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Is There Any Digital Pseudocovering Map?

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Abstract. This paper is devoted to the notion digital pseudocovering map introduced by Han [**?**]. We show that, according to Han's definition, a function is a pseudocovering map if and only if it is a covering map. We give a modified definition of pseudocovering in order to obtain the results Han sought.

Keywords: digital topology, digital covering map, digital pseudocovering map.

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1. Introduction and Motivation

In classical topology, there exist continuous maps $f : X \longrightarrow Y$ that behave like covering maps except in the neighborhood of one point. These maps and their important role in the characterization of fundamental groups have been widely studied and they have led to some generalizations of covering theory such as semicovering theory [**?**] and generalized covering theory [**?**, **?**].

In parallel, there are some non-examples of digital covering maps in digital topology that enjoy many properties of digital covering maps such path lifting property and unique path lifting property. As an example, a digital map obtained by restricting the domain of a digital covering map is not necessarily a digital covering maps but has uniqueness of digital path liftings. This has made it important to generalize the notion digital covering map.

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Han [?] has introduced a generalization of digital covering maps, named digital pseudocovering map, by weakening the local isomorphism condition in the definition of digital covering maps.

At first, we show that his examples are either digital isomorphisms or do not satisfy all conditions of a digital pseudocovering map. Then, we will prove that his conditions in the definition of digital pseudocovering maps are incompatible. In fact, any digital map that applies to these conditions will be the same as the old digital covering map. Finally, by a little modification in one of the conditions, we come to a definition that provides the desired properties.

2. Notations and preliminaries

For a positive integer u with $1 \leq u \leq n$, an adjacency relation of a digital image in $\mathbb{Z}^n$ is defined as follows:
Two distinct points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ in $\mathbb{Z}^n$ are l_u-adjacent [?] if there are at most u distinct indices i such that $|p_i - q_i| = 1$ and for all indices j, $p_j = q_j$ if $|p_j - q_j| \neq 1$. An l_u-adjacency relation on $\mathbb{Z}^n$ can be denoted by the number of points that are l_u-adjacent to a given point $p \in \mathbb{Z}^n$. For example,

- The l_1-adjacent points of $\mathbb{Z}$ are called 2-adjacent.
- The l_1-adjacent points of $\mathbb{Z}^2$ are called 4-adjacent and the l_2-adjacent points in $\mathbb{Z}^2$ are called 8-adjacent.
- The l_1-adjacent, l_2-adjacent and l_3-adjacent points of $\mathbb{Z}^3$ are called 6-adjacent, 18-adjacent, and 26-adjacent, respectively.

More general adjacency relations are studied in [?].

Let κ be an adjacency relation defined on $\mathbb{Z}^n$ and $X \subseteq \mathbb{Z}^n$. Then the pair (X, κ) is said to be a (binary) digital image. A digital image $X \subseteq \mathbb{Z}^n$ is **κ-connected** [?] if and only if for every pair of different points $x, y \in X$, there is a set $x_0, x_1, ..., x_r$ of points of a digital image X such that $x = x_0, y = x_r$ and x_i and x_{i+1} are κ-adjacent where $i = 0, 1, ..., r-1$.

Proposition 2.1. *([?, ?]) Let (X, κ) in $\mathbb{Z}^n$ and (Y, λ) in $\mathbb{Z}^m$ be digital images. A function $f : X \longrightarrow Y$ is (κ, λ)-continuous if and only if for every κ-adjacent points $x_0, x_1 \in X$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are λ-adjacent in Y.*

For $a, b \in \mathbb{Z}$ with $a < b$, a **digital interval** [?] is a set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}.$$

Definition 2.2. By a **digital κ-path** from x to y in a digital image (X, κ), we mean a $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \longrightarrow X$ such that $f(0) = x$ and $f(m) = y$. If $f(0) = f(m)$ then the κ-path is said to be closed, and f is called a κ-loop.

Let $f : [0, m-1]_{\mathbb{Z}} \longrightarrow X \subseteq \mathbb{Z}^n$ be a $(2, \kappa)$-continuous function such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1 \bmod m$. Then f is called a simple κ-path and the set $f([0, m-1]_{\mathbb{Z}})$ is called a simple closed κ-curve containing m points which is denoted by $SC_\kappa^{n,m}$. We say that the length of a simple κ-path is the number m. If f is a constant function, it is called a trivial loop.

If $f : [0, m_1]_{\mathbb{Z}} \longrightarrow X$ and $g : [0, m_2]_{\mathbb{Z}} \longrightarrow X$ are digital κ-paths with $f(m_1) = g(0)$, then define the product [**?**] $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \longrightarrow X$ by

$$(f * g)(t) = \begin{cases} f(t) \quad if \quad t \in [0, m_1]_{\mathbb{Z}}; \\ g(t - m_1) \quad if \quad t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$

Let (E, κ) be a digital image and let $\varepsilon \in N$. The κ-neighborhood [8] of $e_0 \in E$ with radius ε is the set $N_\kappa(e_0, \varepsilon) = \{e \in E |\ l_\kappa(e_0, e) \leq \varepsilon\} \cup \{e_0\}$, where $l_\kappa(e_0, e)$ is the length of a shortest κ-path from e_0 to e in E.

The function $f : X \longrightarrow Y$ is a (κ, λ)-**isomorphism** [**?**], if f is a (κ, λ)-continuous bijection and further $f^{-1} : Y \longrightarrow X$ is (λ, κ)-continuous. In this case, X, Y are called (κ, λ)-isomorphic, denoted by $X \overset{(\kappa,\lambda)}{\approx} Y$. If $n = m$ and $\kappa = \lambda$, then f is called a κ-isomorphism.

Definition 2.3. [**?**] For two digital spaces (X, κ) in $\mathbb{Z}^n$ and (Y, λ) in $\mathbb{Z}^m$, a (κ, λ)-continuous map $h : X \longrightarrow Y$ is called a **local** (κ, λ)**-isomorphism** if for every $x \in X$, $h|_{N_\kappa(x,1)}$ is a (κ, λ)-isomorphism onto $N_\lambda(h(x), 1)$. If $n = m$ and $\kappa = \lambda$, then the map h is called a local κ-isomorphism.

Definition 2.4. [**?**] For two digital spaces (X, κ) and (Y, λ), a map $h : X \longrightarrow Y$ is called a **weakly local** (κ, λ)**-isomorphism** if for every $x \in X$, $h|_{N_\kappa(x,1)}$ is a **weak** (κ, λ)**-isomorphism** that means h maps (κ, λ)-isomorphically $N_\kappa(x, 1)$ onto $h(N_\kappa(x, 1))$.

In the definition of local isomorphism we can remove the condition of the continuity of h, because continuity is a local notion and for every $x \in X$, $h|_{N_\kappa(x,1)}$ is a (κ, λ)-isomorphism and hence h is continuous. Also, it is notable that the difference between local isomorphisms and weakly local isomorphisms is surjectivity of $h|_{N_\kappa(x,1)}$.

Definition 2.5. [**?**, **?**, **?**] Let (E, κ) and (B, λ) be digital images and $p : E \longrightarrow B$ be a (κ, λ)-continuous surjection. The map p is called a (κ, λ)**-covering map** if and only if for each $b \in B$ there exists an index set M such that

(1) $p^{-1}(N_\lambda(b, 1)) = \bigsqcup_{i \in M} N_\kappa(e_i, 1)$ with $e_i \in p^{-1}(b)$;

(2) if $i, j \in M$, $i \neq j$, then $N_\kappa(e_i, 1) \cap N_\kappa(e_j, 1) = \emptyset$; and

(3) the restriction map $p|_{N_\kappa(e_i,1)} : N_\kappa(e_i,1) \longrightarrow N_\lambda(b,1)$ is a (κ,λ)-isomorphism for all $i \in M$.

Moreover, (E,p,B) is said to be a (κ,λ)-covering and (E,κ) is called a digital (κ,λ)-covering space over (B,λ). Also, $N_\lambda(b,1)$ is called an elementary λ-neighborhood of b or a coverable λ-neighborhood of b.

Definition 2.6. [**?**] Let (E,κ), (B,λ), and (X,μ) be digital images, let $p : E \longrightarrow B$ be a (κ,λ)-covering map, and let $f : X \longrightarrow B$ be (μ,λ)-continuous. A **lifting** of f with respect to p is a (μ,κ)-continuous function $\widetilde{f} : X \longrightarrow E$ such that $p \circ \widetilde{f} = f$.

Theorem 2.7. [**?**] *Let (E,κ) be a digital image and $e_0 \in E$. Let (B,λ) be a digital image and $b_0 \in B$. Let $p : E \longrightarrow B$ be a (κ,λ)-covering map such that $p(e_0) = b_0$. Then any λ-path $\alpha : [0,m]_{\mathbb{Z}} \longrightarrow B$ beginning at b_0 has a unique lifting to a path $\widetilde{\alpha}$ in E beginning at e_0.*

Definition 2.8. [**?**] Let $p : (E,\kappa) \to (B,\lambda)$ be a (κ,λ)-continuous surjection map. We say that
(i) p has **digital path lifting property** if for any digital path α in B and any $e \in p^{-1}(\alpha(0))$ there is a lifting $\widetilde{\alpha}$ of α in E such that $\widetilde{\alpha}(0) = e$.
(ii) p has the **uniqueness of digital path lifts property** if any two paths $\alpha, \beta : [0,m]_{\mathbb{Z}} \longrightarrow E$ are equal if $p \circ \alpha = p \circ \beta$ and $\alpha(0) = \beta(0)$.
(iii) p has the **unique path lifting property** (u.p.l, for abbreviation) if it has both the path lifting property and the uniqueness of path lifts property.

Although it was proved that every digital covering map is a local isomorphism and the converse is not true in general ([**?**, **?**]), the author with M. Zakki [**?**] have shown that the given counterexample is not correct and have proved the inverse as follows.

Theorem 2.9. [**?**] *Let $p : (E,\kappa) \to (B,\lambda)$ be a (κ,λ)-continuous surjection map. Then p is a digital covering if and only if it is a local isomorphism.*

In this paper, all the digital spaces assumed to be connected.

3. Nihility of pseudocovering map

In the algebraic topology, there are some examples of maps that are not coverings but have many of their features. For example $exp : (0,+\infty) \longrightarrow S^1$ defined by $exp(t) = e^{2\pi ti}$ is not a covering map but every $x \in S^1$ has a coverable open neighborhood except $(1,0)$.

In digital topology we have also such maps. For example, let

$$f : \mathbb{Z}^+ \longrightarrow SC_\kappa^{n,l} := (c_i)_{i=0}^{l-1};\ l \geq 4,$$

FIGURE 1.

by $f(i) = c_{i \ mod \ l}$, where $\mathbb{Z}^+ = \{k \in \mathbb{Z} | k \geq 0\}$. For every c_i, except c_0, $N_\kappa(c_i, 1)$ is coverable. Also, for c_0, all components of $f^{-1}(N_\kappa(c_0, 1))$ are isomorphic to $N_\kappa(c_0, 1)$ except $N_2(0, 1)$. This motivates us to generalize digital covering map in order to include more classes of maps. Pseudocovering maps, introduced by Han [**?**], seems to be one of these possible generalizations, although we will also mention its defect.

Definition 3.1. ([**?**]) Let (E, κ_0) and (B, κ_1) be digital spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : (E, \kappa_0) \longrightarrow (B, \kappa_1)$ be a surjection. Suppose that for any $b \in B$ the map p has the following properties:

(1) for some index set M, $p^{-1}(N_{\kappa_1}(b, 1)) = \bigsqcup_{i \in M} N_{\kappa_0}(e_i, 1)$ with $e_i \in p^{-1}(b)$,
(2) if $i, j \in M$ and $i \neq j$, then $N_{\kappa_0}(e_i, 1) \cap N_{\kappa_0}(e_j, 1)$ is an empty set; and
(3) the restriction of p on $N_{\kappa_0}(e_i, 1)$ is a weak (k_0, k_1)-isomorphism for all $i \in M$.

Then the map p is called a (k_0, k_1)-pseudocovering map, (E, p, B) is said to be a (k_0, k_1)-pseudocovering and (E, k_0) is called a (k_0, k_1)-pseudocovering space over (B, k_1).

This definition is like the definition of a digital covering map, but $p|_{N_{\kappa_0}(e_i,1)}$ is weak (k_0, k_1)-isomorphism rather than (k_0, k_1)-isomorphism.

In [**?**] it is claimed that the map f, introduced above is a pseudocovering map. In the following we show that it is not true.

Proposition 3.2. *The map* $f : \mathbb{Z}^+ \longrightarrow SC_\kappa^{n,l} := (c_i)_{i=0}^{l-1}$; $l \geq 4$, *defined by* $f(i) = c_{i \ mod \ l}$ *is not a digital pseudocovering map.*

Proof. Consider the point c_{l-1}. Since $c_0 \in N_\kappa(c_{l-1}, 1)$, $0 \in p^{-1}(N_\kappa(c_{l-1}, 1))$. By the condition (1) of the definition of digital pseudocovering map, $p^{-1}(N_\kappa(c_{l-1}, 1)) = \bigsqcup_{k \in \mathbb{N}} N_2((l-1)k, 1)$. But for every $k \in \mathbb{N}$, $0 \notin N_2((l-1)k, 1)$ because $(l-1)k$ is at least 3, as it is shown in Figure 1. This shows that f can not be a pseudocovering map. □

This disability can be found in Example 4.3, [**?**], part (1) and it can easily be checked that maps g, h and p (in [**?**]) are isomorphisms. The following theorem will end all ambiguities.

Theorem 3.3. *Every map satisfying the conditions of pseudocovering map is a covering map.*

Proof. Let (E, κ_0) and (B, κ_1) be digital spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively and $p : (E, \kappa_0) \longrightarrow (B, \kappa_1)$ be a pseudocovering map which is not a covering map. Then there exists $b \in B$ and $e \in p^{-1}(\{b\})$ such that the restriction of p on $N_{\kappa_0}(e,1)$ is a weak (k_0,k_1)-isomorphism and is not onto $N_{\kappa_1}(b,1)$. So there exists $b' \in N_{\kappa_1}(b,1) - p\big(N_{\kappa_0}(e,1)\big)$ such that

$$N_{\kappa_0}(e,1) \cap p^{-1}(b') = \emptyset \tag{3.1}$$

Since b and b' are κ_1-adjacent, $e \in p^{-1}(b) \subset p^{-1}\big(N_{\kappa_1}(b',1)\big)$. Also, $p^{-1}\big(N_{\kappa_1}(b',1)\big) = \bigsqcup_{i\in M} N_{\kappa_0}(e'_i,1)$ which implies that there exists $j \in M$ such that $e \in N_{\kappa_0}(e'_j,1)$ where $e'_j \in N_{\kappa_0}(e,1) \cap p^{-1}(b')$. But this is a contradiction of equation 3.1. □

Now, we can correct the definition of the digital pseudocovering map.

Definition 3.4. Let (E,κ_0) and (B,κ_1) be digital spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : (E,\kappa_0) \longrightarrow (B,\kappa_1)$ be a surjection. Suppose that for any $b \in B$ the map p has the following properties:

(1) for some index set M, $\bigsqcup_{i\in M} N_{\kappa_0}(e_i,1) \subseteq p^{-1}(N_{\kappa_1}(b,1))$ with $e_i \in p^{-1}(b)$,
(2) if $i,j \in M$ and $i \neq j$, then $N_{\kappa_0}(e_i,1) \cap N_{\kappa_0}(e_j,1)$ is an empty set; and
(3) the restriction map $p|_{N_{\kappa_0}(e_i,1)} : N_{\kappa_0}(e_i,1) \longrightarrow p\big(N_{\kappa_0}(e_i,1)\big)$ is a (k_0,k_1)-isomorphism for all $i \in M$.

Then the map p is called a (k_0,k_1)-pseudocovering map.

Han [**?**] proved that his digital pseudocovering maps have the unique lifting property (but not by this name!!!). Although we have changed the definition, but his proof of the following is valid.

Proposition 3.5. *Let (E,κ_0) and (B,κ_1) be digital connected spaces in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, let $p : (E,\kappa_0) \longrightarrow (B,\kappa_1)$ be a digital pseudocovering map and let $f : (X,\lambda) \longrightarrow (B,\kappa_1)$ be a digital continuous map such that (X,λ) is a digital connected space and $f(x_0) = b_0$. Given $e_0 \in p^{-1}(b_0)$, there is at most one digitally continuous map $\tilde{f} : (X,\lambda) \longrightarrow (E,\kappa_0)$ with $p \circ \tilde{f} = f$ and $\tilde{f}(x_0) = e_0$.*

Proof. See [**?**, Theorem 4.9]. □

Since digital intervals are 2-connected, we have the following corollary.

Corollary 3.6. *Digital pseudocovering maps have the uniqueness of digital path lifts property.*

Remark 3.7. It is notable that the *uniqueness of digital path lifts property* in [**?**] was introduced as the unique pseudolifting property. But in order to coordinate with common texts in algebraic topology, we have used the *uniqueness of digital path lifts property*. Also, Han has used the *unique lifting property* for the existence of liftings of paths. It is emphasized that the digital path lifting property means existence of liftings of a path and the *unique path lifting property* means the *digital path lifting property* and the *uniqueness of digital path lifts*.

With this corrected definition, a digital pseudocovering map may not have the *unique path lifting property*. For, if $f : \mathbb{Z}^+ \longrightarrow SC_8^{2,4} := (c_i)_{i=0}^3$, is defined by $f(i) = c_{i\ mod\ 4}$ (as in Figure 1) and $\alpha : [0,1]_{\mathbb{Z}} \longrightarrow SC_8^{2,4}$ is defined by $\alpha(0) = c_0$ and $\alpha(1) = c_3$, then there is no lifting for α beginning at 0.

Corollary 3.8. *Digital pseudocovering maps do not, in general, have the unique path lifting property.*

References

[1] L. Boxer, Digitally continuous functions, *Pattern Recognition Letters*, 15 (1994), 833–839.

[2] L. Boxer, A classical construction for the digital fundamental group, *Journal of Mathematical Imaging and Vision*, 10 (1999), 51–62.

[3] L. Boxer, Digital products, wedges, and covering spaces, *Journal of Mathematical Imaging and Vision*, 25 (2006), 159–171.

[4] J. Brazas, Semicoverings: a generalization of covering space theory, *Homology Homotopy and Applications*, 14 (2012) 33–63.

[5] J. Brazas, Generalized covering space theories, *Theory and Application of Categories*, 30 (2015) 1132-1162.

[6] H. Fischer, A. Zastrow, Generalized universal covering spaces and the shape group, *Fundamenta Mathematicae*, 197 (2007) 167–196.

[7] S.E. Han, Non-product property of the digital fundamental group, *Information Sciences*, 171 (2005), 73–91.

[8] S.E. Han, Digital coverings and their applications, *Journal of Applied Mathematics and Computing*, 18 (2005), 487–495.

[9] S.E. Han, Unique pseudolifting property in digital topology, *Filomat*, 26:4 (2012), 739–746.

[10] G.T. Herman, Oriented surfaces in digital spaces, *CVGIP: Graphical Models and Image Processing*, 55 (1993), 381–396.

[11] E. Khalimsky, Motion, deformation, and homotopy in finite spaces, *Proceedings IEEE International Conference on Systems, Man, and Cybernetics*, (1987), 227–234.

[12] A. Pakdaman and M. Zakki, Equivalent conditions for digital covering maps, *arXiv:1805.03041v4*, to appear in Filomat.

[13] A. Rosenfeld, Digital topology, *American Mathematical Monthly*, 86 (1979), 76–87.

[14] A. Rosenfeld, Continuous functions on digital pictures, *Pattern Recognition Letters*, 4 (1986), 177–184.