Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **11**(1)(2022), 138-160 (Research paper)

A Role of Fuzzy Set-Valued Maps in Integral Inclusions

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Abstract. The aim of this paper is to introduce the concepts of α-continuity, η-admissible pair for fuzzy set-valued maps and define the notion of fuzzy $\eta-(\psi, F)$-contraction. The existence of common fuzzy fixed points for such contraction is investigated in the setting of a complete metric space. The ideas presented herein complement the results of Wardowski, Banach, Heilpern and related results on point-to-point and point-to-set-valued mappings in the literature of classical and fuzzy fixed point theory. A few important of these special consequences are highlighted and discussed. Some nontrivial examples and an application to a system of integral inclusions of Fredholm type are considered to support and illustrate some usefulness of our obtained results herein.

Keywords: fuzzy set, fuzzy fixed point, F-contraction, α-continuous, η-admissible, fuzzy $\eta-(\psi, F)$-contraction, integral inclusion.

2020 Mathematics subject classification: 46S40; 47H10; 54H25; 34A12.

1. Introduction

Several problems in science and engineering defined by nonlinear functional equations can be solved by reducing them to an equivalent fixed-point problem. In fact, an operator equation $\phi x = 0$ may be reformulated as a fixed-point equation $\rho x = x$, where ρ is a self-mapping with a suitable domain. Fixed point theory provides important tools

[1]Corresponding author: shagaris@ymail.com
Received: 06 October 2020
Revised: 17 November 2020
Accepted: 19 November 2020

for solving problems arising in various branches of mathematical analysis, such as split feasibility problems, variational inequality problems, nonlinear optimization problems, equilibrium problems, complementarity problems, selection and matching problems, and problems of proving the existence of solution of integral and differential equations. In particular, the fixed point theorem, generally known as the Banach Contraction Principle (see [7]), appeared in explicit form in Banach Thesis in 1922, where it was originally applied to establish the existence of a solution to an integral equation. Since then, because of its simplicity and usefulness, it has become a significant tool in solving existence problems in many branches of mathematical analysis. Consequently, the principle has gained a number of generalizations and modifications by many authors. Wardowski [33] introduced a new contraction known as F-contraction and established a fixed point result which generalizes the Banach fixed point theorem. Thereafter, the concept of F-contraction has been extended in different directions (see, e.g. [3, 13, 24, 26]). In 2012, Samet et al. [28] introduced the notions of $\eta-\psi$-contractive and η-admissible mappings and extended many existing results, in particular, the contraction mapping principle due to Banach [7]. Meanwhile, the concepts of $\eta - \psi$-contractive and η-admissibility have attracted keen interests of many researchers and thus have been modified in several settings (see, e.g. [16, 17, 22, 27]).

On the other hand, as a natural generalization of the notion of crisp sets, fuzzy set was introduced originally by Zadeh [34]. Since then, to use this concept, many authors have progressively extended the theory and its applications to other branches of sciences, social sciences and engineering. In 1981, Heilpern [12] used the idea of fuzzy set to initiate a class of fuzzy set-valued maps and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem of Nadler [23]. Subsequently, several authors have studied the existence of fixed points of fuzzy set-valued maps, for example, see [1, 5, 6, 14, 19, 20, 29, 32].

By combining the notions of F-contractions and η-admissible mappings, the main aim of this paper is to introduce the concepts of α-continuity, η-admissible pair for fuzzy set-valued maps and define the concept of fuzzy $\eta - (\psi, F)$-contraction. Thereafter, the existence of common fuzzy fixed points for such contraction is investigated in the setting of a complete metric space. The ideas presented herein complement the results of Wardowski [33], Banach [7], Heilpern [12] and others in the comparable literature of classical and fuzzy fixed point theory. A few significant of these consequences of our results are pointed out and discussed. Some examples and an application to a system of integral inclusions of Fredholm type are considered to support our assertions and to illustrate a usability of the results obtained herein.

2. Preliminaries

Throughout this article, the sets $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$, represent the set of real numbers, positive real numbers and natural numbers, respectively. In this section, we collect some basic concepts and results which are relevant to what follows hereafter.

Definition 2.1. [33] Let (X, μ) be a metric space. A mapping $T : X \longrightarrow X$ is called an F-contraction, if there exists $\xi > 0$ such that

$$\mu(Tx, Ty) > 0 \ \Rightarrow \xi + F(\mu(Tx, Ty)) \leq F(\mu(x, y)) \tag{2.1}$$

for all $x, y \in X$, where $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a mapping satisfying the following axioms:

- (F_1) F is strictly increasing; that is, $\tau < \beta \implies F(\tau) < F(\beta)$,
- (F_2) for any sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of positive real numbers, $\lim_{n\longrightarrow\infty} \tau_n = 0$ if and only if $\lim_{n\longrightarrow\infty} F(\tau_n) = -\infty$,
- (F_3) there exists $\varsigma \in (0, 1)$ such that $\lim_{n\longrightarrow\infty} \tau^{\varsigma} F(\tau) = 0$.

We denote the family of functions satisfying $(F_1) - (F_3)$ by Ω.

Example 2.2. [33] The functions $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by

- (i) $F(\tau) = \ln(\tau)$,
- (ii) $F(\tau) = \ln(\tau) + \tau,\ \tau > 0$,
- (iii) $F(\tau) = \frac{-1}{\sqrt{\tau}},\ \tau > 0$,
- (iv) $F(\tau) = \ln(\tau^2 + \tau),\ \tau > 0$,

are elements of Ω.

Denote by $\mathfrak{M}$, the family of nondecreasing functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for each $t > 0$, where $\psi^n(t)$ is the n^{th}-iterate of ψ. The main result of Wardowski [33], which is a generalization of the Banach fixed point theorem [7] is stated as follows.

Theorem 2.3. [33] *Let (X, μ) be a complete metric space and $T : X \longrightarrow X$ be an F-contraction. Then T has a unique fixed point in X.*

Definition 2.4. [28] Let (X, μ) be a metric space and $T : X \longrightarrow X$ be a given mapping. Then T is called an $\eta - \psi$-contractive mapping if there exist two functions $\eta : X \times X \longrightarrow \mathbb{R}_+$ and $\psi \in \mathfrak{M}$ such that for all $x, y \in X$,

$$\eta(x, y)\mu(Tx, Ty) \leq \psi(\mu(x, y)).$$

Definition 2.5. [28] Let $T : X \longrightarrow X$ and $\eta : X \times X \longrightarrow \mathbb{R}_+$ be mappings. Then T is said to be η-admissible if for all $x, y \in X$,

$$\eta(x,y) \geq 1 \implies \eta(Tx,Ty) \geq 1.$$

Example 2.6. [28] Let $X = (0,\infty)$. Define $T : X \longrightarrow X$ and $\eta : X \times X \longrightarrow \mathbb{R}_+$ by

$$Tx = \ln x, \text{ for all } x, y \in X$$

and

$$\eta(x,y) = \begin{cases} 2, & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

Then T is η-admissible.

Example 2.7. [28] Let $X = \mathbb{R}_+$. Define $T : X \longrightarrow X$ and $\eta : X \times X \longrightarrow \mathbb{R}_+$ by $Tx = \sqrt{x}$, for all $x \in X$, and

$$\eta(x,y) = \begin{cases} \exp(x-y), & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

Then T is η-admissible.

Let (X,μ) be a metric space and denote the set of all nonempty compact subsets of X by $\mathcal{K}(X)$. For $A, B \in \mathcal{K}(X)$, the function $H : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathbb{R}_+$ defined by

$$H(A,B) = \begin{cases} \max\left\{\sup_{x\in A} \mu(x,B), \sup_{x\in B} \mu(x,A)\right\}, & \text{if it exists} \\ \infty, & \text{otherwise,} \end{cases}$$

is called Hausdorff metric on $\mathcal{K}(X)$ induced by the metric μ, where

$$\mu(x,A) = \inf_{y\in A} \mu(x,y).$$

Let X be a universal set. A fuzzy set in X is a function with domain X and values in $[0,1] = I$. If A is a fuzzy set in X, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of a fuzzy set A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \begin{cases} \overline{\{x \in X : A(x) > 0\}}, & \text{if } \alpha = 0 \\ \{x \in X : A(x) \geq \alpha\}, & \text{if } \alpha \in (0,1]. \end{cases}$$

where by $\overline{M}$, we mean the closure of the crisp set M. We denote the family of fuzzy sets in X by I^X.

A fuzzy set A in a metric space X is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in X and $\sup_{x\in X} A(x) = 1$. We denote the collection of all approximate quantities in X by $W(X)$. If there exists an $\alpha \in [0,1]$ such that $[A]_\alpha, [B]_\alpha \in \mathcal{K}(X)$, then define

$$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha).$$

$$\mu_\infty(A,B) = \sup_{\alpha} D_\alpha(A,B).$$

Definition 2.8. Let X be a nonempty set and Y a metric space. A mapping $T : X \longrightarrow I^Y$ is called fuzzy set-valued map. A fuzzy set-valued map T is a fuzzy subset of $X \times Y$. The function value $T(x)(y)$ is called the grade of membership of y in $T(x)$.

Definition 2.9. Let X be a nonempty set and $G, T : X \longrightarrow I^X$ be fuzzy set-valued maps. A point $u \in X$ is called a fuzzy fixed point of G if there exists an $\alpha \in (0,1]$ such that $u \in [Gu]_{\alpha(u)}$. An element $u \in X$ is known as a common fuzzy fixed point of G and T, if there exists an $\alpha \in (0,1]$ such that $u \in [Gu]_{\alpha(u)} \cap [Tu]_{\alpha(u)}$.

We represent the set of all fuzzy fixed points of G by $\mathcal{F}_{ix}(G)$ and common fuzzy fixed points of G and T by $\mathcal{F}_{ix}(G,T)$.

Remark 2.10. In $(\mathcal{K}(X), H)$, $u \in X$ is a fuzzy fixed point of T if and only if $\mu(u, [Tu]_\alpha) = 0$.

Definition 2.11. [12] Let (X,μ) be a metric space. A mapping $T : X \longrightarrow W(X)$ is called fuzzy λ-contraction if there exists $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$\mu_\infty(T(x), T(y)) \leq \lambda\mu(x,y).$$

The following result due to Heilpern [12] is the first metric fixed point theorem for fuzzy set-valued maps.

Theorem 2.12. [12] *Every fuzzy λ-contraction on a complete metric space has a fuzzy fixed point.*

Let Ψ denotes the family of functions $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following axioms:

(ψ_1) $\lim_{n\longrightarrow\infty} \frac{\psi^n(t)}{n} < 0$ for all $t > 0$ and $n > 0$;
(ψ_2) $\psi(t) < t$ for all $t \geq 0$;
(ψ_3) ψ is nondecreasing and upper semi-continuous.

Example 2.13. The function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(t) = \begin{cases} t^5 - 3, & \text{if } t < 3 \\ \sqrt{t} - 3, & \text{if } t > 3. \end{cases}$$

belongs to Ψ.

Clearly, any function ψ satisfying (ψ_1) also posses the property that $\lim_{n\longrightarrow\infty} \psi^n(t) = -\infty$ for all $t \in (0,\infty)$.

3. Main Results

Recall that continuity of a set-valued mapping is usually defined in terms of lower and upper semi-continuity via the notion of Hausdorff separation (see, e.g. [31, Chap.1]). We begin this section by initiating the following concept of continuity of fuzzy set-valued maps.

Definition 3.1. Let (X,μ) be a metric space. A fuzzy set-valued map $T : X \longrightarrow I^X$ is said to be α-continuous at $u \in X$ with respect to a mapping $\alpha : X \longrightarrow (0,1]$, if for any sequence $\{x_n\}_{n\geq 1}$ in X,

$$\lim_{n\longrightarrow\infty} \mu(x_n,u) = 0 \implies \lim_{n\longrightarrow\infty} H([Tx_n]_{\alpha(x)},[Tu]_{\alpha(u)}) = 0.$$

We say that T is α-continuous if it is continuous at each point of X.

Definition 3.1 can be reformulated as follows:
A fuzzy set-valued map T is said to be α-continuous at a point $u \in X$ with respect to a mapping $\alpha : X \longrightarrow (0,1]$, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\mu(x,u) < \delta \implies H([Tx]_{\alpha(x)},[Tu]_{\alpha(u)}) < \epsilon.$$

Example 3.2. Let $X = [0,\infty)$ and $\mu(x,y) = |x-y|$ for all $x,y \in X$. Define $T : X \longrightarrow I^X$ by

$$T(x)(t) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq t \leq x+5 \\ \frac{3}{29}, & \text{elsewhere.} \end{cases}$$

Let $\alpha : X \longrightarrow (0,1]$ be given by $\alpha(x) = 0.2$ for all $x \in X$. Then, $[Tx]_{\alpha(x)} = [0,x+5]$. For $\epsilon > 0$, take $\delta = \frac{\epsilon}{8}$, then for all $x,y \in X$, $\mu(x,y) < \delta$ implies $H([Tx]_{\alpha(x)},[Ty]_{\alpha(y)}) = |x-y < \epsilon$. Thus, T is α-continuous on X.

Definition 3.3. Let X be a nonempty set. We say that a pair (G,T) of fuzzy set-valued maps $G,T : X \longrightarrow I^X$ is η-admissible with respect to α, if there exist $\eta : X \times X \longrightarrow \mathbb{R}_+$ and $\alpha : X \longrightarrow (0,1]$ such that

- (a_1) for each $x \in X$ and any $y \in [Gx]_{\alpha(x)}$ with $\eta(x,y) \geq 1$, we have $\eta(y,w) \geq 1$ for all $w \in [Ty]_{\alpha(y)}$;
- (a_2) for each $x \in X$ and $y \in [Tx]_{\alpha(x)}$ with $\eta(x,y) \geq 1$, we have $\eta(y,w) \geq 1$ for all $w \in [Gy]_{\alpha(y)}$.

Recall that a mapping $\eta : X \times X \longrightarrow \mathbb{R}_+$ is called symmetric if $\eta(x,y) \geq 1$ implies $\eta(y,x) \geq 1$ for all $x,y \in X$. Similarly, a pair (G,T) of fuzzy set-valued maps $G,T : X \longrightarrow I^X$ is said to be symmetric η-admissible if there exists a symmetric function $\eta : X \times X \longrightarrow \mathbb{R}_+$ such that (G,T) is η-admissible.

Definition 3.4. Let (X,μ) be a metric space. A pair (G,T) of fuzzy set-valued maps $G,T : X \longrightarrow I^X$ is called fuzzy $\eta-(\psi,F)$-contraction, if there exist two mappings $\eta : X\times X \longrightarrow \mathbb{R}_+$ and $\alpha : X \longrightarrow (0,1]$ with $\psi \in \Psi$ and $F \in \Omega$ such that for all $x,y \in X$,

$$F(H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)})) \leq \psi\left(F\left(\coprod(x,y)\right)\right) \tag{3.1}$$

with $\eta(x,y) \geq 1$ and $H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)}) > 0$, where

$$\coprod(x,y) = \max\left\{\mu(x,y),\mu(x,[Gx]_{\alpha(x)}),\mu(y,[Ty]_{\alpha(y)}),\frac{\mu(x,[Ty]_{\alpha(y)})+\mu(y,[Gx]_{\alpha(x)})}{1+\mu(x,y)}\right\}. \tag{3.2}$$

Now, we present the main result of this paper as follows.

Theorem 3.5. *Let (X,μ) be a complete metric space and $G,T : X \longrightarrow I^X$ be two fuzzy set-valued maps such that the pair (G,T) is a fuzzy $\eta-(\psi,F)$-contraction. Assume that the following conditions are satisfied:*

- (ax_1) *for each $x \in X$, $[Gx]_{\alpha(x)},[Tx]_{\alpha(x)} \in \mathcal{K}(X)$;*
- (ax_2) *there exist $x_0 \in X$ and $x_1 \in [Gx_0]_{\alpha(x_0)}$ such that $\eta(x_0,x_1) \geq 1$;*
- (ax_3) *(G,T) is a symmetric η-admissible pair;*
- (ax_4) *G and T are α-continuous.*

Then G and T have a common fuzzy fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in [Gx_0]_{\alpha(x_0)}$ be such that $\eta(x_0,x_1) \geq 1$. Then, we consider the following cases:

Case 1: If $\coprod(x_0,x_1) = 0$, then, from (3.2), it is easy to see that $x_0 = x_1$ is a common fuzzy fixed point of G and T. So, we presume that $\coprod(x_0,x_1) > 0$. Then,

$$\begin{aligned}\coprod(x_0,x_1) =& \max\Bigg\{\mu(x_0,x_1),(x_0,[Gx_0]_{\alpha(x_0)}),\mu(x_1,[Tx_1]_{\alpha(x_1)}),\\ &\frac{\mu(x_0,[Tx_1]_{\alpha(x_1)})+\mu(x_1,[Gx_0]_{\alpha(x_0)})}{1+\mu(x_0,x_1)}\Bigg\}\\ \leq& \max\left\{\mu(x_0,x_1),\mu(x_1,[Tx_1]_{\alpha(x_1)}),\frac{\mu(x_0,[Tx_1]_{\alpha(x_1)})}{1+\mu(x_0,x_1)}\right\}\\ \leq& \max\{\mu(x_0,x_1),\mu(x_1,[Tx_1]_{\alpha(x_1)})\}.\end{aligned} \tag{3.3}$$

Now, consider the following subcases:

Case 1(i) $\mu(x_1,[Tx_1]_{\alpha(x_1)}) = 0$, that is, $x_1 \in [Tx_1]_{\alpha(x_1)}$. Since the pair (G,T) is symmetric η-admissible, $x_1 \in [Gx_1]_{\alpha(x_1)}$, $\eta(x_0,x_1) \geq 1$ and by (ax_1), we get $\eta(x_1,x_1) \geq 1$. Now, suppose $x_1 \notin [Gx_1]_{\alpha(x_1)}$

so that $\mu(x_1, [Gx_1]_{\alpha(x_1)}) > 0$. Since (G, T) is an $\eta - (\psi, F)$-contraction, we have

$$\begin{aligned}F(\mu(x_1, [Gx_1]_{\alpha(x_1)})) &\leq F(H([Tx_1]_{\alpha(x_1)}, [Gx_1]_{\alpha(x_1)})) \\ &\leq \psi\left(F\left(\coprod(x_1, x_1)\right)\right) \\ &< F\left(\coprod(x_1, x_1)\right) = F(\mu(x_1, [Gx_1]_{\alpha(x_1)})),\end{aligned}$$

a contradiction. It follows that $x_1 \in [Gx_1]_{\alpha(x_1)}$, and hence $x_1 \in [Gx_1]_{\alpha(x_1)} \cap [Tx_1]_{\alpha(x_1)}$.

Case 1(ii) $\mu(x_1, [Tx_1]_{\alpha(x_1)}) > 0$. Since $\eta(x_0, x_1) \geq 1$ and (G, T) is an $\eta - (\psi, F)$-contraction, we get

$$\begin{aligned}& F(\mu(x_1, [Tx_1]_{\alpha(x_1)})) \quad (3.4)\\ \leq\ & F([Gx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) \\ \leq\ & \psi(F\left(\coprod(x_0, x_1)\right)) \\ =\ & \psi\left(F\left(\max\{\mu(x_0, x_1), \mu(x_1, [Tx_1]_{\alpha(x_1)})\}\right)\right). \quad (3.5)\end{aligned}$$

If $\max\{\mu(x_0, x_1), \mu(x_1, [Tx_1]_{\alpha(x_1)})\} = \mu(x_1, [Tx_1]_{\alpha(x_1)})$, then

$$\begin{aligned}F(\mu(x_1, [Tx_1]_{\alpha(x_1)})) &\leq \psi(F(\mu(x_1, [Tx_1]_{\alpha(x_1)}))) \\ &< F(\mu(x_1, [Tx_1]_{\alpha(x_1)})),\end{aligned}$$

is a contradiction. Therefore,

$$F(\mu(x_1, [Tx_1]_{\alpha(x_1)})) \leq \psi(F(\mu(x_0, x_1))). \quad (3.6)$$

Since $[Tx_1]_{\alpha(x_1)} \in \mathcal{K}(X)$, there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$ such that

$$\mu(x_1, x_2) = \mu(x_1, [Tx_1]_{\alpha(x_1)}). \quad (3.7)$$

Putting (3.7) into (3.6), we have

$$F(\mu(x_1, x_2)) \leq \psi(F(\mu(x_0, x_1))). \quad (3.8)$$

Case 2: If $\coprod(x_1, x_2) = 0$, then $x_1 = x_2$ is a common fuzzy fixed point of G and T. Assume that $\coprod(x_1, x_2) > 0$. Then,

$$\begin{aligned}\coprod(x_1, x_2) &= \max\left\{\mu(x_1, x_2), \mu(x_2, [Gx_2]_{\alpha(x_2)}), \mu(x_1, [Tx_1]_{\alpha(x_1)}),\right. \\ &\qquad \left.\frac{\mu(x_1, [Gx_2]_{\alpha(x_2)}) + \mu(x_2, [Tx_1]_{\alpha(x_1)})}{1 + \mu(x_1, x_2)}\right\} \\ &\leq \max\{\mu(x_1, x_2), \mu(x_2, [Gx_2]_{\alpha(x_2)})\}.\end{aligned}$$

Now, we consider the following subcases:

Case 2 (i) $\mu(x_2, [Gx_2]_{\alpha(x_2)}) = 0$, that is, $x_2 \in [Gx_2]_{\alpha(x_2)}$. Since (G, T) is a symmetric η-admissible pair, $x_2 \in [Tx_1]_{\alpha(x_1)}$, $\eta(x_1, x_2) \geq 1$ and by (ax_2), we get $\eta(x_2, x_2) \geq 1$. Suppose that $\mu(x_2, [Tx_2]_{\alpha(x_2)}) > 0$. Then given that the pair (G, T) is an $\eta - (\psi, F)$-contraction, we have

$$\begin{aligned} F(\mu(x_2, [Tx_2]_{\alpha(x_2)})) &\leq F(H([Gx_2]_{\alpha(x_2)}, [Tx_2]_{\alpha(x_2)})) \\ &\leq \psi(F\left(\coprod(x_2, x_2)\right)) \\ &< F(\mu(x_2, [Tx_2]_{\alpha(x_2)})), \end{aligned}$$

a contradiction. Thus, $x_2 \in [Tx_2]_{\alpha(x_2)}$. It follows that $x_2 \in [Gx_2]_{\alpha(x_2)} \cap [Tx_2]_{\alpha(x_2)}$.

Case 2(ii) $\mu(x_2, [Gx_2]_{\alpha(x_2)}) > 0$. Since $\eta(x_1, x_2) \geq 1$ and the pair (G, T) is an $\eta - (\psi, F)$-contraction, we have

$$F(\mu(x_2, [Gx_2]_{\alpha(x_2)})) \tag{3.9}$$

$$\begin{aligned} &\leq F(H([Gx_2]_{\alpha(x_2)}, [Tx_1]_{\alpha(x_1)})) \\ &\leq \psi(F\left(\coprod(x_2, x_1)\right)) \\ &= \psi(F\left(\max\{\mu(x_1, x_2), \mu(x_2, [Gx_2]_{\alpha(x_2)})\}\right)). \end{aligned} \tag{3.10}$$

If $\max\{\mu(x_1, x_2), \mu(x_2, [Gx_2]_{\alpha(x_2)})\} = \mu(x_2, [Gx_2]_{\alpha(x_2)})$, then

$$\begin{aligned} F(\mu(x_2, [Gx_2]_{\alpha(x_2)})) &\leq \psi(F(\mu(x_2, [Gx_2]_{\alpha(x_2)}))) \\ &< F(\mu(x_2, [Gx_2]_{\alpha(x_2)})) \end{aligned}$$

yields a contradiction. Therefore,

$$F(\mu(x_2, [Gx_2]_{\alpha(x_2)})) \leq \psi(F(\mu(x_1, x_2))). \tag{3.11}$$

Moreover, since $[Gx_2]_{\alpha(x_2)} \in \mathcal{K}(X)$, there exists $x_3 \in [Gx_2]_{\alpha(x_2)}$ such that

$$\mu(x_2, x_3) = \mu(x_2, [Gx_2]_{\alpha(x_2)}). \tag{3.12}$$

Substituting (3.12) in (3.11), gives

$$F(\mu(x_2, x_3)) \leq \psi(F(x_1, x_2)). \tag{3.13}$$

Combining (3.8) and (3.13), we have $F(\mu(x_2, x_3)) \leq \psi^2(F(\mu(x_0, x_1)))$. Proceeding recursively, we generate a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{2n+1} \in [Gx_{2n}]_{\alpha(x_{2n})}$, $x_{2n+2} \in [Tx_{2n+1}]_{\alpha(x_{2n+1})}$, $\mu(x_n, x_{n+1}) > 0$, $\eta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and

$$F(\mu(x_n, x_{n+1})) \leq \psi^n(F(\mu(x_0, x_1))),\ n \in \mathbb{N}. \tag{3.14}$$

Take $\delta_n = \mu(x_n, x_{n+1})$. Then, from the above inequality, we get

$$F(\delta_n) \leq \psi^n(F(\delta_0)) \longrightarrow -\infty \text{ as } n \longrightarrow \infty. \tag{3.15}$$

Therefore, by (F_2), $\lim_{n\longrightarrow\infty}\delta_n = 0$. From (3.15), for all $n \in \mathbb{N}$, there exists $\zeta \in (0,1)$ such that

$$\delta_n^{\zeta}(F(\delta_n)) \leq \delta_n^{\zeta}\psi^n(F(\delta_0)). \tag{3.16}$$

As $n \longrightarrow \infty$ in (3.16), we have $\delta_n^{\zeta}\psi^n(F(\delta_n)) = 0$. Moreover, from (ψ_1), there exists $\lambda > 0$ such that $\lambda < \left|\frac{\psi^n(F(\delta_0))}{n}\right|$; from which we have

$$n\delta_n^{\zeta}\lambda \leq n\delta_n^{\zeta}\left|\frac{\psi^n(F(\delta_0))}{n}\right| = \left|\delta_n^{\zeta}\psi^n(F(\delta_0))\right|. \tag{3.17}$$

As $n \longrightarrow \infty$ in (3.17), we get $\lim_{n\longrightarrow\infty} n\delta_n^{\zeta}\lambda = 0$; that is, $\lim_{n\longrightarrow\infty} n\delta_n^{\zeta} = 0$. It follows that there exists $n_0 \in \mathbb{N}$ such that $\delta_n \leq \frac{1}{n^{\frac{1}{\zeta}}}$, for all $n \geq n_0$. Now, for $m, n \in \mathbb{N}$ with $n < m$, we obtain

$$\begin{aligned} \mu(x_n, x_m) &\leq \sum_{i=n}^{m-1}\delta_i \leq \sum_{i=n}^{m-1}\frac{1}{i^{\frac{1}{\zeta}}} \\ &\leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\zeta}}}. \end{aligned}$$

By Cauchy root test, it is verifiable that the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\zeta}}}$ is convergent; and hence, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. The completeness of X implies that there exists $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, since G and T are α-continuous, then

$$\begin{aligned} \mu(u, [Gu]_{\alpha(u)}) &= \lim_{n\longrightarrow\infty}\mu(x_{2n+1}, [Gu]_{\alpha(u)}) \\ &\leq \lim_{n\longrightarrow\infty} H([Gx_{2n}]_{\alpha(x_{2n})}, [Gu]_{\alpha(u)}) = 0. \end{aligned}$$

It follows that $u \in [Gu]_{\alpha(u)}$. Similarly, one can show that $\mu(u, [Tu]_{\alpha(u)}) = 0$. Consequently, u is the expected common fuzzy fixed point of G and T. $\square$

Corollary 3.6. *Let (X, μ) be a complete metric space and $G, T : X \longrightarrow I^X$ be two fuzzy set-valued maps satisfying the contractive condition:*

$$\xi + F(H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)})) \leq F\left(\coprod(x,y)\right)$$

for all $x, y \in X$ with $H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) > 0$ where $\alpha : X \longrightarrow (0,1]$ is a mapping, $F \in \Omega$ and $\coprod(x,y)$ is given by (3.2). Let $\eta : X \times X \longrightarrow \mathbb{R}_+$ be a function and assume that the following conditions hold:

(i) *there exists $x_0 \in X$ and $x_1 \in [Gx_0]_{\alpha(x_0)}$ with $\eta(x_0, x_1) \geq 1$;*
(ii) *(G, T) is a symmetric η-admissible pair;*
(iii) *G and T are α-continuous;*
(iv) *for each $x \in X$, $[Gx]_{\alpha(x)}$ and $[Tx]_{\alpha(x)}$ are nonempty compact subsets of X.*

Then, G and T have a common fuzzy fixed point in X.

Proof. Put $\psi(t) = t - \xi,\ \xi > 0$ in Theorem 3.5. □

Corollary 3.7. *Let (X, μ) be a complete metric space and $G, T : X \longrightarrow I^X$ be two fuzzy set-valued maps satisfying the contractive condition:*

$$\frac{H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)})}{\coprod(x,y)} \exp\Big(H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) - \coprod(x,y)\Big) \leq \exp(-\xi) \tag{3.18}$$

for all $x, y \in X$ with $H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) > 0$, where $\alpha : X \longrightarrow (0,1]$ is a mapping, and $\coprod(x,y)$ is given by (3.2)*. Let $\eta : X \times X \longrightarrow \mathbb{R}_+$ be a function and assume that the following conditions hold:*

(i) *there exists $x_0 \in X$ and $x_1 \in [Gx_0]_{\alpha(x_0)}$ such that $\eta(x_0, x_1) \geq 1$;*
(ii) *(G, T) is a symmetric η-admissible pair;*
(iii) *G and T are α-continuous;*
(iv) *for each $x \in X$, $[Gx]_{\alpha(x)}$ and $[Tx]_{\alpha(x)}$ are nonempty compact subsets of X.*

Then, G and T have a common fuzzy fixed point in X.

Proof. Set $F(t) = \ln t + t,\ t > 0$ in Corollary 3.6. □

Corollary 3.8. *Let (X, μ) be a complete metric space and $G, T : X \longrightarrow I^X$ be two fuzzy set-valued maps satisfying the contractive condition:*

$$H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \frac{1}{\Big(1 + \xi\left(\coprod(x,y)\right)^{\frac{1}{2}}\Big)^2} \coprod(x,y),$$

for all $x, y \in X$ with $H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) > 0$, where $\alpha : X \longrightarrow (0,1]$ is a mapping and $\coprod(x,y)$ is given by (3.2)*. Assume that conditions (i)-(iv) of Corollary 3.6 hold. Then, G and T have a common fuzzy fixed point in X.*

Proof. Put $F(t) = \frac{-1}{\sqrt{t}},\ t > 0$ in Corollary 3.6. □

Corollary 3.9. *Let (X, μ) be a complete metric space and $G, T : X \longrightarrow I^X$ be two fuzzy set-valued maps satisfying the contractive condition:*

$$\frac{H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)})\left(1 + H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)})\right)}{\coprod(x,y)\left(1 + \coprod(x,y)\right)} \leq \exp(-\xi),$$

for all $x, y \in X$ with $H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) > 0$, where $\alpha : X \longrightarrow (0,1]$ is a mapping and $\coprod(x,y)$ is given by (3.2)*. Assume that conditions (i)-(iv) of Corollary 3.6 hold. Then, G and T have a common fuzzy fixed point in X.*

Proof. Put $F(t) = \ln(t^2 + t),\ t > 0$ in Corollary 3.6. □

Corollary 3.10. *Let (X,μ) be a complete metric space and $G,T : X \longrightarrow I^X$ be two fuzzy set-valued maps satisfying the contractive condition:*

$$F\left(H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)})\right) \leq \psi(F\left(\coprod(x,y)\right))$$

for all $x,y \in X$ with $H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)}) > 0$, where $\alpha : X \longrightarrow (0,1]$ is a mapping, $\psi \in \Psi$, $F \in \Omega$ and $\coprod(x,y)$ is given by (3.2). Assume that the following axioms hold:

(i) *for each $x \in X$, $[Gx]_{\alpha(x)}$ and $[Tx]_{\alpha(x)}$ are nonempty compact subsets of X;*
(ii) *G and T are α-continuous.*

Then, G and T have a common fuzzy fixed point in X.

Proof. For all $x,y \in X$, put $\eta(x,y) = 1$ in Theorem 3.5. □

Corollary 3.11. *Let (X,μ) be a complete metric space and $G,T : X \longrightarrow I^X$ be two fuzzy set-valued maps satisfying the contractive condition:*

$$\xi + F\left(H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)})\right) \leq F\left(\coprod(x,y)\right)$$

for all $x,y \in X$ with $H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)}) > 0$, where $\alpha : X \longrightarrow (0,1]$ is a mapping, $F \in \Omega$ and $\coprod(x,y)$ is given by (3.2). Assume that the following axioms hold:

(i) *for each $x \in X$, $[Gx]_{\alpha(x)}$ and $[Tx]_{\alpha(x)}$ are nonempty compact subsets of X;*
(ii) *G and T are α-continuous;*

Then, G and T have a common fuzzy fixed point in X.

Proof. Put $\psi(t) = t - \xi,\ \xi > 0$ in Corollary 3.10. □

Corollary 3.12. *Let (X,μ) be a complete metric space and $G,T : X \longrightarrow I^X$ be two fuzzy set-valued maps satisfying the contractive condition:*

$$\frac{H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)})}{\coprod(x,y)} \exp\left(H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)}) - \coprod(x,y)\right) \leq \exp(-\xi)$$

for all $x,y \in X$ with $H([Gx]_{\alpha(x)},[Ty]_{\alpha(y)}) > 0$, where $\alpha : X \longrightarrow (0,1]$ is a mapping and $\coprod(x,y)$ is given by (3.2). Assume that the following axioms hold:

(i) *for each $x \in X$, $[Gx]_{\alpha(x)}$ and $[Tx]_{\alpha(x)}$ are nonempty compact subsets of X;*
(ii) *G and T are α-continuous;*

Then, G and T have a common fuzzy fixed point in X.

Proof. Set $F(t) = \ln t + t,\ t > 0$ in Corollary 3.11. □

Corollary 3.13. *Let (X,μ) be a complete metric space and $G : X \longrightarrow I^X$ be an α-continuous fuzzy set-valued map. Assume that for $x \in X$, there exist $\zeta \in (0,1)$ and a mapping $\alpha : X \longrightarrow (0,1]$ such that $[Gx]_{\alpha(x)}$ is a nonempty compact subset of X, and*

$$H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)}) \leq \zeta\mu(x,y), \tag{3.19}$$

for all $x, y \in X$ with $H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)}) > 0$. Then, G has a fuzzy fixed point in X.

Proof. Taking ln on both sides of (3.21), gives

$$-\ln\zeta + \ln\left(H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)})\right) \leq \ln(\mu(x,y)) \tag{3.20}$$

for all $x, y \in X$ with $\mu(x,y) > 0$. Setting $\xi = -\ln\zeta$ and $\ln t = F(t),\ t > 0$ in (3.20), gives the condition:

$$H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)}) > 0 \Longrightarrow \xi + F\left(H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)})\right) \leq F(\mu(x,y)).$$

Consequently, all the assumptions of Corollary 3.11 are satisfied with $G = T$. Hence, G has at least one fuzzy fixed point in X. $\square$

We provide the following Example 3.14 to support the hypotheses of Corollary 3.7.

Example 3.14. Let $X = \{1,3,5,7,13\}$ and $\mu(x,y) = |x-y|$ for all $x, y \in X$. Then (X,μ) is a complete metric space. Define $\eta : X \times X \longrightarrow \mathbb{R}_+$ by

$$\eta(x,y) = \begin{cases} 1, & \text{if } x, y \in \{3,5,13\} \\ 0, & \text{if } x, y \in \{1,7\}. \end{cases}$$

For each $x \in X$, consider two fuzzy set-valued maps $G(x), T(x) : X \longrightarrow [0,1]$ defined by

$$G(x)(t) = \begin{cases} \frac{3}{8}, & \text{if } t = 1 \\ \frac{2}{9}, & \text{if } t = 3 \\ \frac{5}{6}, & \text{if } t \in \{5,7,13\}, \end{cases}$$

and

$$T(x)(t) = \begin{cases} \frac{2}{3}, & \text{if } t = 1 \\ \frac{7}{9}, & \text{if } t \in \{5,7\} \\ 0, & \text{if } t = 13. \end{cases}$$

Suppose that the mapping $\alpha : X \longrightarrow (0,1]$ is defined as $\alpha(x) = 0.5$ for all $x \in X$. Then, we have

$$\begin{aligned} [Gx]_{\alpha(x)} &= \{t \in X : G(x)(t) \geq \alpha(x)\} \\ &= \{5,7,13\}. \end{aligned}$$

Similarly, $[Tx]_{\alpha(x)} = \{1,3,5,7\}$. To see that the contraction condition 3.18 hold, let $x, y \in X$ with $\eta(x,y) \geq 1$, then $x, y \in \{3,5,13\}$. Consider the following cases:

Case 1: For $x = 3$ and $y = 13$, we have

$$\begin{aligned} H([G3]_{\alpha(3)}, [T13]_{\alpha(13)}) &= H(\{5,7,13\},\{1,3,5,7\}) \\ &= \max\{6,4,2\} = 6, \end{aligned}$$

and

$$\begin{aligned} \coprod(3,13) = \ & \max\Bigg\{\mu(3,13), \mu(3,[G3]_{\alpha(3)}), \mu(13,[T13]_{\alpha(13)}), \\ & \frac{\mu(3,[T13]_{\alpha(13)}) + \mu(13,[G3]_{\alpha(3)})}{1+\mu(3,13)}\Bigg\} \\ = \ & \max\{10,2,6\} = 10. \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{H([G3]_{\alpha(3)}, [T13]_{\alpha(13)})}{\coprod(3,13)} \exp\Big(H([G3]_{\alpha(3)}, [T13]_{\alpha(13)}) - \coprod(3,13)\Big) \\ = \ & \frac{3}{5}\exp(-4) < \exp(-4). \end{aligned}$$

Case 2: For $x = 13$ and $y = 3$, we have

$$\begin{aligned} H([G13]_{\alpha(13)}, [T3]_{\alpha(3)}) &= H(\{5,7,13\},\{1,3,5,7\}) \\ &= \max\{6,4,2\} = 6, \end{aligned}$$

and

$$\begin{aligned} \coprod(13,3) = \ & \max\Bigg\{\mu(13,3), \mu(13,[G13]_{\alpha(13)}), \mu(3,[T3]_{\alpha(3)}), \\ & \frac{\mu(13,[T3]_{\alpha(3)}) + \mu(3,[G13]_{\alpha(13)})}{1+\mu(13,3)}\Bigg\} \\ = \ & \max\left\{10, \frac{8}{11}\right\} = 10. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{H([G13]_{\alpha(13)}, [T3]_{\alpha(3)})}{\coprod(13,3)} \exp\Big(H([G13]_{\alpha(13)}, [T3]_{\alpha(3)}) - \coprod(13,3)\Big) \\ = \ & \frac{3}{5}\exp(-4) < \exp(-4). \end{aligned}$$

Case 3: For $x = 5$ and $y = 13$, we have

$$\begin{aligned} H([G5]_{\alpha(5)}, [T13]_{\alpha(13)}) &= H(\{5,7,13\}, \{1,3,5,7\}) \\ &= \max\{6,4,2\} = 6, \end{aligned}$$

and

$$\begin{aligned} \coprod(5,13) = & \max\Big\{ \mu(5,13), \mu(5,[G5]_{\alpha(5)}), \mu(13,[T13]_{\alpha(13)}), \\ & \frac{\mu(5,[T13]_{\alpha(13)}) + \mu(13,[G5]_{\alpha(5)})}{1+\mu(5,13)} \Big\} \\ = & \max\{8,6\} = 8. \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{H([G5]_{\alpha(5)}, [T13]_{\alpha(13)})}{\coprod(5,13)} \exp\Big(H([G5]_{\alpha(5)}, [T13]_{\alpha(13)}) - \coprod(5,13) \Big) \\ = & \ \frac{3}{4}\exp(-2) < \exp(-2). \end{aligned}$$

Case 4: For $x = 13$ and $y = 5$, we have

$$\begin{aligned} H([G13]_{\alpha(13)}, [T5]_{\alpha(5)}) &= H(\{5,7,13\}, \{1,3,5,7\}) \\ &= \max\{6,4,2\} = 6, \end{aligned}$$

and

$$\begin{aligned} \coprod(13,5) = & \max\Big\{ \mu(13,5), \mu(13,[G13]_{\alpha(13)}), \mu(5,[T5]_{\alpha(5)}), \\ & \frac{\mu(13,[T5]_{\alpha(5)}) + \mu(5,[G13]_{\alpha(13)})}{1+\mu(13,5)} \Big\} \\ = & \max\left\{10, \frac{1}{3}\right\} = 8. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{H([G13]_{\alpha(13)}, [T5]_{\alpha(5)})}{\coprod(13,5)} \exp\Big(H([G13]_{\alpha(13)}, [T5]_{\alpha(5)}) - \coprod(13,5) \Big) \\ = & \ \frac{3}{4}\exp(-2) < \exp(-2). \end{aligned}$$

Thus, for all $x, y \in X$ with $H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) > 0$, there exists $\xi \in \{2,4\}$ such that the inequality 3.18 is true. Moreover, it is clear that the pair (G,T) is symmetric η-admissible. And, for $\epsilon > 0$, we can find a $\delta > 0$ such that for all $x, y \in X$, $\mu(x,y) < \delta$ implies $H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)}) < \epsilon$ and $H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) < \epsilon$, that is G and T are α-continuous. If we take $x_0 = 5$ and $x_1 = 13$, then $x_1 \in [Gx_0]_{\alpha(x_0)}$ and $\eta(x_0, x_1) =$

$\eta(5,13) \geq 1$. It is also obvious that G and T are nonempty compact subsets of X. It follows that all the hypotheses of Corollary 3.7 are satisfied. Consequently, G and T have a common fuzzy fixed point in X, given by $\mathcal{F}_{ix}(G,T) = \{5,7\}$.

In what follows, we apply Corollary 3.13 to study fuzzy fixed point result in connection with μ_∞-metric for fuzzy sets. It is noteworthy that fuzzy fixed point results in the setting of μ_∞-metric are very significant in evaluating Hausdorff dimensions. These dimensions help us to understand the concepts of ε^∞-space which is of tremendous importance in higher energy physics (see, e.g. [9, 10]).

Theorem 3.15. *Let (X,μ) be a complete metric space and $G : X \longrightarrow W(X)$ be an α-continuous fuzzy set-valued map. Assume that for each $x \in X$, there exist $\zeta \in (0,1)$ and a mapping $\alpha : X \longrightarrow (0,1]$ such that for all $x, y \in X$,*

$$\mu_\infty(Gx, Gy) \leq \zeta\mu(x,y), \tag{3.21}$$

for all $x, y \in X$ with $H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)}) > 0$. Then, G has at least one fuzzy fixed point in X.

Proof. Since $H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)}) \leq \mu_\infty(Gx, Gy)$ for all $x, y \in X$, then Corollary 3.13 can be applied to find $u \in X$ such that $u \in [Gu]_{\alpha(u)}$. $\square$

The following example shows that Theorem 3.15 cannot be followed from the main result of Heilpern [12].

Example 3.16. Let $X = [0,1]$ and $\mu(x,y) = |x-y|$, for all $x, y \in X$. Then, (X,μ) is a complete metric space. Define a fuzzy set-valued map $G : X \longrightarrow W(X)$ by

$$G(x)(t) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq t \leq \frac{1}{\rho(n+1)^3},\ n \in \mathbb{N}, \rho \geq 2 \\ \frac{4}{15}, & \text{if } \frac{1}{\rho(n+1)^3} < t \leq 1. \end{cases}$$

Suppose that $\alpha(x) = 0.3$ for all $x \in X$. Then, $[Gx]_{\alpha(x)} = \left[0, \frac{1}{\rho(n+1)^3}\right]$. Now, for $x, y \in X$ with $x \neq y$, without loss of generality, take $x = \frac{1}{n^2}$ and $y = \frac{1}{m^2}$ with $n, m \in \mathbb{N}$ $(n \neq m)$. To see that G is α-continuous, given $\epsilon > 0$, take $\delta = \frac{\epsilon}{7}$. Then, $\mu(x,y) < \delta$ implies

$$\begin{aligned} \mu_\infty(Gx, Gy) &= \sup_\alpha H([Gx]_\alpha, [Gy]_\alpha) \\ &= \left| \frac{1}{\rho(n+1)^2} - \frac{1}{\rho(m+1)^2} \right| \\ &\leq \frac{1}{\rho} \left| \frac{1}{n^2} - \frac{1}{m^2} \right| < \epsilon. \end{aligned}$$

Hence, G is α-continuous on X. Moreover, for $\zeta = \frac{1}{\rho} \in (0,1)$, we have

$$\begin{aligned} \mu_\infty(Gx, Gy) &\leq \frac{1}{\rho}\left|\frac{1}{n^2} - \frac{1}{m^2}\right| \\ &= \zeta\mu(x,y). \end{aligned}$$

Also notice that $[Gx]_{\alpha(x)}$ is compact for each $x \in X$. Thus, all the hypotheses of Theorem 3.15 are satisfied and 0 is a fuzzy fixed point of G. However, G is not a fuzzy λ-contraction, since

$$\sup_{x,y\in X, x\neq y} \frac{H([Gx]_{\alpha(x)}, [Gy]_{\alpha(y)})}{\mu(x,y)} = 1.$$

Consequently, Theorem 2.12 due to Heilpern [12] is not applicable to this example.

Remark 3.17.

(i) If we consider a multivalued mapping $\Lambda : X \longrightarrow \mathcal{K}(X)$ defined as $\Lambda x = [Tx]_1$ for all $x \in X$, where X is a complete metric space, then all the results discussed in this section can be reduced to their corresponding crisp set-valued mappings.

(ii) By taking $G = T$ and define the cut set of $T : X \longrightarrow I^X$ by $[Tx]_{\alpha(x)} = \{g(x)\}$ for all $x \in X$, where $g : X \longrightarrow X$ is a self-mapping, then clearly, $\{g(x)\} \in \mathcal{K}(X)$ for all $x \in X$. Consequently, Corollary 3.10 can be applied to deduce the main result of Wardowski [33].

(iii) It is obvious that when we consider in Definition 3.4, various types of mappings in Ω, then more independent consequences of our results can be derived by using the contractive inequality (3.1). But, we skip obtaining such corollaries due to the length of the paper.

4. An Application to a System of Integral Inclusions

Integral inclusions arise in several problems in mathematical physics, control theory, critical point theory for non-smooth energy functionals, differential variational inequalities, economics, fuzzy set arithmetic, traffic theory, and in several other macrosystem dynamics. (see, for instance, [2, 4, 8, 11]). Usually, the first most concerned problem in the investigation of integral inclusions is the conditions for existence of its solutions. In this context, several authors have applied different fixed point approaches and topological methods to obtain existence results of integral inclusions in abstract spaces, see, for example, Appele et al. [2], Cardinali and Papageorgiou [8], Kannan and O'Regan [15], Pathak et al. [25], Sintamarian [30], and the references therein.

Following the above developments, in this section, we apply one of the results in the previous section to study some sufficient conditions for existence of solutions to a system of Fredholm integral inclusions. For basic concepts of integral and differential inclusions, we refer the interested reader to Appelle [2] and Smirnov [31].

Hereafter, $|.|$ represents either absolute value or the vector norm in $\mathbb{R}^n$, which of the two of these being evident from the context. The notation $\|.\|$ is used to denote the sup norm in a specified function space.

Theorem 4.1. *Consider the system of Fredholm integral inclusions:*

$$\begin{aligned} x(t) &\in f(t) + \int_a^b K(t,s,x(s))\mu s,\ t \in [a,b] \\ x(t) &\in f(t) + \int_a^b L(t,s,x(s))\mu s,\ t \in [a,b]. \end{aligned} \tag{4.1}$$

Assume that the following conditions hold:

(i) *the multivalued $K, L : [a,b] \times [a,b] \times \mathbb{R}^n \longrightarrow \mathcal{K}(X)$ are such that for each $x \in C([a,b], \mathbb{R}^n)$, the maps $K_x(t,s) := K(t,s,x(s))$ and $L_x(t,s) := L(t,s,x(s))$, $(t,s) \in [a,b] \times [a,b]$ are lower semicontinuous;*

(ii) *there exists $\xi > 0$ and a continuous function $\vartheta : [a,b] \longrightarrow \mathbb{R}_+$ with $\sup_{t\in[a,b]} \left(\int_a^b \vartheta(t)dt \right) \leq 1$ such that*

$$H(K(t,s,x(s)), L(t,s,x(s))) \leq \frac{\vartheta(s)|x(s)-y(s)|}{\xi^3\|x-y\| + 3\xi^2\left(\sqrt[3]{\|x-y\|}\right)^2 + 3\xi\sqrt[3]{\|x-y\|} + 1} \tag{4.2}$$

where $x, y \in C([a,b], \mathbb{R}^n)$ and $s, t \in [a,b]$. Then, the system of integral inclusions (4.1) *have a common solution in $C([a,b], \mathbb{R}^n)$.*

Proof. Let $X = C([a,b], \mathbb{R}^n)$ and $\mu : X \times X \longrightarrow \mathbb{R}_+$ be defined by

$$\mu(x,y) = \sup_{t\in[a,b]} (|x(t) - y(t)|) = \|x - y\|, \text{ for all } x, y \in X.$$

Then, (X, μ) is a complete metric space. For each $x \in X$, consider two functions $\Lambda_x, \Theta_x : [a,b] \longrightarrow \mathbb{R}^n$, respectively defined as $\Lambda_x = f(t) + \int_a^b K(t,s,x(s))\mu s$ and $\Theta_x = f(t) + \int_a^b L(t,s,x(s))\mu s$. Then define two fuzzy set-valued maps $G, T : X \longrightarrow I^X$ as:

$$G(x)(\omega) = \begin{cases} \frac{2}{3}, & \text{if } \omega = \Lambda_x \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$T(x)(\omega) = \begin{cases} \frac{4}{5}, & \text{if } \omega = \Theta_x \\ 0, & \text{elsewhere.} \end{cases}$$

Take $\alpha : X \longrightarrow (0,1]$ as $\alpha(x) = 0.5$ for all $x \in X$. Then,

$$\begin{aligned} [Gx]_{\alpha(x)} &= \{\omega \in X : G(x)(\omega) \geq \alpha(x)\} \\ &= \left\{\omega \in X : \omega(t) = f(t) + \int_a^b K(t,s,x(s))\mu s, \ t \in [a,b]\right\}. \end{aligned}$$

Similarly,

$$[Tx]_{\alpha(x)} = \left\{\omega \in X : \omega(t) = f(t) + \int_a^b L(t,s,x(s))\mu s, \ t \in [a,b]\right\}.$$

We are to show that G and T have a common fuzzy fixed point in X, and that corresponds to a common solution of(4.1) on $[a,b]$. Now, since the maps K_x and L_x are lower semicontinuous, then, by Michael's selection theorem [18, Theorem 1], it follows that there exist continuous operators $\gamma_x, \pi_x : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ such that $\gamma_x \in K_x(t,s)$ and $\pi_x \in L_x(t,s)$, for each $(t,s) \in [a,b] \times [a,b]$. Therefore, $f(t) + \int_a^t \gamma_x(t,s)\mu s \in [Gx]_{\alpha(x)}$ and $f(t) + \int_a^t \pi_x(t,s)\mu s \in [Tx]_{\alpha(x)}$ for each $x \in X$. Hence, $[Gx]_{\alpha(x)}$ and $[Tx]_{\alpha(x)}$ are nonempty. Let $x, y \in X$ and $\omega \in [Gx]_{\alpha(x)}$. Then, consistent with [30], we have that there exists $\pi_y \in L_y(t,s)$ such that

$$|\gamma_x(t,s) - \pi_y(t,s)| \leq \frac{\vartheta(s)|x(s) - y(s)|}{\xi^3\|x-y\| + 3\xi^2\left(\sqrt[3]{\|x-y\|}\right)^2 + 3\xi\sqrt[3]{\|x-y\|} + 1}.$$

Take $\varpi = f(t) + \int_a^b \pi_x(t,s)\mu s$, then $\varpi \in [Tx]_{\alpha(x)}$ for each $x \in X$. Hence,

$$\begin{aligned} |\omega(t) - \varpi(t)| &\leq \left|\int_a^b \gamma_x(t,s)\mu s - \int_a^t \pi_y(t,s)\mu s\right| \\ &\leq \int_a^b |\gamma_x(t,s) - \pi_y(t,s)|\,\mu s \\ &\leq \int_a^b \frac{\vartheta(s)|x(s) - y(s)|\mu s}{\xi^3\|x-y\| + 3\xi^2\left(\sqrt[3]{\|x-y\|}\right)^2 + 3\xi\sqrt[3]{\|x-y\|} + 1}. \end{aligned} \tag{4.3}$$

Taking sup over $t \in [a,b]$ in (4.3), gives

$$\|\omega - \varpi\| \leq \frac{\mu(x,y)}{\xi^3\mu(x,y) + 3\xi^2(\sqrt[3]{\mu(x,y)})^2 + 3\xi\sqrt[3]{\mu(x,y)} + 1}. \tag{4.4}$$

The expression (4.4) implies that for each $x, y \in X$,

$$H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \frac{\mu(x,y)}{\xi^3\mu(x,y) + 3\xi^2(\sqrt[3]{\mu(x,y)})^2 + 3\xi\sqrt[3]{\mu(x,y)} + 1}.$$

Therefore,

$$\sqrt[3]{H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)})} \leq \frac{\sqrt[3]{\mu(x,y)}}{\xi\sqrt[3]{\mu(x,y)} + 1} \tag{4.5}$$

for all $x, y \in X$ with $x \neq y$. From (4.5), we have

$$\xi + \frac{-1}{\sqrt[3]{H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)})}} \leq \frac{-1}{\sqrt[3]{\mu(x,y)}}. \tag{4.6}$$

Setting $F(t) = \frac{-1}{\sqrt[3]{t}},\ t > 0$ in (4.6), yields

$$\begin{aligned} \xi + F\left(H([Gx]_{\alpha(x)}, [Ty]_{\alpha(y)})\right) &\leq F(\mu(x,y)) \\ &\leq F\left(\coprod(x,y)\right), \end{aligned}$$

for all $x, y \in X$, where $\coprod(x,y)$ is given by (3.2). Hence, all the conditions of Corollary 3.11 are satisfied. Consequently, Problem (4.1) has a common solution in X. □

Example 4.2. Let $K, L : [-1,30] \times [-1,30] \times \mathbb{R} \longrightarrow \mathcal{K}(X)$ be defined by

$$K(t,s,x(s)) = \begin{cases} \left[\frac{-1}{8}, 1\right], & \text{if } x(s) \neq 0, (s,t) \in [-1,5) \times [-1,5) \\ \{0\}, & \text{if } x(s) = 0, (s,t) \in [5,30] \times [5,30]. \end{cases}$$

and

$$L(t,s,x(s)) = \begin{cases} \left[\frac{-1}{8}, 7\right], & \text{if } x(s) \neq 0, (s,t) \in [-1,5) \times [-1,5) \\ \{0\}, & \text{if } x(s) = 0, (s,t) \in [5,30] \times [5,30]. \end{cases}$$

By taking $f(t) = \cos t$ and $\vartheta(t) = \frac{|t|}{1+|t|}$, for all $t \in [-1,30]$, then all the assumptions of Theorem 4.1 are verifiable. Therefore, there exists a common solution to the system of Fredholm integral inclusions:

$$\begin{aligned} x(t) &\in \cos t + \int_{-1}^{30} K(t,s,x(s))\mu s,\ t \in [-1,30] \\ x(t) &\in \cos t + \int_{-1}^{30} L(t,s,x(s))\mu s,\ t \in [-1,30]. \end{aligned}$$

5. Conclusion

First in this article, a new form of continuity for fuzzy set-valued maps, α-continuity is inaugurated. Thereafter, we defined the concept of η-admissibility pair for fuzzy set-valued maps and a generalized quasi-contraction of Wardowski-type, thereby, establishing a common fuzzy fixed point theorem under suitable hypotheses. A few consequences of our main results are pointed out and analyzed. Moreover, as an application, an existence theorem for a system of Fredholm integral inclusions is established. The the idea of this paper, being obtained in the context of metric space, is fundamental. Thus, it can be improved upon when presented in the setting of some generalized metric spaces such as b-metric, G-metric, F-metric and other pseudometric or quasi metric spaces. Similarly, the fuzzy mapping's component can be extended to L-fuzzy, intuitionistic, soft set-valued maps and related non-crisp mappings.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgement

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments that helped to improve this manuscript.

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