}}{2}, -1, \frac{1-i\sqrt{3}}{2}\}$ and $\sigma(C) = \{21, -3-3i\sqrt{3}, -3-i\sqrt{3}, -3, -3+i\sqrt{3}, -3+3i\sqrt{3}\}$.Then by theorem **??** and **??** we have:

$$S = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{bmatrix}$$

whose that C is a circulant matrix and S is a skew-circulant matrix. Then the matrix A obtained from S and C with the techniques above:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 2 & 4 & 5 & 6 \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 & 2 & 1 & 4 & 5 & 6 \\ \frac{7}{2} & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 3 & 4 & 5 \\ \frac{5}{2} & \frac{7}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 3 & 4 & 5 \\ \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 2 & 3 & 4 \\ \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{7}{2} & \frac{1}{2} & \frac{1}{2} & 2 & 3 & 4 \\ 4 & 0 & 5 & 0 & 6 & 0 & 1 & 2 & 3 \\ 3 & 0 & 4 & 0 & 5 & 0 & 6 & 1 & 2 \\ 2 & 0 & 3 & 0 & 4 & 0 & 5 & 6 & 1 \end{bmatrix}$$

All eigenvalues of matrix M is

$$\sigma(A) = \{21, -3-3i\sqrt{3}, -3-i\sqrt{3}, -3, -3+i\sqrt{3}, -3+3i\sqrt{3}, \frac{1+i\sqrt{3}}{2}, -1, \frac{1-i\sqrt{3}}{2}\} = \sigma(C) \cup \sigma(S)$$

Example 3.4. Let $\sigma(S) = \{3+i, 3-i\}$ and $\sigma(C) = \{15, 2+5i, 1, 2-5i\}$. Then we can compute matrix S and C by theorems **??** and **??**. So we have:

$$S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 6 & 3 & 1 \\ 1 & 5 & 6 & 3 \\ 3 & 1 & 5 & 6 \\ 6 & 3 & 1 & 5 \end{bmatrix},$$

where C is a circulant and S is a skew-circulant matrix. Then the matrix A obtained from S and C with the techniques above:

$$A = \begin{bmatrix} 4 & 1 & \frac{5}{2} & \frac{7}{2} & 3 & 1 \\ 1 & 4 & \frac{7}{2} & \frac{5}{2} & 3 & 1 \\ 1 & 0 & 4 & 1 & 6 & 3 \\ 0 & 1 & 1 & 4 & 6 & 3 \\ 3 & 0 & 1 & 0 & 5 & 6 \\ 6 & 0 & 3 & 0 & 1 & 5 \end{bmatrix}.$$

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