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CONE NORMED SPACES

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ABSTRACT. In this paper, we introduce the cone normed spaces and cone bounded linear mappings. Among other things, we prove the Baire category theorem and the Banach–Steinhaus theorem in cone normed spaces.

Keywords : One metric space; Baire category theorem; Banach Steinhaus theorem.
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1. INTRODUCTION

Let E be a Banach space and P be a subset of E. P is called a cone whenever
(1) P is a closed, non-empty set and $P \neq \{0\}$,
(2) $ax + by \in P$ for all $x, y \in P$ and $a, b \geq 0$,
(3) $P \cap (-P) = \{0\}$.
For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in intP$, where $intP$ denoted the interior of P. The cone P is normal if there is a number $M > 0$ such that for all $x, y \in E$

$$0 \leq x \leq y \quad \Longrightarrow \quad \|x\| \leq M\|y\|.$$

The least positive number satisfying the above is called the normal constant of P [1]. It is clear that $M \geq 1$. In the following we always suppose that E is a real Banach space and P is a cone in E with $intP \neq \varnothing$ and $\leq$ is a partial ordering with respect to P.

Definition 1.1. ([1]) Let X be non-empty set . Suppose that the mapping $d : X \times X \to E$ satisfies:
(1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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Then d is called a cone metric on X, and (X, d) is called cone metric space.

Example 1.2. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then P is a normal cone with normal constant $M = 1$ and (X, d) is a cone metric space[2].

Definition 1.3. ([1]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(1) $\{x_n\}$ is said to convergent to x whenever for every $c \in E$, with $0 \ll c$ there is a positive integer N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) $\{x_n\}$ is said to be a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(3) (X, d) is called a complete cone metric space if every Cauchy sequence is convergent.

Let us recall [1] that if P is a normal cone, then $\{x_n\} \subseteq X$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. Furthermore, $\{x_n\} \subseteq X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.4. Let (X, d) be a cone metric space and $B \subseteq X$.

(1) A point $b \in B$ is called an interior point of B whenever there exists a point p, $0 \ll p$, such that $B_p(b) \subseteq B$ where $B_p(b) := \{y \in X : d(b, y) \ll p\}$.

(2) A subset $B \subseteq X$ is called open if each element of B is an interior point of B. The family $\beta = \{B_e(x) : x \in X, 0 \ll e\}$ is a sub-basis for a topology on X. We denote this cone topology by τ_c. The topology τ_c is a Hausdorff and first countable [2].

In this paper we suppose that P is a normal cone with normal constant M and fixed c_0 with $0 \ll c_0$.

2. CONE NORMED SPACES

Definition 2.1. Let X be real vector space. Suppose that the mapping $\|.\|_p : X \to E$ satisfies:

(1) $\|x\|_p \geq 0$ for all $x \in X$ and $\|x\|_p = 0$ if and only if $x = 0$,

(2) $\|\alpha x\|_p = |\alpha| \ \|x\|_p$, for all $x \in X$ and $\alpha \in \mathbb{R}$,

(3)$\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in X$.

Then $\|.\|_p$ is called a cone norm on X and $(X, \|.\|_p)$ is called a cone normed space.

It is easy to see that every normed space is a cone normed space by putting $E := \mathcal{R}$, $P := [0.\infty)$.

Example 2.2. Let $E = l_1$, $P = \{\{x_n\} \in E : x_n \geq 0, for\ all\ n\}$ and $(X, \|.\|)$ be a normed space and $\|.\|_p : X \to E$ defined by $\|x\|_p = \{\frac{\|x\|}{2^n}\}$. Then P is a normal cone with constant normal $M = 1$ and $(X, \|.\|_p)$ is a cone normed space.
Let $(X, \|.\|_p)$ be a cone normed space. Set $d(x, y) = \|x - y\|_p$. It is easy to see that (X, d) is a cone metric space. d is called getting cone metric of cone norm $\|.\|_p$.

Definition 2.3. We say that the cone normed space $(X, \|.\|_p)$ is a cone Banach space when getting cone metric of $\|.\|_p$ is complete.

Lemma 2.4. *Let $(X, \|.\|_p)$ and $(Y, \|.\|_p)$ be cone normed spaces, and let T be a linear map from X into Y. If T has any one of the five following properties, it has all five of them:*
(a) (continuity at a point) For some fixed $x_0 \in X$ we have: Given $0 \ll c$ there is a $0 \ll t$ such that $\|Tx - Tx_0\|_p \ll c$ whenever $\|x - x_0\|_p \ll t$.
(b) (continuity at zero) For $0 \ll c$ there is a $0 \ll t$ such that $\|Tx\|_p \ll c$ whenever $x \in X$ and $\|x\|_p \ll t$.
(c) (continuity at every point of x) For any $x \in X$ we have: Given $0 \ll c$ there is a $0 \ll t$ such that $\|Tx - Ty\|_p \ll c$ whenever $y \in X$ and $\|y - x\|_p \ll t$.
(d) (uniform continuity) Given $0 \ll c$ there is a $0 \ll t$ such that $\|Tx - Ty\|_p \ll c$ whenever $x, y \in X$ and $\|x - y\|_p \ll t$.
(e) (sequential continuity) Given any sequence $\{x_n\} \subseteq X$ which is convergent to a point $x_0 \in X$, the sequence $\{Tx_n\} \subseteq Y$ is convergent to the point $Tx_0 \in Y$.

Proof: First assume that T has property (a). So for some $x_0 \in X$ and any $0 \ll c$ we can choose $0 \ll t$ such that $\|Tx - Tx_0\|_p \ll c$ whenever $\|x - x_0\|_p \ll t$. Then for any $w \in X$ with $\|w\|_p \ll t$ we have $\|T(w+x_0) - Tx_0\| \ll c$ because $\|(w+x_0) - x_0\|_p \ll t$. But T is linear this says that $\|Tw\|_p \ll c$ whenever $\|w\|_p \ll t$ and we have shown that (a) implies (b).
Now suppose that T has property (b), let $x \in X$ and $0 \ll c$ be given. There is a $0 \ll t$ such that $\|Tw\|_p \ll c$ whenever $\|w\|_p \ll t$. We have $\|T(y - x)\|_p \ll c$ whenever $\|y - x\|_p \ll t$; just use $y - x$ in place of w. Again recalling that T is linear we see that (b) implies (c). Clearly (c) implies (a). Thus (a), (b) and (c) are equivalent.
Let us show that (b) implies (d). Given $0 \ll c$ we may choose $0 \ll t$ so that $\|w\|_p \ll t$ implies $\|Tw\|_p \ll c$. Now if $x, y \in X$ and $\|x - y\|_p \ll t$ then $\|T(x - y)\|_p \ll c$. Since T is linear (b) implies (d) and clearly (d) implies (b). Thus (a) through (d) are equivalent.
We will complete the proof by showing that (b) and (e) are equivalent. First suppose that T has property (b) and let $0 \ll c$ be given. Then we have choose $0 \ll t$ that $\|Tw\|_p \ll c$ whenever $\|w\|_p \ll t$. Now suppose $\{x_n\} \subseteq X$ converges x_0. Then we can find positive integer N such that $\|x_n - x_0\|_p \ll t$ whenever $n \geq N$. Thus $\|T(x_n - x_0)\|_p \ll c$ whenever $n \geq N$. Clearly this says that $\{Tx_n\}$ converges to Tx_0. Now assume (e) and negation of (b). So we are supposing that there is a $0 \ll c$ such that for any $0 \ll t$ we can find $w_t \in X$ with $\|w_t\|_p \ll t$ and $\|Tw_t\|_p \not\ll c$. Thus for this c we can find $\{w_n\} \subseteq X$ such that $\|w_n\|_p \ll \frac{c}{2^n}$ and $\|Tw_n\|_p \not\ll c$ for all n. But $\{w_n\}$ is converges to 0 because $\| \, \|w_n\|_p \| \leq \frac{M\|c\|}{2^n} \to 0$ as $n \to \infty$. By (e) $\{Tw_n\}$ must converge to $T0 = 0$ and this impossible. $\square$

Proposition 2.5. *Let $(X, \|.\|_p)$ be a cone normed space, and $x \in X$, $0 \ll c$. Then*

$$y \in \overline{B_c(x)} \iff (\exists \{z_n\} \subseteq B_c(x)\,; \quad z_n \to y).$$

Proof: Let $y \in \overline{B_c(x)}$. Then for any positive integer n, $z_n \in B_{\frac{c}{2^n}}(y) \cap B_c(x) \neq \varnothing$. We obtain $z_n \to y$ as $n \to \infty$. Now we suppose that $\{z_n\} \in B_c(x)$ is a sequence that $z_n \to y$ as $n \to \infty$. Let W be a open set such that W consists of y. There is $0 \ll p$ such that $B_p(y) \subseteq W$. We choose the positive integer n such that $\|z_n - y\|_p \ll p$.

Hence, $z_n \in B_p(y)$ and $W \cap B_c(x) \neq \varnothing$. So $y \in \overline{B_c(x)}$. □

We need the following lemma to prove the Baire category theorem.

Lemma 2.6. *Let $(X, \|.\|_p)$ be a cone normed space, $x \in X$ and $0 \ll c$. Then $\overline{B_{\frac{c}{2}}(x)} \subseteq B_c(x)$.*

Proof: Let $y \in \overline{B_{\frac{c}{2}}(x)}$. Then there is a sequence $\{z_n\} \subseteq B_{\frac{c}{2}}(x)$ such that $z_n \to y$ as $n \to \infty$. We can choose the positive integer n such that $\|z_n - y\|_p \ll \frac{c}{2}$. We obtain that $\|x-y\|_p \leq \|x-z_n\|_p + \|z_n - y\|_p \ll \frac{c}{2} + \frac{c}{2} = c$. Letting $a = (\|x-z_n\|_p + \|z_n - y\|_p) - \|x-y\|_p$, by attention that action $+$ is continuous in E, we have

$$c - \|x-y\|_p = (c - (\|x-z_n\|_p + \|z_n-y\|_p) + \ a \in a + intP = int(a+P) \subseteq intP.$$

So $\|x-y\|_p \ll c$ and thus $y \in B_c(x)$. □

Theorem 2.7. *(Baire Category Theorem) Let $(X, \|.\|_p)$ be a cone Banach space. Then every countable intersection of dense and open sets is dense.*

Proof: Let $\{A_n\} \subseteq X$ be a sequence of dense and open sets. Suppose $x \in X$, and W is a open set such that W consists of x. Then there is $0 \ll r$ such that $B_r(x) \subseteq W$. Since A_1 is dense in X, we obtain $z_1 \in A_1 \cap B_r(x) \neq \varnothing$. But $A_1 \cap B_r(x)$ is open and hence there exists $0 \ll r'$ such that $B_{r'}(z_1) \subseteq A_1 \cap B_r(x)$. We can choose the positive real number k_1 such that $k_1 < min\{\frac{1}{2}, \frac{1}{2\|r'_1\|}\}$. By setting $r_1 = k_1 r'_1$, we have $r_1 \ll \frac{r'}{2}$ and $\overline{B_{r_1}(z_1)} \subseteq B_{r'_1}(z_1)$. Since A_2 is dense in X, we have $z_2 \in A_2 \cap B_{r_1}(z_1) \neq \varnothing$, but this set is open. Thus there exists $0 \ll r'_2$ such that $B_{r'_2}(z_2) \subseteq A_2 \cap B_{r_1}(z_1)$. By choosing $0 < k_2 < min\{\frac{1}{2}, \frac{1}{2^2\|r'_2\|}\}$, we have $r_2 = k_2 r'_2 \ll \frac{r'_2}{2}$ and hence $\overline{B_{r_2}(z_2)} \subseteq B_{r'_2}(z_2)$. Since A_3 is dense in X then let $z_3 \in A_3 \cap B_{r_2}(z_2)$. Since A_3 is open, there is a $0 \ll r'_3$ such that $B_{r'_3}(z_3) \subseteq A_3 \cap B_{r_2}(z_2)$. We can choose the real number $0 < k_3$ for which $k_3 < min\{\frac{1}{2}, \frac{1}{2^3\|r'_3\|}\}$. Setting $r_3 = k_3 r'_3$, we conclude that $r_3 \ll \frac{r'_3}{2}$ and $\overline{B_{r_3}(z_3)} \subseteq B_{r'_3}(z_3)$. Repeating the above argument we obtain

$$...\overline{B_{r_3}(z_3)} \subseteq \overline{B_{r_2}(z_2)} \subseteq \overline{B_{r_1}(z_1)}.$$

We claim $r_n \to 0$, because $r_n = k_n r'_n \ll \frac{1}{2^n\|r'_n\|}\ r'_n$. So $\|r_n\| \leq \frac{M}{2^n\|r'_n\|}\|r'_n\| = \frac{M}{2^n} \to 0$ as $n \to \infty$. Moreover , we can show that $\{z_n\}$ is Cauchy sequence in X. To see this, let $\epsilon > 0$ is given, there is a positive integer N such that $M\|r_n\| < \epsilon$ for all $n \geq N$. So $\|\|z_m - z_n\|_p\| \leq M\|r_n\| < \epsilon$ for all $m > n \geq N$. This means that $\{z_n\}$ is Cauchy sequence. Since X is cone Banach space, there is $z \in X$ such that $\lim_{n\to\infty} z_n = z$. For any positive integer N, if $n > N$ then $z_n \in B_{r_N}(z_N)$, so $z \in \overline{B_{r_N}(z_N)}$ and hence

$$z \in \bigcap_{N=1}^{\infty} B_{r_N}(z_N) \subseteq \bigcap_{N=1}^{\infty} A_N \bigcap B_r(x) \subseteq \bigcap_{N=1}^{\infty} A_N \bigcap W.$$

This says that $\bigcap_{N=1}^{\infty} A_N$ is dense in X. □

Corollary 2.8. *Every cone Banach space is second category.*

Definition 2.9. A subset $A \subseteq E$ is upper bounded, if there exists $0 \leq t$ such that $a \leq t$ for all $a \in A$, t is an upper bound for A. We say that P has supremum property, if for every upper bounded set A in P least upper bound exists in P, we show this element to $supA$.

Example 2.10. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$. P is a normal cone with normal constant $M = 1$, and P has supremum property. Because if $A \subseteq P$ is upper bounded set, then $supA = (sup_{z\in A}\pi_1(z), sup_{z\in A}\pi_2(z))$. Where π_1 and π_2 are projections on first and second components, respectively.

From now on, we suppose that P has supremum property.

Definition 2.11. Let $(X, \|.\|_p)$ be a cone normed space. A subset A in X is cone bounded, if $\{\|x\|_p; x \in A\}$ is upper bounded.

Definition 2.12. Let $(X, \|.\|_p)$ and $(Y, \|.\|_p)$ be cone normed spaces, and $\Lambda : X \to Y$ be a linear mapping. Λ is cone bounded if the set $\Lambda(B_{c_0}(0))$ is a cone bounded set. Let $B(X, Y)$ denotes the set of all cone bounded linear mappings from X into Y. It is easy to see that $\|\Lambda\|_p = sup\{\|\Lambda(x)\|_p : \|x\|_p \ll c_0\}$ is a cone norm on $B(X, Y)$.

In the following, we obtain a linear mapping that is not cone bounded.

Example 2.13. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $c_0 = (1, 1)$ and let X be the set of all real- valued polynomials on interval $[0, 1]$ and $\|.\|_u$ is supremum norm on X, that is $\|f\|_u = sup\{|f(x)| : x \in [0, 1]\}$ for all $f \in X$. Let $\|.\|_p : X \to E$ defined by $\|f\|_p = \|f\|_u c_0$. It is easy to see that $(X, \|.\|_p)$ is a cone normed space. Suppose $D : X \to X$ defined by $D(f) = f'$. Then D is a linear mapping that is not cone bounded.

Theorem 2.14. *(The Uniform Boundness Principle) Let $(X, \|.\|_p)$ be a cone Banach space and $(Y, \|.\|_p)$ be a cone normed space. Suppose $A \subseteq B(X, Y)$ is pointwise bounded, that is for each $x \in X$, the set $\{Tx : T \in A\}$ is cone bounded. Then A is a cone bounded set in $B(X, Y)$.*

Proof: Let
$$E_n = \{x \in X : \|Tx\|_p \leq nc_0 \; forall \; T \in A\}$$
for all $n \in \mathbb{N}$. It is easy too see that the set $\{y \in Y : \|y\|_p \leq nc_0\}$ is closed. Hence, for each positive integer n, E_n is a closed set. We claim $X = \bigcup_{n=0}^{\infty} E_n$. Too see this, since $0 \ll c_0$, choose $0 < \delta$ such that
$$c_0 + \{x \in E : \|x\| \leq \delta\} \subseteq P.$$
If $x \in X$ is given. Setting $\alpha_x = sup\{\|Tx\|_p : T \in A\}$. Choose a positive integer n such that $\|\frac{\alpha_x}{n}\| < \delta$. So $c_0 - \frac{\alpha_x}{n} \in c_0 + \{x \in E : \|x\| \leq \delta\} \subseteq P$ and $\alpha_x \ll nc_0$. This shows that $x \in E_n$. So $X = \bigcup_{n=0}^{\infty} E_n$. But X is a cone Banach space and haence X is second category. Then there is a positive integer k such that $intE_k = int\overline{E_k} \neq \varnothing$. Suppose $y \in intE_k$. We can choose $0 \ll r$ such that $B_r(y) \subseteq E_k$. Repeating the above method, there exists a positive integer m such that $\dfrac{c_0}{m} \ll r$. Now, we let $T \in A$ and $x \in X$, $\|x\|_p \ll c_0$. Then $\|(y + \dfrac{x}{m}) - y\|_p = \|\dfrac{x}{m}\|_p \ll \dfrac{c_0}{m} \ll r$. This means that

$y + \frac{x}{m} \in B_r(y)$. We have

$$\begin{aligned}\|Tx\|_p = m\|T\frac{x}{m}\|_p &= m\|T(y + \frac{x}{m}) - Ty\|_p \\ &\leq m(\|T(y + \frac{x}{m})\|_p + \|Ty\|_p) \\ &\leq m(k + k) = 2mk.\end{aligned}$$

Thus $\|T\|_p \leq 2mk$. In the other words A is a cone bounded set in $B(X, Y)$. □

The question arises here is whether Hahn Banach theorem and open mapping theorem can be extended similar to cone normed spaces or not?

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