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Extension of Topological Derived Set Operator and Topological Closure Set Operator Via a Class of Sets to Construct Generalized Topologies

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Abstract. In this paper, we intend to extract some types of generalized topologies from a topological space. To do this, we first generalize the derived set operator and the closure operator of a topological space using a class of subsets of the space, this collection is called the hereditary family since it is closed under the operation subset. The generalized closure operator induces a structure that is our desired generalized topology.
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1. Introduction

The purpose of this paper is to create a special type of generalized topological spaces from ordinary topological spaces. So first let us state some of the required concepts related to topology and generalized topology.

Following we recall some of notions concerning topological spaces from [**?**]. A topological space is a pair of (X, τ) in which X is a set and τ is a collection of subsets of X including $\emptyset$ and X which is closed under arbitrary union and finite intersection. Those subsets of X, which are

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members of τ, are called open (sub)set in the space X. A subset $F \subseteq X$ is closed in (X, τ), if its complement $X - F$ is an open set.

In any topological space (X, τ) associated with the topology τ is the topological closure operator, denoted by cl_τ (in short, cl), which gives for any subset $A \subseteq X$, the smallest closed set containing A. Obviously, a set A is closed in (X, τ) if and only if $cl_\tau(A) = A$.

Denote $\mathcal{P}(X)$ as the powerset of X. Then cl as defined above can be viewed as an operator $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ that satisfies the following properties (for arbitrary $A, B \subseteq X$):

(1) $cl(\emptyset) = \emptyset$,
(2) $A \subseteq cl(A)$,
(3) $cl(cl(A)) = cl(A)$,
(4) $cl(A \cup B) = cl(A) \cup cl(B)$.

Indeed, any operator cl on $\mathcal{P}(X)$ that satisfies the above four axioms (called Kuratowski Closure Axioms) defines a topological closure operator. So that the collection $\{X - A : cl(A) = A\}$ gives rise a set system that will properly be a topology. In this sense, we can say that an operator satisfying the Kuratowski Closure Axioms $(1) - (4)$ defines a topological space (X, τ).

Here we ask permission to describe briefly the history of the formation of the notion of a generalized topology.

The beginning of the story of generalized topologies dates back to 1963. At that time N. Levein in an article entitled "Semi-open sets and semi-continuity in topological spaces" [?], tried to generalize topology by replacing semi-open sets with open sets, which became the starting point for forming a ground for research to generalize topology and a variety of generalized open sets.

Some of these efforts led to the introduction of important and valuable types of generalized open sets, for example in 1965 α-open sets [?], in 1979 feebly open sets [?], in 1982, pre-open sets [?] and in 1983 β-open sets[?] were introduced. These efforts continued until, in 1997, Á. Császár finally succeeded in organizing them in the form of a new concept called γ-open sets([?]). Császár found that γ-open sets closed under arbitrary union and empty set is γ-open. Some time later, the concept of γ-open sets led Császár to introduce the concept of generalized topology in 2002 [?].

What we present below are concepts related to a generalized topology, which is taken from [?], [?], [?], [?], [?]and [?].

Simply put, a structure that results from the removal of the underlying set and finite intersection condition from a topology is called a general topology. In fact, a generalized topology on a set X is a subset μ of $\mathcal{P}(X)$ that contains $\emptyset$ and any union of elements of μ belongs to μ. In other words, we replace the family of open sets with a larger one. Therefore every topology is a generalized topology. A set X with a generalized topology μ on it, is called a generalized topological space and is denoted by (X, μ). Subsets of X, which are members of μ, are called generalized open (sub)set in the space X and a subset $F \subseteq X$ is called generalized closed in (X, μ) if its complement $X - F$ is an element of μ. A generalized topology is named strong if $X \in \mu$.

In addition, Császár showed that corresponding to any generalized topology, there is an operator called the envelope operator whose fixed points are exactly the same as the generalized closed sets of space. In the sense of Császár an operator $\lambda : \mathcal{P}(X) \to \mathcal{P}(X)$ is an envelope operator on X if it satisfies the following properties (for arbitrary $A, B \subseteq X$):

(1) $A \subseteq \lambda(A)$ (property of enlarging),
(2) $\lambda(A) \subseteq \lambda(B)$ whenever $A \subseteq B$ (property of monotonicity),
(3) $\lambda(\lambda(A)) = \lambda(A)$ (property of idempotency).

The envelope operator can be considered as a generalization of the Kuratowski closure operator.

In this paper using mathematical tool called hereditary family, we intend to present a special kind of envelope operator on any topological space to extract some generalized topologies from that space.

By definition ([**?**]) a collection $\mathfrak{H}$ of subsets of a set X is called hereditary if every subset of a member of $\mathfrak{H}$ is also a member of $\mathfrak{H}$.

The collection of all finite subsets of X(especially, the set $\{\emptyset\}$), the collection of all countable subsets of X and the collection of all subsets of X with empty interior have the property of hereditary.

2. Main result

In this section using hereditary families, we define a specific type of envelope operator, but before that, we define another operator that can be considered as a type of generalization of the derived set operator. We emphasize that our pattern in this regard is [**?**]. This section is presented in three subsections.

2.1. Operator $\Phi_{\mathfrak{H}}$ as a special extension of the derived set operator.

We start by defining the desired operator.

Definition 2.1. Let (X, τ) be a topological space and $\mathfrak{H}$ be an arbitrary hereditary family on it. We define an operator $\Phi_{\mathfrak{H}} : \mathcal{P}(X) \to \mathcal{P}(X)$ as the following:

$$\Phi_{\mathfrak{H}}(A) = \{x \in X : A \cap U \notin \mathfrak{H},\ \forall\, U \in \tau(x)\}, \tag{2.1}$$

where $\tau(x) = \{U \in \tau : x \in U\}$.

Note: Considering the collection $\mathfrak{H} = \mathfrak{F} =$ *the set of all finite subsets* *of* X, as a hereditary family in Equation **??**, we will have;

$$\Phi_{\mathfrak{F}}(A) = \{x \in X : A \cap U \neq \textit{ finite set}, \ \forall\, U \in \tau(x)\} = A^d,$$

where A^d is the derived set of A that contains all cluster points of A.

With a glanced at the definition **??**, we will have the following lemma.

Lemma 2.2. *Let (X,τ) be a topological space and $\mathfrak{H}$ be an arbitrary hereditary family on it, then the operator $\Phi_{\mathfrak{H}}$ defined in Definition* **??** *has the following properties;*

(1) *For any subset A of X, $x \notin \Phi_{\mathfrak{H}}(A)$ if and only if there exists some $U \in \tau(x)$ such that $U \cap A \in \mathfrak{H}$.*
(2) *For any subset A of X and $U \in \tau$, $U \cap \Phi_{\mathfrak{H}}(A) \in \mathfrak{H}$ whenever, $U \cap A \in \mathfrak{H}$. In particular, $U \cap \Phi_{\mathfrak{H}}(A) = \emptyset$.*

Proof. (1) It is obvious from definition.

(2) Let $U \in \tau$ with $U \cap A \in \mathfrak{H}$. If $U \cap \Phi_{\mathfrak{H}}(A) \notin \mathfrak{H}$, then from $U \cap \Phi_{\mathfrak{H}}(A) \notin \mathfrak{H}$, it follows that $U \cap \Phi_{\mathfrak{H}}(A) \neq \emptyset$ (because the empty set is a member of any hereditary family). So there exists an element $y \in U \cap \Phi_{\mathfrak{H}}(A)$. Since U is an open set containing y and $y \in \Phi_{\mathfrak{H}}(A)$, from the definition of $\Phi_{\mathfrak{H}}(A)$, $U \cap A \notin \mathfrak{H}$ and it is a contradiction. Consequently, $U \cap \Phi_{\mathfrak{H}}(A) \in \mathfrak{H}$.

For the proof of the last statement, assume that $U \cap \Phi_{\mathfrak{H}}(A) \neq \emptyset$ for some $U \in \tau$. Then there exists an element $y \in U \cap \Phi_{\mathfrak{H}}(A)$ and so $U \cap A \notin \mathfrak{H}$, again it is a contradiction. □

From the above theorem, we have the following result;

Corollary 2.3. *Let (X,τ) be a T_1 topological space. If $U \cap A = \emptyset$ for some $U \in \tau$, then $U \cap A^d = \emptyset$.*

Proof. Placing $\mathfrak{H} = \mathfrak{F}$ (as the hereditary family of finite subsets of X), in part (2) of the Lamma **??** completes the proof. □

According to [**?**], we know that the derived set operator associated with a space (X,τ) for every $A, B \subseteq X$ satisfies the following;

(1) $\emptyset^d = \emptyset$,
(2) $A \subseteq B (\subseteq X)$ implies $A^d \subseteq B^d$,
(3) $A^d \subseteq cl_\tau A$,
(4) $(A^d)^d \subseteq A^d$,
(5) A^d is closed.

Corresponding to the properties of derived set operator(mentioned above), we state the following theorem concerning $\Phi_{\mathfrak{H}}$.

Theorem 2.4. *Let $\mathfrak{H}$ be a hereditary family on a topological space (X,τ). Then for $A \subseteq X$,*

(1) *If* $A \in \mathfrak{H}$ *then* $\Phi_{\mathfrak{H}}(A) = \emptyset$*, and especially* $\Phi_{\mathfrak{H}}(\emptyset) = \emptyset$*,*
(2) $\Phi_{\mathfrak{H}}$ *is an enlarging operator,*
(3) $\Phi_{\mathfrak{H}}(A) \subseteq clA$*,*
(4) $\Phi_{\mathfrak{H}}(\Phi_{\mathfrak{H}}(A)) \subseteq \Phi_{\mathfrak{H}}(A)$*,*
(5) $\Phi_{\mathfrak{H}}(A)$ *is closed in* (X, τ)*.*

Proof. (1) It is obvious from the definition.
(2) It is easily obtained from the property of a hereditary family.
(3) Suppose that $x \notin clA$, then there exists $U \in \tau(x)$ such that $U \cap A = \emptyset$. Now, $U \cap A = \emptyset$ implies $U \cap A \in \mathfrak{H}$ and so from part (1) of this theorem, we have $x \notin \Phi_{\mathfrak{H}}(A)$.
(4) Let $x \in \Phi_{\mathfrak{H}}(\Phi_{\mathfrak{H}}(A))$, then for every $U \in \tau(x)$, $\Phi_{\mathfrak{H}}(A) \cap U \notin \mathfrak{H}$ and therefore $\Phi_{\mathfrak{H}}(A) \cap U \neq \emptyset$. Since $\Phi_{\mathfrak{H}}(A) \cap U \neq \emptyset$, so there exists an element $z \in \Phi_{\mathfrak{H}}(A) \cap U$, then $z \in \Phi_{\mathfrak{H}}(A)$ and U is also an open set containing z. Now, from the definition of operator $\Phi_{\mathfrak{H}}$, we have $A \cap U \notin \mathfrak{H}$, and so $x \in \Phi_{\mathfrak{H}}(A)$.
(5) Here it is enough to show $cl(\Phi_{\mathfrak{H}}(A)) = \Phi_{\mathfrak{H}}(A)$. Clearly $cl(\Phi_{\mathfrak{H}}(A)) \supseteq \Phi_{\mathfrak{H}}(A)$ so we show $cl(\Phi_{\mathfrak{H}}(A)) \subseteq \Phi_{\mathfrak{H}}(A)$. Let $x \in cl(\Phi_{\mathfrak{H}}(A))$, so for each $U \in \tau(x)$, $U \cap \Phi_{\mathfrak{H}}(A) \neq \emptyset$. Now, according to part (2) of Lemma **??** we have, $U \cap A \in \mathfrak{H}$ and therefore $x \in \Phi_{\mathfrak{H}}(A)$.

□

The following example shows that reverse inclusion in part (5) of Theorem **??** may not be hold in general.

Example 2.5. Suppose $X = \{a, b, c, d\}$. Let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathfrak{H} = \{\emptyset, \{a\}, \{c\}, \{d\}, \{c, d\}\}$ be a topology and a hereditary family, respectively on X. For $A = \{a, c, d\}$, we have $\Phi_{\mathfrak{H}}(A) = \{c, d\}$ and $\Phi_{\mathfrak{H}}(\Phi_{\mathfrak{H}}(A)) = \emptyset$. So, $\Phi_{\mathfrak{H}}(\Phi_{\mathfrak{H}}(A)) \neq \Phi_{\mathfrak{H}}(A)$.
Note: In the previous example we have $\Phi_{\mathfrak{H}}(X) = \{b, c, d\} \neq X$. So equality $\Phi_{\mathfrak{H}}(X) = X$ is not generally valid. In addition, for $A = \{a, b\}$, we have $\Phi_{\mathfrak{H}}(A) = \{b, c, d\} \neq A$ which indicates that there is no inclusion relation between $\Phi_{\mathfrak{H}}(A)$ and A, in general.

Theorem 2.6. *Let* $\mathfrak{H}$ *be a hereditary family on a topological space* (X, τ)*. Then for* $A \subseteq X$*,* $\Phi_{\mathfrak{H}}(A \cup \Phi_{\mathfrak{H}}(A)) = \Phi_{\mathfrak{H}}(A)$*.*

Proof. Due to the enlarging property of the operator $\Phi_{\mathfrak{H}}$ (expressed in part (2) of Theorem **??**) we will have

$$\Phi_{\mathfrak{H}}(A) \subseteq \Phi_{\mathfrak{H}}(A \cup \Phi_{\mathfrak{H}}(A)).$$

For the other inclusion, let $x \notin \Phi_{\mathfrak{H}}(A)$. So, there exists an open set U containing x such that $U \cap A \in \mathfrak{H}$, and thus from Lemma **??**, $U \cap \Phi_{\mathfrak{H}}(A) = \emptyset$. Hence, $U \cap (A \cup \Phi_{\mathfrak{H}}(A)) = (U \cap A) \cup (U \cap \Phi_{\mathfrak{H}}(A)) = U \cap A \in \mathfrak{H}$, and therefore, $x \notin \Phi_{\mathfrak{H}}(A \cup \Phi_{\mathfrak{H}}(A))$. □

As mentioned earlier, by putting $\mathfrak{H} = \mathfrak{F}$ in the results which are stated about the operator $\Phi_{\mathfrak{H}}$, some classical results concerning the derived set operator can be extract.
By placing $\mathfrak{H} = \mathfrak{F}$ in the previous theorem, we will have the following corollary;

Corollary 2.7. *For any T_1-topological space (X, τ) and $A \subseteq X$, we have $(A \cup A^d)^d = A^d$.*

Lemma 2.8. *Let $\mathfrak{H}$ be a hereditary family on a topological space (X, τ). If $U \in \tau$, then $U \cap \Phi_{\mathfrak{H}}(A) = U \cap \Phi_{\mathfrak{H}}(U \cap A)$.*

Proof. According to part (2) of Theorem **??**, clearly we have;

$$U \cap \Phi_{\mathfrak{H}}(U \cap A) \subseteq U \cap \Phi_{\mathfrak{H}}(A).$$

For the other inclusion, let $x \in U \cap \Phi_{\mathfrak{H}}(A)$ and $(U \neq)V \in \tau(x)$. Then $x \in U \cap V$ and $x \in \Phi_{\mathfrak{H}}(A)$. So $(U \cap A) \cap V = (U \cap V) \cap A \notin \mathfrak{H}$, which implies $x \in \Phi_{\mathfrak{H}}(U \cap A)$ therefore, $x \in U \cap \Phi_{\mathfrak{H}}(U \cap A)$. $\square$

By putting $\mathfrak{H} = \mathfrak{F}$ in the previous lemma, the following result is obtained;

Corollary 2.9. *Let (X, τ) be a T_1-topological space. If $U \in \tau$, then we have $U \cap A^d = U \cap (U \cap A)^d$.*

In example **??**, we saw that there is no any inclusion relation between $\Phi_{\mathfrak{H}}(A)$ and A in general. The following theorem introduces the conditions under which we can establish inclusion relation between $\Phi_{\mathfrak{H}}(A)$ and A for certain subsets of (X, τ).

Theorem 2.10. *Let (X, τ) is a topological space and $\mathfrak{H}$ is a hereditary family on it. If $\tau \cap \mathfrak{H} = \{\emptyset\}$ and $U \in \tau$, then $U \subseteq \Phi_{\mathfrak{H}}(U)$.*

Proof. Considering the condition $\tau \cap \mathfrak{H} = \{\emptyset\}$, in the definition of the operator $\Phi_{\mathfrak{H}}$ leads to equality $\Phi_{\mathfrak{H}}(X) = X$.
Using equality $\Phi_{\mathfrak{H}}(X) = X$ in Lemma **??** leads to $U \subseteq \Phi_{\mathfrak{H}}(U)$, because

$$U = U \cap \Phi_{\mathfrak{H}}(X) = U \cap \Phi_{\mathfrak{H}}(U \cap X) = U \cap \Phi_{\mathfrak{H}}(U) \subseteq \Phi_{\mathfrak{H}}(U).$$

$\square$

Corollary 2.11. *Let τ be a T_1-topology on X. If τ does not contain any finite subset of X except $\emptyset$, then for $U \in \tau$ we have $U \subseteq U^d$.*

Proof. Just put $\mathfrak{H} = \mathfrak{F}$ in Theorem **??**. $\square$

For any $A \subseteq X$, from part (3) of Theorem **??** we have $\Phi_{\mathfrak{H}}(A) \subseteq clA$. The following theorem presents a condition under which we will have $\Phi_{\mathfrak{H}}(A) = clA$

Theorem 2.12. *Let $\mathfrak{H}$ be a hereditary family on a topological space (X, τ). If $\tau \cap \mathfrak{H} = \{\emptyset\}$, then $clU = \Phi_{\mathfrak{H}}(U)$ for $U \in \tau$.*

Proof. Here it is enough to prove $clU \subseteq \Phi_{\mathfrak{H}}(U)$. Let $U \in \tau$ and $x \notin \Phi_{\mathfrak{H}}(U)$, then there exists an open set G containing x such that $U \cap G \in \mathfrak{H}$. Now since $U \cap G$ is an open, so according to the assumption $\tau \cap \mathfrak{H} = \{\emptyset\}$, we must have $U \cap G = \emptyset$, that is $x \notin clU$. □

Considering $\mathfrak{H} = \mathfrak{F}$ in Theorem **??** yields the following corollary;

Corollary 2.13. *Let τ be a T_1-topology on X. If τ does not contain any finite subset of X except the empty set, then $clU = U^d$ for any $U \in \tau$.*

Proof. From Corollary **??** we have $U \subseteq U^d$, so $clU = U \cup U^d = U^d$. □

2.2. **Operator $\Psi_{\mathfrak{H}}$ as a special extension of the closure operator.**

Definition 2.14. Let (X, τ) be a topological space and let $\mathfrak{H}$ be a hereditary family on the space. For any $A \subseteq X$, we define an operator

$$\Psi_{\mathfrak{H}} : \mathcal{P}(X) \to \mathcal{P}(X)$$

by;

$$\Psi_{\mathfrak{H}}(A) = A \cup \Phi_{\mathfrak{H}}(A). \tag{2.2}$$

Remark 2.15. *In T_1 space (X, τ), the operator $\Psi_{\mathfrak{H}}$ with respect to hereditary family of finite sets of X coincides with the closure set operator, that is, for any $A \subseteq X$; $\Psi_{\mathfrak{F}}(A) = clA$*

We know from [**?**] that the closure operator has the following properties;

1. $cl(\emptyset) = \emptyset$,
2. $A \subseteq clA$,
3. $cl(clA) = clA$,
4. $cl(A \cup B) = clA \cup clB$,
5. $A \subseteq B$ implies $clA \subseteq clB$.
6. clA is closed in (X, τ).

In the next theorem, we examine the validity of the above properties for the operator $\Psi_{\mathfrak{H}}$.

Theorem 2.16. *Let (X, τ) be a topological space and $\mathfrak{H}$ be a hereditary family on it. For $A \subseteq X$;*

1. *$\Psi_{\mathfrak{H}}(\emptyset) = \emptyset$,*
2. *$A \subseteq \Psi_{\mathfrak{H}}(A)$; moreover $\Psi_{\mathfrak{H}}(X) = X$,*
3. *$A \subseteq B \subseteq X$ implies $\Psi_{\mathfrak{H}}(A) \subseteq \Psi_{\mathfrak{H}}(B)$, that is, $\Psi_{\mathfrak{H}}$ is a monotonic operator,*
4. *$\Psi_{\mathfrak{H}}(\Psi_{\mathfrak{H}}(A)) = \Psi_{\mathfrak{H}}(A)$, that is, $\Psi_{\mathfrak{H}}$ is an idempotent operator,*
5. *For $A, B \subseteq X, \Psi_{\mathfrak{H}}(A \cup B) \supseteq \Psi_{\mathfrak{H}}(A) \cup \Psi_{\mathfrak{H}}(B)$.*

Proof. (1) From Theorem **??** we have $\Phi_{\mathfrak{H}}(\emptyset) = \emptyset$, so by Definition **??**, $\Psi_{\mathfrak{H}}(\emptyset) = \emptyset \cup \Phi_{\mathfrak{H}}(\emptyset) = \emptyset$.
(2) Obvious.
(3) For $A \subseteq B \subseteq X$, from part (2) of Theorem **??**, we have $\Phi_{\mathfrak{H}}(A) \subseteq \Phi_{\mathfrak{H}}(B)$, so according to Definition **??**, $\Psi_{\mathfrak{H}}(A) = A \cup \Phi_{\mathfrak{H}}(A) \subseteq B \cup \Phi_{\mathfrak{H}}(B) = \Psi_{\mathfrak{H}}(B)$.
(4) By definition of the operator $\Psi_{\mathfrak{H}}$ we have, $\Psi_{\mathfrak{H}}(\Psi_{\mathfrak{H}}(A)) = \Psi_{\mathfrak{H}}(A \cup \Phi_{\mathfrak{H}}(A)) = (A \cup \Phi_{\mathfrak{H}}(A)) \cup \Phi_{\mathfrak{H}}(A \cup \Phi_{\mathfrak{H}}(A))(by\ Theorem\ \mathbf{??}) = A \cup \Phi_{\mathfrak{H}}(A) \cup \Phi_{\mathfrak{H}}(A) = A \cup \Phi_{\mathfrak{H}}(A) = \Psi_{\mathfrak{H}}(A)$.
(5) It is obvious from part (3) of the theorem. □

In the following example, we show that in case (5) of the above theorem, equality does not occur in the general case.

Example 2.17. Remember Example **??** and put $A = \{a\}$ and $B = \{b, c\}$. Note that $\Phi_{\mathfrak{H}}(A) = \emptyset$ and $\Phi_{\mathfrak{H}}(B) = \{b, c, d\}$ and $\Phi_{\mathfrak{H}}(A \cup B) = \Phi_{\mathfrak{H}}(\{a, b, c\}) = \{b, c, d\}$. So $\Psi_{\mathfrak{H}}(A) = \{a\}$ and $\Psi_{\mathfrak{H}}(B) = \{b, c, d\}$ and thus $\Psi_{\mathfrak{H}}(A) \cup \Psi_{\mathfrak{H}}(B) = \{b, c, d\}$ but $\Psi_{\mathfrak{H}}(A \cup B) = \{a, b, c, d\}$. So $\Psi_{\mathfrak{H}}(A) \cup \Psi_{\mathfrak{H}}(B) \neq \Psi_{\mathfrak{H}}(A \cup B)$.

Corollary 2.18. *The properties* (2), (3) *and* (4) *of the theorem* **??** *show that the operator* $\Psi_{\mathfrak{H}}$ *is an envelope operator in the sense of Császár, but as Example* **??** *shows this operator cannot be a Kuratowski closure operator.*

Theorem 2.19. *Let* (X, τ) *be a topological space. Then for any hereditary family* $\mathfrak{H}$ *on* X *the operator* $\Psi_{\mathfrak{H}} : \mathcal{P}(X) \to \mathcal{P}(X)$ *induces a generalized topology on* X*.*

Proof. Császár in [**?**] proved that any envelope operator can construct a generalized topology. □

Remark 2.20. *As shown in the example* **??***, the equality in* (5) *of Theorem* **??** *is not true in general and thus the operator* $\Psi_{\mathfrak{H}}$ *cannot be a Kuratowski closure operator, so* $\Psi_{\mathfrak{H}}$ *cannot induce a topology on* X*.*

2.3. $\tau\mathfrak{H}$ as a strong generalized topology on X.

Definition 2.21. Let (X, τ) be a topological space and $\mathfrak{H}$ be a hereditary family on X. We define

$$\tau\mathfrak{H} = \{X - A : A \subseteq X,\ \Psi_{\mathfrak{H}}(A) = A\} \tag{2.3}$$

From Theorem **??** we know that the collection $\tau\mathfrak{H}$ is a generalized topology on X, but in the next theorem we prove this directly, that is, we prove that the collection $\tau\mathfrak{H}$ is closed under any arbitrary unions of its elements.

Theorem 2.22. *Let $\mathfrak{H}$ be a hereditary family on a topological space (X, τ). Then $\tau_{\mathfrak{H}}$ is closed under each of the union of its elements.*

Proof. Suppose Λ be an arbitrary indexing set, and $\{U_\alpha\}_{\alpha\in\Lambda} \subseteq \tau_{\mathfrak{H}}$. Let $\alpha_0 \in \Lambda$, then acccording to the enlarging property of $\Phi_{\mathfrak{H}}$ we have $\Phi_{\mathfrak{H}}(X - \cup_{\alpha\in\Lambda}U_\alpha) \subseteq \Phi_{\mathfrak{H}}(X - U_{\alpha_0})$. Now, $U_{\alpha_0} \in \tau_{\mathfrak{H}}$ implies that $\Phi_{\mathfrak{H}}(X - U_{\alpha_0}) \subseteq X - U_{\alpha_0}$, so $\Phi_{\mathfrak{H}}(X - \cup_{\alpha\in\Lambda}U_\alpha) \subseteq X - U_{\alpha_0}$. Because $\alpha_0 \in \Lambda$ be an arbitrary, then $\Phi_{\mathfrak{H}}(X - \cup_{\alpha\in\Lambda}U_\alpha) \subseteq \cap_{\alpha\in\Lambda}(X - U_\alpha) = X - \cup_{\alpha\in\Lambda}U_\alpha$. So $\Psi_{\mathfrak{H}}(X - \cup_{\alpha\in\Lambda}U_\alpha) = (X - \cup_{\alpha\in\Lambda}U_\alpha) \cup \Phi_{\mathfrak{H}}(X - \cup_{\alpha\in\Lambda}U_\alpha) = X - \cup_{\alpha\in\Lambda}U_\alpha$, and hence $\cup_{\alpha\in\Lambda}U_\alpha \in \tau_{\mathfrak{H}}$. $\square$

Remark 2.23. *We have the following two facts about $\tau_{\mathfrak{H}}$;*

(1) *$\emptyset \in \tau_{\mathfrak{H}}$, because according to the contract, if $\Lambda = \emptyset$ then we have $\cup_{\alpha\in\Lambda}U_\alpha = \emptyset$.*
(2) *$X \in \tau_{\mathfrak{H}}$, because from the definition* **??**, *$A \in \tau_{\mathfrak{H}}$ iff $\Psi_{\mathfrak{H}}(X - A) = X - A$. Considering $A = X$ we have, $X - A = \emptyset$, so from part* (1) *of Theorem* **??** *we have $\Psi_{\mathfrak{H}}(X - A) = \Psi_{\mathfrak{H}}(\emptyset) = \emptyset = X - A$, thus X belongs to $\tau_{\mathfrak{H}}$. So $\tau_{\mathfrak{H}}$ is a strong generalized topology on X.*

In Remark **??** we stated that the operator $\Psi_{\mathfrak{H}}$ is not a Kuratowski closure operator and therefore its induced structure can not be a topology, below we express this directly in the form of an example.

Example 2.24. Let $X = \{a, b, c, d, e\}$. If $\mathfrak{H} = \{\emptyset, \{e\}, \{c\}, \{c, e\}\}$ and $\tau = \{\emptyset, \{a\}, \{b, c, d\}, \{a, b, c, d\}, X\}$ are respectively, a hereditary family and a topology on X, then for $U_1 = \{a, b, c, d\}$ and $U_2 = \{a, b, d\}$, we have $\Phi_{\mathfrak{H}}(X - U_1) = \Phi_{\mathfrak{H}}(\{e\}) = \emptyset$ and $\Phi_{\mathfrak{H}}(X - U_2) = \Phi_{\mathfrak{H}}(\{c, e\}) = \emptyset$. So, $\Psi_{\mathfrak{H}}(X - U_1) = X - U_1$ and $\Psi_{\mathfrak{H}}(X - U_2) = X - U_2$, that is, $U_1, U_2 \in \tau_{\mathfrak{H}}$. But for $U_1 \cap U_2 = \{e\}$, $\Phi_{\mathfrak{H}}(X - (U_1 \cap U_2)) = \Phi_{\mathfrak{H}}(\{a, b, c, d\}) = \{a, b, c, d\}$ and $\Psi_{\mathfrak{H}}(X - (U_1 \cap U_2)) = X$. Now because $\Psi_{\mathfrak{H}}(X - (U_1 \cap U_2)) \neq X - (U_1 \cap U_2)$, so we have $U_1 \cap U_2 \notin \tau_{\mathfrak{H}}$.

Below, some facts of classical topology will be presented concerning the generalized topology $\tau_{\mathfrak{H}}$.

Definition 2.25. Let (X, τ) be a topological space and $\mathfrak{H}$ be a hereditary family on it. Then the elements of $\tau_{\mathfrak{H}}$ are said to be $\tau_{\mathfrak{H}}$-open, and also the complement of any memeber of $\tau_{\mathfrak{H}}$, is called $\tau_{\mathfrak{H}}$-closed.

Definition 2.26. Let $\mathfrak{H}$ be a hereditary family on a topological space (X, τ). For any $A \subseteq X$, the $\tau_{\mathfrak{H}}$-interior and $\tau_{\mathfrak{H}}$-closure of A are respectively;

$$i_{\tau_{\mathfrak{H}}}(A) = \cup\{U \subseteq X : U \in \tau_{\mathfrak{H}} \text{ and } U \subseteq A\}$$

$$c_{\tau_{\mathfrak{H}}}(A) = \cap\{F \subseteq X : X - F \in \tau_{\mathfrak{H}} \text{ and } A \subseteq F\}$$

The following two theorems are obviously obtained from the definitions of $c_{\tau_{\mathfrak{H}}}$ and $i_{\tau_{\mathfrak{H}}}$ and so their obvious proofs are omitted.

Theorem 2.27. *Let (X,τ) be a topological space and $\mathfrak{H}$ be a hereditary family on it. For $A,B \subseteq X$*

(1) $i_{\tau_{\mathfrak{H}}}(\emptyset) = \emptyset$,
(2) $i_{\tau_{\mathfrak{H}}}(A) \subseteq A$,
(3) *If* $A \subseteq B$, *then* $i_{\tau_{\mathfrak{H}}}(A) \subseteq i_{\tau_{\mathfrak{H}}}(B)$,
(4) $i_{\tau_{\mathfrak{H}}}(i_{\tau_{\mathfrak{H}}}(A)) = i_{\tau_{\mathfrak{H}}}(A)$.

Theorem 2.28. *Let $\mathfrak{H}$ be a hereditary family on topological space (X,τ). For $A,B \subseteq X$;*

(1) $c_{\tau_{\mathfrak{H}}}(X) = X$,
(2) $c_{\tau_{\mathfrak{H}}}(A) \supseteq A$,
(3) *If* $A \subseteq B$, *then* $c_{\tau_{\mathfrak{H}}}(A) \subseteq c_{\tau_{\mathfrak{H}}}(B)$,
(4) $c_{\tau_{\mathfrak{H}}}(c_{\tau_{\mathfrak{H}}}(A)) = c_{\tau_{\mathfrak{H}}}(A)$.

Theorem 2.29. *Let $\mathfrak{H}$ be a hereditary family on a topological space (X,τ). For $A \subseteq X$, A is $\tau_{\mathfrak{H}}$-closed iff $\Psi_{\mathfrak{H}}(A) = A$.*

Proof. A is $\tau_{\mathfrak{H}}$-closed iff $X - A$ is $\tau_{\mathfrak{H}}$-open iff $\Psi_{\mathfrak{H}}(X-(X-A)) = X-(X-A)$. So we have the statement. □

Theorem 2.30. *Let $\mathfrak{H}$ be a hereditary family on a topological space (X,τ). For $A \subseteq X$;*

(1) $x \in i_{\tau_{\mathfrak{H}}}(A)$ *iff there exists a $\tau_{\mathfrak{H}}$-open set U containing x such that $U \subseteq A$'*
(2) $x \in c_{\tau_{\mathfrak{H}}}(A)$ *iff for each $\tau_{\mathfrak{H}}$-open set V containing x, $A \cap V \neq \emptyset$.*

Proof. Obvious. □

Some important facts about operatores $i_{\tau_{\mathfrak{H}}}$ and $c_{\tau_{\mathfrak{H}}}$ are presented in the next theorems.

Theorem 2.31. *Let (X,τ) be a topological space and $\mathfrak{H}$ be a hereditary family on it. If $x \in i_{\tau_{\mathfrak{H}}}(A)$, then there exists some $W \in \tau(x)$ satisfying $A^c \cap W \in \mathfrak{H}$.*

Proof. For $x \in i_{\tau_{\mathfrak{H}}}(A)$, by Theorem **??**, there exists a $\tau_{\mathfrak{H}}$-open set U containing x such that $U \subseteq A$. From $X - U = \Psi_{\mathfrak{H}}(X-U)$, we have $x \notin \Phi_{\mathfrak{H}}(X-U)$, and so there exists an open set W containing x such that $(X-U) \cap W \in \mathfrak{H}$. Since $\mathfrak{H}$ is a hereditary family and $A^c \subseteq U^c$, we have $A^c \cap W \in \mathfrak{H}$. □

Theorem 2.32. *Let $\mathfrak{H}$ be a hereditary family on a topological space (X,τ). Then for $A \subseteq X$;*

(1) $c_{\tau_{\mathfrak{H}}}(A) = \Psi_{\mathfrak{H}}(A)$,

(2) *If* $A \notin \mathfrak{H}$, *then* $X - A \in \tau_{\mathfrak{H}}$,
(3) $\Phi_{\mathfrak{H}}(A)$ *is* $\tau_{\mathfrak{H}}$*-closed.*

Proof. (1) First, by Theorem **??** (5) and Theorem **??**, $\Psi_{\mathfrak{H}}(A)$ is $\tau_{\mathfrak{H}}$-closed. From $A \subseteq \Psi_{\mathfrak{H}}(A)$, it follows $A \subseteq c_{\tau_{\mathfrak{H}}}(A) \subseteq \Psi_{\mathfrak{H}}(A)$. Furthermore, since $A \subseteq c_{\tau_{\mathfrak{H}}}(A)$, from Theorem **??** (3) and Theorem **??**, it follows $\Psi_{\mathfrak{H}}(A) \subseteq \Psi_{\mathfrak{H}}(c_{\tau_{\mathfrak{H}}}(A)) = c_{\tau_{\mathfrak{H}}}(A)$. Consequently, $c_{\tau_{\mathfrak{H}}}(A) = \Psi_{\mathfrak{H}}(A)$.

(2) If $A \in \mathfrak{H}$, then by Lemma **??**, we know that $\Phi_{\mathfrak{H}}(A) = \emptyset$. So $\Psi_{\mathfrak{H}}(X - (X - A)) = \Psi_{\mathfrak{H}}(A) = A \cup \Phi_{\mathfrak{H}}(A) = A = X - (X - A)$, that is, $X - A \in \tau_{\mathfrak{H}}$.

(3) For $A \subseteq X$, from Theorem **??** (4), $\Psi_{\mathfrak{H}}(\Phi_{\mathfrak{H}}(A)) = \Phi_{\mathfrak{H}}(A) \cup \Phi_{\mathfrak{H}}(\Phi_{\mathfrak{H}}(A)) = \Phi_{\mathfrak{H}}(A)$. By Theorem **??**, $\Phi_{\mathfrak{H}}(A)$ is $\tau_{\mathfrak{H}}$-closed

□

Theorem 2.33. *Let* $\mathfrak{H}$ *be a hereditary family on a topological space* (X, τ). *Then*

(1) *for* $U \in \tau$ *and* $A \in \mathfrak{H}$, $U - A \in \tau_{\mathfrak{H}}$.
(2) $\tau \subseteq \tau_{\mathfrak{H}}$.

Proof. (1) First, we show that $\Phi_{\mathfrak{H}}(U^c \cup A) \subseteq U^c \cup A$ for $U \in \tau$ and $A \in \mathfrak{H}$. Assume $x \in \Phi_{\mathfrak{H}}(U^c \cup A)$, then for every $G \in \tau(x), G \cap (U^c \cup A) = (G \cap U^c) \cup (G \cap A) \notin \mathfrak{H}$. Now, we claim that $G \cap U^c \neq \emptyset$, since otherwise if $G \cap U^c = \emptyset$ then from $(G \cap U^c) \cup (G \cap A) \notin \mathfrak{H}$, we have $G \cap A \notin \mathfrak{H}$ and thus $A \notin \mathfrak{H}$. It contradicts to the fact $A \in \mathfrak{H}$. So $G \cap U^c \neq \emptyset$ for every $G \in \tau(x)$ and this implies $x \in cl(U^c) = U^c \subseteq U^c \cup A$. Hence $\Phi_{\mathfrak{H}}(U^c \cup A) \subseteq U^c \cup A$ and from the fact, it follows $\Psi_{\mathfrak{H}}(X - (U - A)) = X - (U - A) \cup \Phi_{\mathfrak{H}}(X - (U - A)) = (X - (U - A))$. Hence $U - A \in \tau_{\mathfrak{H}}$.

(2) It is obvious from part (1) of the Theorem and $\emptyset \in \mathfrak{H}$.

□

In the next definition the concept of a base for generalized topological spaces is stated.

Definition 2.34. Let (X, μ) be a generalized topological space. A subcollection $\beta \subseteq \mathcal{P}(X)$ is a base for μ if $\mu = \{\cup\beta' : \beta' \subseteq \beta\}$.

The next theorem demonstrates that the set $\mathcal{B} = \{U - A : U \in \tau,\ A \in \mathfrak{H}\}$ is a base for the generalized topological space $(X, \tau_{\mathfrak{H}})$.

Theorem 2.35. *Let* $\mathfrak{H}$ *be a hereditary family on a topological space* (X, τ). *Then the set* $\{U - A : U \in \tau,\ A \in \mathfrak{H}\}$ *is a base for the generalized topological space* $(X, \tau_{\mathfrak{H}})$.

Proof. It is enough to show any $W \in \tau_{\mathfrak{H}}$ can be written in the form $W = \cup(U_\alpha - A_\alpha)$ for some $U_\alpha \in \tau$ and $A_\alpha \in \mathfrak{H}$.

Let $W \in \tau_{\mathfrak{H}}$ and $x \in W$. Then $\Psi_{\mathfrak{H}}(X-W) = (X-W)\cup\Phi_{\mathfrak{H}}(X-W) = X-W$ and $x \notin X-W$. So $x \notin \Phi_{\mathfrak{H}}(X-W)$ and there exists some $U_x \in \tau(x)$ such that $(X-W)\cap U_x \in \mathfrak{H}$. Put $A_x = (X-W)\cap U_x$. Then $x \notin A_x$ and $A_x \in \mathfrak{H}$. Moreover, we have $x \in U_x - A_x \subseteq W$. So the proof is completed. □

Definition 2.36. Let (X,τ) be a topological space. A subset A of X is said a preopen set in (X,τ), whenever $A \subseteq int(cl(A))$.

The collection of all preopen subsets of the space (X,τ) is indicated by the symbol of $PO(X,\tau)$ (and for convenience with $PO(X)$).

We are now in the position to illustrate the potential of our proposed method by extracting the famous collection $PO(X)$ from the space (X,τ).

Remark 2.37. *As mentioned at the beginning of the paper, the collection of all preopen subsets of any topological space (X,τ) forms a generalized topology on X. Here, we show that by using a suitable hereditary family in the proposed method, it is possible to create the set of all preopen subsets of each space.*

Let (X,τ) be a topological space and put $\mathfrak{H} = \{A \subseteq X : intA = \emptyset\}$. Clearly $\mathfrak{H}$ is a hereditary family on (X,τ). By doing a simple calculation we will have;

$$\Phi_{\mathfrak{H}}(A) = \{x \in X : intA \cap U \neq \emptyset\} = cl(int(A)).$$

Thus $\Psi_{\mathfrak{H}}(A) = A \cup cl(int(A))$ and therefore

$$\begin{aligned}\tau_{\mathfrak{H}} &= \{A \subseteq X : (X-A)\cup cl(int(X-A)) = X-A\}\\ &= \{A \subseteq X : A \subseteq int(cl(A))\}\\ &= \textit{the collection of all preopen subsets of } X.\end{aligned}$$

As an example of the application of the presented method in this paper, we derive the following results from the contents of the paper.
Results: *Let (X,τ) be a topological space.*

(1) *: For any $A \subseteq X$ and any $U \in \tau$, $U \cap intA = \emptyset$ implies $U \cap clintA = \emptyset$. Especially for $U,V \in \tau$, if $U \cap V = \emptyset$ then $U \cap clV = \emptyset = clU \cap V$.*

(2) *: If we put $i_{PO}(A) = \cup\{U \in PO(X) : U \subseteq A\}$, then $x \in i_{PO}(A)$ if and only if there exists $W \in \tau$ such that $W \subseteq clA$.*

(3) *: For any $A \subseteq X$ if $intA = \emptyset$ then $X - A \in PO(X)$.*

(4) *: For any $A \subseteq X$ with $intA = \emptyset$ and any $U \in \tau$, we have $U - A \in PO(X)$. According to this, we will have $\tau \subseteq PO(X)$*

(5) *: Any $W \in PO(X)$ can be written in the form $W = \cup(U_\alpha - A_\alpha)$ for some $U_\alpha \in \tau$ and $A_\alpha \subseteq X$ with $intA_\alpha = \emptyset$.*

Remark 2.38. [**?**] *A nonempty collection $\mathcal{I}$ of subsets of a set X is said to be an ideal on X, if it satisfies the following two conditions:*

(1) *If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (heredity),*
(2) *If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finitea dditivity).*

So, an ideal on a set X is a hereditary family $\mathcal{I}$ on X with the propery of finite additivity.

As an important note, we require to mention that the result of replacing the role of a hereditry family in our argument by any ideal $\mathcal{I}$, isn't a strong generalized topology, in fact what is created is a topology equal or finer than τ, see [**?**].

For example, the ideal $\mathcal{I}_f = \mathfrak{F}$ of all finite subsets and the ideal $\mathcal{I}_c$ of countable subsets of a topological space (X, τ) induce respectively the topologies τ_f and τ_c on X taht $\tau_f = \tau$ and $\tau_c \supsetneq \tau$, see [**?**].

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