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Some properties of certain subclass of meromorphic functions associated with q-derivative

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Abstract. In this paper, by making use of q-derivative we introduce a new subclass of meromorphically univalent functions. Precisely, we give a necessary and sufficient coefficient condition for functions in this class. Coefficient estimates, extreme points, convex linear combination Radii of starlikeness and convexity and finally partial sum property are investigated.

Keywords: q-derivative, Meromorphic function, Coefficient bound, Extreme point, Convex set, Partial sum.

2000 Mathematics subject classification: xxxx, xxxx; Secondary xxxx.

1. Introduction

The q-theory has important role in various branches of mathematics and physics as for example, in the areas of special functions, ordinary fractional calculus, optimal control problems, q-difference, q-integral equations, q-transform analysis and in quantum physics (see for instance, [1], [2],[6], [11]).
The theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus as the

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q-integral and q-derivative were used to construct several subclasses of analytic functions (see, for example,[4], [5], [8],[9], [10]).
Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k z^{k-1}, \tag{1.1}$$

which are analytic in the punctured unit disk

$$\triangle^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\}.$$

Gasper and Rahman [3] defined the q- derivative of a function $f(z)$ of the form (1.1) by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}. \tag{1.2}$$

where $z \in \triangle^*$ and $0 < q < 1$.
From (1.2) for a function $f(z)$ given by (1.1) we get

$$D_q f(z) = \frac{-1}{qz^2} + \sum_{k=1}^{\infty} [k-1]_q a_k z^{k-2} \ , \ z \in \triangle^*, \tag{1.3}$$

where

$$[k-1]_q := \frac{1 - q^{k-1}}{1-q} = 1 + q + q^2 + \cdots + q^{k-2}. \tag{1.4}$$

also $[k-1]_q \to k-1$ as $q \to \overline{1}$. So we conclude

$$\lim_{q \to \overline{1}} D_q f(z) = f^{'}(z) \ , \ z \in \triangle^*,$$

see also [7, 12].
For $0 < q < 1, 0 \leq \lambda \leq 1, 0 < \alpha \leq 1$ and $\beta > 0$.
Let $\sum_q(\lambda, \alpha, \beta)$ be the subclass of $\sum$ consisting of functions f of the form (1.1) and satisfying the condition

$$\left| \frac{z^4 \left(D_q f(z)\right)^{''} + z^3 \left(D_q f(z)\right)^{'} + \dfrac{4}{q}}{\lambda z^2 \left(D_q f(z)\right) - \dfrac{1}{q} + \dfrac{(1+\lambda)\alpha}{q}} \right| < \beta. \tag{1.5}$$

2. Main result

Unless otherwise mentioned, we suppose throughout this paper that $0 < q < 1, 0 \leq \lambda < 1, 0 < \alpha < 1$ and $\beta > 0$. First we state coefficient estimates on the class $\sum_q(\lambda, \alpha, \beta)$.

Theorem 2.1. *Let $f(z) \in \sum$, then $f(z) \in \sum_q(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{k=1}^{+\infty}[k-1]_q\left((k-2)^2+\lambda\beta\right)a_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{q}, \tag{2.1}$$

and the result is sharp for $G(z)$ given by

$$G(z) = \frac{1}{z} + \frac{\beta(1+\lambda)(1-\alpha)}{q[k-1]_q\left((k-2)^2+\lambda\beta\right)}z^k. \tag{2.2}$$

Proof. Let $f(z) \in \sum_q(\lambda, \alpha, \beta)$, then (1.5) holds true. So by replacing (1.3) in (1.5) we have

$$\left|\frac{\sum_{k=1}^{+\infty}[k-1]_q(k-2)(k-3)a_kz^k + \sum_{k=1}^{+\infty}[k-1]_q(k-2)a_kz^k}{-\dfrac{\lambda}{q} + \sum_{k=1}^{+\infty}\lambda[k-1]_qa_kz^k - \dfrac{1}{q} + \dfrac{(1+\lambda)\alpha}{q}}\right| < \beta,$$

or

$$\left|\frac{\sum_{k=1}^{+\infty}[k-1]_q(k-2)^2a_kz^k}{\dfrac{(1+\lambda)}{q}(1-\alpha) - \sum_{k=1}^{+\infty}\lambda[k-1]_qa_zz^k}\right| < \beta.$$

Since $Re(z) \leq |z|$ for all z, therefore

$$Re\left\{\frac{\sum_{k=1}^{+\infty}[k-1]_q(k-2)^2a_kz^k}{(\dfrac{1+\lambda}{q})(1-\alpha) - \sum_{k=1}^{+\infty}\lambda[k-1]_qa_kz^k}\right\} < \beta.$$

By letting $z \to \overline{1}$ through real values, we have

$$\sum_{k=1}^{+\infty}[k-1]_q\left((k-2)^2+\lambda\beta\right)a_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{q}.$$

Conversely, Let (2.1) holds true, it is enough to show that

$$X(f) = \left|\frac{z^4\left(D_qf(z)\right)'' + z^3\left(D_qf(z)\right)' + \dfrac{4}{q}}{\lambda z^2\left(D_qf(z)\right) - \dfrac{1}{q} + \dfrac{(1+\lambda)\alpha}{q}}\right| < \beta.$$

or

$$X(f) = \left|z^4\left(D_qf(z)\right)'' + z^3\left(D_qf(z)\right)' + \frac{4}{q}\left|-\beta\right|\lambda z^2\left(D_qf(z)\right) - \frac{1}{q} + \frac{(1+\lambda)\alpha}{q}\right| < 0.$$

But for $0<|z|=r<1$ we have

$$\begin{aligned}X(f)=&\left|\sum_{k=1}^{+\infty}[k-1]_q(k-2)^2a_kz^k\right|-\beta\left|\frac{(1+\lambda)}{q}(1-\alpha)-\lambda\sum_{k=1}^{+\infty}[k-1]_qa_kz^k\right|\\ \leq&\sum_{k=1}^{+\infty}[k-1]_q(k-2)^2|a_k|r^k-\frac{\beta(1+\lambda)(1-\alpha)}{q}+\sum_{k=1}^{+\infty}\lambda\beta[k-1]_q|a_k|r^k\\ \leq&\sum_{k=1}^{+\infty}[k-1]_q\Big((k-2)^2+\lambda\beta\Big)|a_k|r^k-\frac{\beta(1+\lambda)(1-\alpha)}{q}.\end{aligned}$$

Since the above inequality holds for all r $(0<r<1)$, by letting $r\to\overline{1}$ and using (2.1) we obtain $X(f)\leq 0$, and this completes the proof. $\square$

Next we obtain extreme points and convex linear combination property for
$f(z)\in\sum_q(\lambda,\alpha,\beta)$.

Theorem 2.2. *The function $f(z)$ of the form* (1.1) *belongs to $\sum_q(\lambda,\alpha,\beta)$ if and only if it can be expressed by $f(z)=\sum_{k=0}^{\infty}\sigma_kf_k(z),\sum_{k=0}^{\infty}\sigma_k=1,\sigma_k\geq 0$, where*
$f_0(z)=\dfrac{1}{z}$
and
$f_k(z)=\dfrac{1}{z}+\dfrac{\beta(1+\lambda)(1-\alpha)}{q[k-1]_q[(k-2)^2+\lambda\beta]}z^k,\ k=1,2,\cdots.$

Proof. Let

$$\begin{aligned}f(z)=&\sum_{k=0}^{\infty}\sigma_kf_k(z),\\ =&\sigma_0f_0(z)+\sum_{k=1}^{\infty}\sigma_k\Big[\frac{1}{z}+\frac{\beta(1+\lambda)(1-\alpha)}{q[k-1]_q\big[(k-2)^2+\lambda\beta)\big]}z^k\Big]\\ =&\frac{1}{z}+\sum_{k=1}^{\infty}\frac{\beta(1+\lambda)(1-\alpha)}{q[k-1]_q\big[(k-2)^2+\lambda\beta)\big]}\sigma_kz^k.\end{aligned}$$

Now by using Theorem 2.1 we conclude that $f(z)\in\sum_q(\lambda,\alpha,\beta)$.
Conversely, if $f(z)$ given by (1.1) belongs to $\sum_q(\lambda,\alpha,\beta)$, by letting $\sigma_0=1-\sum_{k=1}^{+\infty}\sigma_k$, where

$$\sigma_k=\frac{q[k-1]_q\big[(k-2)^2+\lambda\beta\big]}{\beta(1+\lambda)(1-\alpha)}a_k,\ k=1,2,\cdots.$$

we conclude the required result. $\square$

Theorem 2.3. *Let for* $n = 1, 2, \cdots, m,\ f_n(z) = \dfrac{1}{z} + \sum_{k=1}^{+\infty} a_{k,n} z^k$ *belongs to* $\sum_q(\lambda, \alpha, \beta)$*, then* $F(z) = \sum_{n=1}^{m} \sigma_n f_n(z)$*, is also in the same class, where* $\sum_{n=1}^{m} \sigma_n = 1$*. (Hence* $\sum_q(\lambda, \alpha, \beta)$ *is a convex set.)*

Proof. According to Theorem 2.1 for every $n = 1, 2, \cdots, m$, we have

$$\sum_{n=1}^{+\infty} [k-1]_q \big((k-2)^2 + \lambda\beta\big) a_{k,n} \leq \frac{\beta(1+\lambda)(1-\alpha)}{q}.$$

But

$$\begin{aligned} F(z) &= \sum_{n=1}^{m} \sigma_n f_n(z) \\ &= \sum_{n=1}^{m} \sigma_n \left(\frac{1}{z} + \sum_{k=1}^{\infty} a_{k,n} z^k \right) \\ &= \frac{1}{z} \sum_{n=1}^{m} \sigma_n + \sum_{k=1}^{\infty} \left(\sum_{n=1}^{m} \sigma_n a_{k,n} \right) z^k \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \big(\sum_{n=1}^{m} \sigma_n a_{k,n} \big) z^k. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{k=1}^{\infty} [k-1]_q \big((k-2)^2 + \lambda\beta\big) \big(\sum_{n=1}^{m} \sigma_n a_{k,n} \big) \\ &= \sum_{n=1}^{m} \sigma_n \Big(\sum_{k=1}^{\infty} [k-1]_q \big((k-2)^2 + \lambda\beta\big) a_{k,n} \\ &\leq \sum_{n=1}^{m} \sigma_n \big(\frac{\beta(1+\lambda)(1-\alpha)}{q} \big) \\ &= \frac{\beta(1+\lambda)(1-\alpha)}{q} \sum_{n=1}^{m} \sigma_n = \frac{\beta(1+\lambda)(1-\alpha)}{q}, \end{aligned}$$

then by Theorem 2.1 the proof is complete. □

3. Radii Condition and partial sum property

In this section we obtain radii of starlikeness and convexity and investigate about partial sum property.

Theorem 3.1. *If* $f(z) \in \sum_q(\lambda, \alpha, \beta)$*, then* f *is meromorphically univalent starlike of order* γ *in disk* $|z| < R_1$*, and it is meromerphically* *univalent convex of order γ in disk $|z| < R_2$ where*

$$R_1 = \inf_k \left\{ \frac{q[k-1]_q \left((k-2)^2 + \lambda\beta\right)(1-\gamma)}{\beta(1+\lambda)(1-\alpha)(k+2+\gamma)} \right\}^{\frac{1}{k+1}}, \tag{3.1}$$

and

$$R_2 = \inf_k \left\{ \frac{q[k-1]_q \left((k-2)^2 + \lambda\beta\right)(1-\gamma)}{\beta k(1+\lambda)(1-\alpha)(k+2+\gamma)} \right\}^{\frac{1}{k+1}}. \tag{3.2}$$

Proof. For starlikeness it is enough to show that

$$\left|\frac{zf'}{f} + 1\right| < 1-\gamma,$$

but

$$\left|\frac{zf'}{f} + 1\right| = \left|\frac{\sum_{k=1}^{+\infty}(k+1)a_k z^{k+1}}{1+\sum_{k=1}^{+\infty} a_k z^{k+1}}\right| \le \frac{\sum_{k=1}^{+\infty}(k+1)a_k |z|^{k+1}}{1-\sum_{k=1}^{+\infty} a_k |z|^{k+1}} \le 1-\gamma,$$

or

$$\sum_{k=1}^{+\infty}(k+1)a_k|z|^{k+1} \le 1-\gamma-(1-\gamma)\sum_{k=1}^{+\infty} a_k|z|^{k+1},$$

or

$$\sum_{k=1}^{+\infty} \frac{k+2+\gamma}{1-\gamma} a_k |z|^{k+1} \le 1.$$

By using (2.1) we obtain

$$\sum_{k=1}^{+\infty} \frac{k+2+\gamma}{1-\gamma} a_k |z|^{k+1} \le$$
$$\sum_{k=1}^{+\infty} \frac{\beta(1+\lambda)(1-\alpha)(k+2+\alpha)}{q[k-1]_q\left((k-2)^2+\lambda\beta\right)(1-\alpha)} |z|^{k+1} \le 1.$$

So, it is enough to suppose

$$|z|^{k+1} \le \frac{q[k-1]_q\left((k-2)^2+\lambda\beta\right)(1-\alpha)}{\beta(1+\lambda)(1-\alpha)(k+2+\alpha)}.$$

Hence we get the required result (3.1). For convexity, by using the Alexander,s Theorem(If f be an analytic function in the unit disk and normalized by $f(0) = f'(0) - 1 = 0$, then $f(z)$ is convex if and only if $zf'(z)$ is starlike.) and applying an easy calculation we conclude the required result (3.2). So the proof is complete. $\square$

Theorem 3.2. *Let $f(z) \in \sum$, and define*

$$S_1(z) = \frac{1}{z}\ ,\ S_m(z) = \frac{1}{z} + \sum_{k=1}^{m-1} a_k z^k\ ,\ (m = 2, 3, \cdots). \tag{3.3}$$

Also suppose $\sum_{k=1}^{+\infty} x_k a_k \le 1$, where

$$x_k = \frac{q[k-1]_q \left((k-2)^2 + \lambda\beta\right)}{\beta(1+\lambda)(1-\alpha)}, \tag{3.4}$$

then

$$Re\left(\frac{f(z)}{S_m(z)}\right) > 1 - \frac{1}{x_m}\ ,\ Re\left(\frac{S_m(z)}{f(z)}\right) > \frac{x_m}{1+x_m}. \tag{3.5}$$

Proof. Since $\sum_{k=1}^{+\infty} x_k a_k \le 1$, then by Theorem 2.1, $f(z) \in \sum_q(\lambda, \alpha, \beta)$. Also by (1.4) we have $\dfrac{[k-1]_q}{1-\alpha} \ge 1$, so

$$x_k > \frac{q\left((k-2)^2 + \lambda\beta\right)}{\beta(1+\lambda)}, \tag{3.6}$$

and $\{x_k\}$ is an increasing sequence, therefore we obtain

$$\sum_{k=1}^{m-1} a_k + x_m \sum_{k=m}^{+\infty} a_k \le 1. \tag{3.7}$$

Now by putting

$$X(z) = x_m \left[\frac{f(z)}{S_m(z)} - (1 - \frac{1}{x_m})\right], \tag{3.8}$$

and making use of (3.7) we obtain

$$\begin{aligned} Re\left(\frac{X(z)-1}{X(z)+1}\right) \le \left|\frac{X(z)-1}{X(z)+1}\right| &= \left|\frac{x_m f(z) - x_m S_m(z)}{x_m f(z) - x_m S_m(z) + 2S_m(z)}\right| \\ &= \left|\frac{x_m \sum_{k=m}^{+\infty} a_k z^k}{x_m \sum_{k=m}^{+\infty} a_k z^k + 2(\frac{1}{z} + \sum_{k=1}^{m=1}) a_k z^k}\right| \\ &\le \frac{x_m \sum_{k=m}^{+\infty} |a_k|}{2 - \sum_{k=1}^{m-1} |a_k| - x_m \sum_{k=m}^{+\infty} |a_k|} \le 1. \end{aligned}$$

By a simple calculation we get

$$Re\,(X(z)) > 0, \text{therefore } Re\left(\frac{X(z)}{x_m}\right) > 0,$$

or equivalently $Re\left[\frac{f(z)}{S_m(z)}-(1-\frac{1}{x_m})\right]>0$, and this gives the first inequality in (3.5).
For the second inequality we consider

$$Y(z)=(1+x_m)\big[\frac{S_m(z)}{f(z)}-\frac{x_m}{1+x_m}\big],$$

and by using (3.7) we have $\left|\frac{Y(z)-1}{Y(z)+1}\right|\leq 1$, and Hence $Re\,(Y(z))>0$, therefore $Re\left(\frac{Y(z)}{1+x_m}\right)>0$, or equivalently $Re\left[\frac{S_m(z)}{f(z)}-\frac{x_m}{1+x_m}\right]>0$, and this shows the second inequality in (3.5). So the proof is complete. $\square$

4. Some properties of $\sum_q(\lambda,\alpha,\beta)$

Theorem 4.1. *Let $f(z),g(z)\in\sum_q(\lambda,\alpha,\beta)$ and given by $f(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}a_kz^{k-1}$, $g(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}b_kz^{k-1}$. Then the function $h(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}(a_k^2+b_k^2)z^{k-1}$ is also in $\sum_q(\gamma,\alpha,\beta)$ where $\gamma\leq\frac{\lambda}{2}-\frac{(k-2)^2}{2\beta}$.*

Proof. Since $f(z),g(z)\in\sum_q(\lambda,\alpha,\beta)$ therefore we have

$$\begin{aligned}\sum_{k=1}^{+\infty}\left[[k-1]_q\left((k-2)^2+\lambda\beta\right)\right]^2a_k^2&\leq\left[\sum_{k=1}^{+\infty}[k-1]_q\left((k-2)^2+\lambda\beta\right)a_k\right]^2\\&\leq\left[\frac{\beta(1+\lambda)(1-\alpha)}{q}\right]^2,\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\sum_{k=1}^{+\infty}\left[[k-1]_q\left((k-2)^2+\lambda\beta\right)\right]^2b_k^2&\leq\left[\sum_{k=1}^{+\infty}[k-1]_q\left((k-2)^2+\lambda\beta\right)b_k\right]^2\\&\leq\left[\frac{\beta(1+\lambda)(1-\alpha)}{q}\right]^2.\end{aligned}\tag{4.2}$$

The above inequalities yield us

$$\sum_{k=1}^{+\infty}\frac{1}{2}\left[[k-1]_q\left((k-2)^2+\lambda\beta\right)\right]^2(a_k^2+b_k^2)\leq\left[\frac{\beta(1+\lambda)(1-\alpha)}{q}\right]^2.\tag{4.3}$$

Now we must show

$$\sum_{k=1}^{+\infty}\left[[k-1]_q\left((k-2)^2+\gamma\beta\right)\right]^2(a_k^2+b_k^2)\le\left[\frac{\beta(1+\lambda)(1-\alpha)}{q}\right]^2. \quad (4.4)$$

But above inequalities holds if

$$[k-1]_q\left((k-2)^2+\gamma\beta\right)\le\tfrac{1}{2}\left[[k-1]_q\left((k-2)^2+\lambda\beta\right)\right],$$

or equivalently

$$2(k-2)^2+2\gamma\beta\le(k-2)^2+\lambda\beta,$$

or

$$\gamma\le\tfrac{\lambda}{2}-\tfrac{(k-2)^2}{2\beta}.$$

□

Theorem 4.2. *The class $\sum_q(\lambda,\alpha,\beta)$ is a convex set.*

Proof. Let

$$f(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}a_kz^{k-1},$$

and

$$g(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}b_kz^{k-1},$$

be in the class $\sum_q(\lambda,\alpha,\beta)$. For $t\in(0,1)$, it is enough to show that the function $h(z)=(1-t)f(z)+tg(z)$ is in the class $\sum_q(\lambda,\alpha,\beta)$. Since

$$h(z)=\frac{1}{z}+\sum_{k=1}^{+\infty}((1-t)a_k+tb_k)z^{k-1}, \quad (4.5)$$

then

$$\sum_{k=1}^{\infty}\left[[k-1]_q\left((k-2)^2+\lambda\beta\right)\right]((1-t)a_k+tb_k)\le\frac{\beta(1+\lambda)(1-\alpha)}{q}, \quad (4.6)$$

so $h(z)\in\sum_q(\lambda,\alpha,\beta)$. □

Corollary 4.3. *Let $f_j(z)$ $(j=1,2,\cdots,n)$, defined by $f_j(z)=\dfrac{1}{z}+\sum_{k=1}^{+\infty}a_{k,j}z^{k-1}$ be in the class $\sum_q(\lambda,\alpha,\beta)$, then the function $F(z)=\sum_{j=1}^{n}c_jf_j(z)$ is also in $\sum_q(\lambda,\alpha,\beta)$ where $\sum_{j=1}^{n}c_j=1$.*

5. Hadamard product

Theorem 5.1. *If $f(z), g(z) \in \sum_q(\lambda, \alpha, \beta)$ then Hadamard product of f and g defined by*

$$f * g(z) = \frac{1}{z} + \sum_{k=1}^{+\infty} a_k b_k z^{k-1},$$

is in the class $\sum_q(\gamma, \alpha, \beta)$ where

$$\gamma \leq \left(\frac{[k-1]_q q \left((k-2)^2 + \lambda\beta\right)^2}{\beta^2(1+\lambda)(1-\alpha)} - \frac{(k-2)^2}{\beta} \right).$$

Proof. Since $f(z), g(z) \in \sum_q(\lambda, \alpha, \beta)$, so by 2.1

$$\sum_{k=1}^{+\infty} [k-1]_q \left((k-2)^2 + \lambda\beta\right) a_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{q}, \tag{5.1}$$

and

$$\sum_{k=1}^{+\infty} [k-1]_q \left((k-2)^2 + \lambda\beta\right) b_k \leq \frac{\beta(1+\lambda)(1-\alpha)}{q}. \tag{5.2}$$

We must find the smallest γ such that

$$\sum_{k=1}^{+\infty} [k-1]_q \left((k-2)^2 + \gamma\beta\right) a_k b_k \leq \frac{\beta(1+\gamma)(1-\alpha)}{q}. \tag{5.3}$$

By using the Cauchy-Schwarts inequality we have

$$\sum_{k=1}^{+\infty} [k-1]_q \left((k-2)^2 + \lambda\beta\right) \sqrt{a_k b_k} \leq \frac{\beta(1+\lambda)(1-\alpha)}{q}. \tag{5.4}$$

Now it is enough to show that

$$[k-1]_q \left((k-2)^2 + \gamma\beta\right) a_k b_k \leq [k-1]_q \left((k-2)^2 + \gamma\beta\right) \sqrt{a_k b_k}, \tag{5.5}$$

or equivalently

$$\sqrt{a_k b_k} \leq \tfrac{(k-2)^2 + \lambda\beta}{(k-2)^2 + \gamma\beta}.$$

But from

$$\sqrt{a_k b_k} \leq \tfrac{\beta(1+\lambda)(1-\alpha)}{q[k-1]_q((k-2)^2+\lambda\beta)},$$

so it is enough that

$$\tfrac{\beta(1+\lambda)(1-\alpha)}{q[k-1]_q((k-2)^2+\lambda\beta)} \leq \tfrac{(k-2)^2+\lambda\beta}{(k-2)^2+\gamma\beta},$$

or

$$\gamma \leq \left(\tfrac{[k-1]_q q \left((k-2)^2+\lambda\beta\right)^2}{\beta^2(1+\lambda)(1-\alpha)} - \tfrac{(k-2)^2}{\beta} \right).$$

□

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