Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **11**(1)(2022), 191-202 (Research paper)

Fixed point results for Geraghty contractive type operators in uniform spaces

Joy Chinyere Umudu [1] Johnson Olajire Olaleru [2] and Adesanmi Alao Mogbademu [2]
[1] Department of Mathematics, Faculty of Natural Sciences, University of Jos, Nigeria
[2] Department of Mathematics, Faculty of Science, University of Lagos, Nigeria

Abstract. In this paper, we consider a generalization of α-ϕ-Geraghty contractive type operators and investigate the conditions for the existence and uniqueness of fixed point in a S-complete Hausdorff uniform space equipped with a E-distance. Our results extend, improve and generalize some related works in the literature. We illustrate the validity of the results with examples.

Keywords: α-ϕ-Geraghty contractive type operator, generalized α-ϕ-Geraghty contractive type operator, fixed point, uniform spaces.

2020 Mathematics subject classification: 47H10; Secondary 54H25.

1. Introduction

In 1922, Banach [4] presented a very famous fixed point result, namely Banach contraction principle. This result plays a fundamental role and brought a great revolution and applications in the field of fixed point theory. Afterwards, several generalizations and improvements of this result have been obtained among which include quasi-contraction operators [1, 7, 13]. Interesting results have also been investigated in other spaces of study aside metric spaces [14, 17, 18, 19, 20, 21, 22, 26, 27]. Uniform

[1] Corresponding author: umuduj@unijos.edu.ng
Received: 10 October 2019
Revised: 30 September 2020
Accepted: 06 October 2020

space is known to generalize the metric and pseudometric spaces. Weil [28] was the first to characterise uniform spaces in terms of a family of pseudometrics and Bourbaki [5] provided the definition of a uniform structure in terms of entourages. Aamri and El Moutawakil [1] gave some results on common fixed point for some contractive and expansive operators in uniform spaces and provided the definition of A-distance and E-distance. Olisama et al. [15] introduced the concept of J_{AV}-distance (an analogue of b-metric), ϕ_p-proximal contraction, and ϕ_p-proximal cyclic contraction for non-self operators in Hausdorff uniform spaces and investigated the existence and uniqueness of best proximity points for these modified contractive operators.

In another development, Geraghty [8] introduced the generalized contraction self-operator in metric spaces using comparison functions. In 2013, Cho et al. [6] defined the concept of α-Geraghty contraction type operators in the setting of a metric space while Karapinar [9, 10] introduced the notion of α-ϕ-Geraghty contractive operators and proved the existence and uniqueness of a fixed point of such operators in the context of a complete metric space. For other results on Geraghty contractions see [3, 6, 8, 9, 10, 11, 12, 16, 25, 26, 27, 29].

Motivated by the results above, we extend the concept of α-ϕ-Geraghty contractive type operator in metric spaces to Hausdorff uniform spaces and obtain the unique fixed point for the contractive type operators using a E-distance function.

The following definitions are fundamental to our work.

Definition 1.1. [5] A uniform space (X, Γ) is a nonempty set X equipped with a uniform structure which is a family Γ of subsets of Cartesian product $X \times X$ which satisfy the following conditions:

(i) If $U \in \Gamma$, then U contains the diagonal $\Delta = \{(x, x) : x \in X\}$.
(ii) If $U \in \Gamma$, then $U^{-1} = \{(y, x) : (x, y) \in U\}$ is also in Γ.
(iii) If $U, V \in \Gamma$, then $U \cap V \in \Gamma$.
(iv) If $U \in \Gamma$ and $V \subseteq X \times X$, which contains U, then $V \in \Gamma$.
(v) If $U \in \Gamma$, then there exists $V \in \Gamma$ such that whenever (x, y) and (y, z) are in V, then (x, z) is in U.

Γ is called the uniform structure or uniformity of U and its elements are called entourages, neighbourhoods, surroundings, or vicinities.

Definition 1.2. [1] Let (X, Γ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be a

(a) A-distance if, for any $V \in \Gamma$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.
(b) E-distance if p is a A-distance and $p(x, z) \leq p(x, y) + p(y, z)$, $\forall x, y, z \in X$.

Definition 1.3. [5] Let (X, Γ) be a uniform space and p a A-distance on X.

(a) If $V \in \Gamma, (x, y) \in V$, and $(y, x) \in V$, x and y are said to be V-close. A sequence $\{x_n\}$ is a Cauchy sequence for Γ if for any $V \in \Gamma$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. The sequence $\{x_n\} \in X$ is a p-Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $p(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

(b) X is S-complete if for any p-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$.

(c) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.

(d) X is p-bounded if $\delta_p(X) = \sup\{p(x, y) : x, y \in X\} < \infty$.

To guarantee the uniqueness of the limit of the Cauchy sequence for Γ, the uniform space (X, Γ) needs to be Hausdorff.

Definition 1.4. [5] A uniform space (X, Γ) is said to be Hausdorff if and only if the intersection of all the $V \in \Gamma$ reduces to the diagonal Δ of X, $\Delta = \{(x, x), x \in X\}$. In other words, $(x, y) \in V$ for all $V \in \Gamma$ implies $x = y$.

Popescu [16] introduced the concepts of α-orbital admissible and triangular α-orbital admissible operators as improvements of α -admissible operator defined in [24] and triangular α-admissible operator defined in [11] respectively.

Definition 1.5. [16] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then T is said to be α-orbital admissible if $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.

Definition 1.6. [16] Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then T is said to be triangular α-orbital admissible if T is α-orbital admissible and $\alpha(x, y) \geq 1$, $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

Lemma 1.7. [16] *Let $T : X \to X$ be a triangular α - orbital admissible operator. Assume that there exists $x_1 \in X$ such that $\alpha(x, Tx) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Let F be the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition $\lim_{n\to\infty} \beta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Let Φ denote the class of the functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(i) ϕ is non decreasing;

(ii) ϕ is continuous;
(iii) $\phi(t) = 0 \iff t = 0$.

2. Main result

Now, we introduce the following concepts in a uniform space.

Definition 2.1. Let (X, Γ) be a S-complete Hausdorff uniform space such that p is a E-distance on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. A self operator $T : X \to X$ is called a generalized α-ϕ-Geraghty contractive type operator if there exists $\beta \in F$ such that for all $x, y \in X$,

$$\alpha(x, y)\phi(p(Tx, Ty)) \leq \beta(\phi(M_T(x, y)))\phi(M_T(x, y)), \tag{2.1}$$

where $M_T(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\right\}$.

If $M_T(x, y) = p(x, y)$, inequality (2.1) reduces to the following.

Definition 2.2. Let (X, Γ) be a S-complete Hausdorff uniform space such that p is a E-distance on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. A self operator $T : X \to X$ is called an α-ϕ-Geraghty contractive type operator if there exists $\beta \in F$ such that for all $x, y \in X$,

$$\alpha(x, y)\phi(p(Tx, Ty)) \leq \beta(\phi(p(x, y)))\phi(p(x, y)), \tag{2.2}$$

where $\phi \in \Phi$.

Example 2.3. Let $X = [0, +\infty)$ and $T : X \to X$ an operator such that p is a E-distance. Suppose T is defined by

$$T(x) = \begin{cases} \frac{1}{2}x, & \text{if} \quad x \in [0, 2] \\ 1, & \quad \textit{otherwise}. \end{cases}$$

Consider the function p and ϕ defined: $p(x, y) = y$, $\phi(t) = \frac{t}{2}$. We see that p is a E-distance, X is S-complete and T is p-continuous. Taking $\beta : [0, \infty) \to [0, 1)$ defined by $\beta = \frac{1}{1+t}$, T is a generalized α-ϕ-Geraghty contractive type operator.

Theorem 2.4. *Let (X, Γ) be a S-complete Hausdorff uniform space such that p is a E-distance on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Suppose the following conditions are satisfied:*

(i) *T is a generalized α-ϕ-Geraghty contractive type operator;*
(ii) *T is triangular α-orbital admissible operator;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is continuous.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n \geq 1$. If $x_{n_0} = x_{n_{0+1}}$ for some $1 \leq i \leq n-1$, then obviously T has a fixed point. Thus, we suppose that $x_n \neq x_{n+1}$ for all $n \geq 1$. By Lemma 1.7, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2.3}$$

for all $n \geq 1$. By (2.1) we get,

$$\begin{aligned} \phi(p(x_{n+1}, x_{n+2})) &= \phi(p(Tx_n, Tx_{n+1})) \\ &\leq \alpha(x_n, x_{n+1})\phi(p(Tx_n, Tx_{n+1})) \\ &\leq \beta(\phi(M_T(x_n, x_{n+1})))\phi(M_T(x_n, x_{n+1})) \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} M_T(x_n, x_{n+1}) &= \max\{p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \\ &\quad \tfrac{p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)}{2}\} \\ &= \max\left\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \tfrac{p(x_n, x_{n+2}}{2}\right\} \\ &\leq \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \\ &\quad \tfrac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}}{2}\} \\ &= \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Note that $M_T(x_n, x_{n+1}) = p(x_{n+1}, x_{n+2})$ is impossible due to the definition of β. Indeed,

$$\begin{aligned} \phi(p(x_{n+1}, x_{n+2})) &\leq \beta(\phi(M_T(x_n, x_{n+1})))\phi(M_T(x_n, x_{n+1})) \\ &\leq \beta(\phi(M_T(x_{n+1}, x_{n+2})))\phi(M_T(x_{n+1}, x_{n+2})) \\ &< \phi(p(x_{n+1}, x_{n+2})) \end{aligned}$$

is a contradiction. Therefore, $\phi(p(x_{n+1}, x_{n+2})) \leq \phi(p(x_n, x_{n+1}))$ for all $n \in \mathbb{N}$. Thus, the sequence $\{p(x_n, x_{n+1})\}$ is non-negative and non increasing. Consequently, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r.$$

We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Then we have

$$\frac{\phi(p(x_{n+1}, x_{n+2}))}{\phi(p(x_n, x_{n+1}))} \leq \beta(\phi(M_T(x_n, x_{n+1}))) < 1.$$

Therefore,

$$\lim_{n\to\infty} \beta(\phi(M_T(x_n, x_{n+1}))) = 1.$$

Since $\beta \in F$,

$$\lim_{n\to\infty} \phi(M_T(x_n, x_{n+1})) = 0 \tag{2.5}$$

and

$$r=\lim_{n\to\infty}p(x_n,x_{n+1})=0$$

which is a contradiction.
Next, to show that $\{x_n\}$ is a p-Cauchy sequence. Suppose, to the contrary that $\{x_n\}$ is not p-Cauchy. Then there exists $\epsilon>0$ such that for all $n\geq 1$, there exist $m>n$ with $p(x_n,x_m)\geq\epsilon$. Therefore,

$$p(x_n,x_m) \leq p(x_n,x_{n+1)})+p(x_{n+1},x_{m+1})+p(x_{m+1},x_m). \quad (2.6)$$

Combining (2.3) and (2.6) with the properties of ϕ, we get

$$\begin{aligned}\phi(p(x_n,x_m)) &\leq \phi(p(x_n,x_{n+1})+p(Tx_n,Tx_m)+p(x_{m+1},x_m))\\ &\leq \phi(p(x_n,x_{n+1}))+\phi(p(Tx_n,Tx_m))+\phi(p(x_{m+1},x_m))\\ &\leq \phi(p(x_n,x_{n+1}))+\phi(p(x_{m+1},x_m))+\\ &\quad \beta(\phi(M_T(x_n,x_m)))\phi(M_T(x_n,x_m)). \quad (2.7)\end{aligned}$$

From (2.7), we deduce that

$$\begin{aligned}\lim_{m,n\to\infty}\phi(p(x_n,x_m)) &\leq \lim_{m,n\to\infty}\beta(\phi(M_T(x_n,x_m)))\lim_{m,n\to\infty}\phi(M_T(x_n,x_m))\\ &\leq \lim_{m,n\to\infty}\beta(\phi(M_T(x_n,x_m)))\lim_{m,n\to\infty}\phi(p(x_n,x_m)).\end{aligned}$$

This implies,

$$\lim_{m,n\to\infty}\beta(\phi(M_T(x_n,x_m)))=1.$$

Consequently, $\lim_{m,n\to\infty}M_T(x_n,x_m)=0$, a contradiction. Therefore, $\{x_n\}$ is a p-Cauchy sequence. Recalling S-completeness of X, we conclude that there exists $x^*=\lim_{n\to\infty}x_n\in X$. By continuity of T, $\lim_{n\to\infty}Tx_n=Tx^*$ and so $x^*=Tx^*$, which means x^* is a fixed point of T.

The continuity of the operator T can be replaced with an appropriate condition.

Theorem 2.5. *Let (X,Γ) be a S-complete Hausdorff uniform space such that p is a E-distance on X and let $\alpha: X\times X\to\mathbb{R}^+$ be a function. Suppose the following conditions are satisfied:*

(i) *T is a generalized α-ϕ-Geraghty contractive type operator;*
(ii) *T is triangular α-orbital admissible operator;*
(iii) *there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq 1$;*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $n\to+\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x)\geq 1$ for all k.*

Then T has a fixed point $x^*\in X$ and $\{T^nx_1\}$ converges to x^*.

Proof. Following the proof of Theorem 2.4, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ converges to some $x^* \in X$. By condition of (iv), we deduce that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_n) \geq 1$ for all k. Applying (2.1), for all k, we get that

$$\begin{aligned} \alpha(x_{n_k}, x^*)\phi(p(x_{n_{k+1}}, Tx^*)) &= \alpha(x_{n_k}, x^*)\phi(p(Tx_{n_k}, Tx^*)) \\ &\leq \beta(\phi(M_T(x_{n_k}, x^*))\phi(M_T(x_{n_k}, x^*)) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} M_T(x_{n_k}, x^*) &= \max\{p(x_{n_k}, x^*), p(x_{n_k}, Tx_{n_k}), p(x^*, Tx^*), \\ &\quad \tfrac{p(x_{n_k}, Tx^*)+p(x^*, Tx_{n_k})}{2}\} \\ &= \max\{p(x_{n_k}, x^*), p(x_{n_k}, x_{n_{k+1}}), p(x^*, Tx^*), \\ &\quad \tfrac{p(x_{n_k}, Tx^*)+p(x^*, x_{n_{k+1}})}{2}\}. \end{aligned}$$

Thus,

$$\lim_{k\to\infty} \phi(M_T(x_{n_k}, x^*) = \phi(p(x^*, Tx^*)).$$

From (2.8), we have

$$\alpha(x_{n_k}, x^*)\frac{\phi(p(x_{n_{k+1}}, Tx^*))}{\phi(M_T(x_{n_k}, x^*))} \leq \beta(\phi(M_T(x_{n_k}, x^*))) < 1.$$

As $k \to \infty$, we conclude that $\lim\limits_{k\to\infty} \beta(\phi(M_T(x_{n_k}, x^*))) = 1$, and so $\lim\limits_{k\to\infty} \phi(M_T(x_{n_k}, x^*)) = \phi(p(x^*, Tx^*)) = 0$. This is a contradiction. Therefore, $Tx^* = x^*$.

For the uniqueness of a fixed point of a generalized α-ϕ-Geraghty contractive type operator, we replace condition (iii) with the following hypothesis called the (H) property.

(H) For all $x \neq y \in Fix(T)$, there exists $w \in X$ such that $\alpha(x, w) \geq 1$, $\alpha(y, w) \geq 1$ and $\alpha(w, Tw) \geq 1$. $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.6. *Adding condition (H) to the hypothesis of Theorem 2.4 (respectively Theorem 2.5), we obtain that x^* is the unique fixed point of T.*

Proof. Due to Theorem 2.4 (respectively Theorem 2.5), we have a fixed point, say $x^* \in X$. Assume by contradiction that x^* and y^* are two fixed points of T such that $x^* \neq y^*$. Then by (H), there exists $w \in X$ such that $\alpha(x, w) \geq 1, \alpha(y, w) \geq 1$ and $\alpha(w, Tw) \geq 1$. Since T is a triangular α-orbital admissible operator we get that $\alpha(x^*, T^n w) \geq 1$

and $\alpha(y^*, T^n w) \geq 1$ for all $n \geq 1$. We then have,

$$p(x^*, T^{n+1}w) \leq \alpha(x^*, T^n w)p(Tx^*, T^{n+1}w) \leq \beta(M_T(x^*, T^n w))M_T(x^*, T^n w)$$

for all $n \geq 1$ where,

$$\begin{aligned} M_T(x^*, T^n w) &= \max\{p(x^*, T^n w), p(x^*, Tx^*), p(T^n w, T^{n+1}w), \\ &\quad \tfrac{p(x^*,T^{n+1}w)+p(Tx^*,T^n w)}{2}\} \\ &= \max\{p(x^*, T^n w), p(T^n w, T^{n+1}w), \\ &\quad \tfrac{p(x^*,T^{n+1}w)+p(x^*,T^n w)}{2}\}. \end{aligned}$$

By Theorem 2.4 and 2.5, we deduce that the sequence $T^n w$ converges to a fixed point z^* of T. Let $n \to \infty$ in the above equation, we get

$$\lim_{n\to\infty} M_T(x^*, T^n w) = p(x^*, z^*).$$

Suppose $x^* \neq z^*$ then $\frac{p(x^*,T^{n+1}w)}{M_T(x^*,T^n w)} \leq \beta(M_T(x^*, T^n w))$ and as $n \to \infty$, we have $\lim_{n\to\infty} \beta(M_T(x^*, T^n w)) = 1$.
Thus $\lim_{n\to\infty} M_T(x^*, T^n w) = 0$ and $p(x^*, z^*) = 0$, which is a contradiction. Therefore $x^* = z^*$. Similarly, we have $T^n w = y^*$ which implies $x^* = y^*$, a contradiction. Hence, the fixed point is unique.

Corollary 2.7. *Let (X, Γ) be a S-complete Hausdorff uniform space such that p is a E-distance on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Suppose the following conditions are satisfied:*

(i) *T is an α-ϕ-Geraghty contractive type operator;*
(ii) *T is triangular α-orbital admissible operator;*
(iii) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*
(iv) *T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x_*.

Proof. It follows from Theorem 2.4 and Theorem 2.5 respectively if $\max\{M_T(x, y)\} = p(x, y)$.

Corollary 2.8. *Adding condition (H) to the hypotheses of Corollary 2.7, we obtain that x^* is the unique fixed point of T.*

Corollary 2.9. *Let (X, Γ) be a S-complete Hausdorff uniform space such that p is a E-distance on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Suppose the following conditions are satisfied:*

(i) *T is a generalized α-Geraghty contractive type operator;*
(ii) *T is triangular α-orbital admissible operator;*

(iii) *there exists* $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$;
(iv) T *is continuous or if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to +\infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_1\}$ *converges to* x_*.

Proof. It follows from Theorem 2.4 and Theorem 2.5 if $\phi(t) = t$.

Corollary 2.10. *Let* (X, Γ) *be a* S*-complete Hausdorff uniform space such that* p *is a* E*-distance on* X *and let* $\alpha : X \times X \to \mathbb{R}^+$ *be a function. Suppose the following conditions are satisfied:*

(i) T *is an* α*-Geraghty contractive type operator;*
(ii) T *is triangular* α*-orbital admissible operator;*
(iii) *there exists* $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$;
(iv) T *is continuous or if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to +\infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then T *has a fixed point* $x^* \in X$ *and* $\{T^n x_1\}$ *converges to* x_*.

Proof. If $\phi(t) = t$ and $\max\{M_T(x, y)\} = p(x, y)$ in Theorem 2.4 and Theorem 2.5 then the proof follows.

We give an example to illustrate Theorem 2.4.

Example 2.11. Let (X, Γ) be a uniform space such that p is a E-distance. Consider $X = [0, \infty^+)$ and a E-distance p defined by

$$p(x, y) = \begin{cases} y, & \text{if} \quad x \leq y, \\ \\ 1, & otherwise. \end{cases}$$

Suppose $\beta(t) = \frac{1}{1+t}$, $\phi(t) = \frac{t}{2}$ and the operator $T : X \to X$ is defined by $T(x) = \frac{1}{3}x$, $\forall x \in X$. We also define a function $\alpha : X \times X \to \mathbb{R}+$ in the following way

$$\alpha(x, y) = \begin{cases} 1, & \text{if} \quad (0 \leq x, y \leq 3), \\ \\ 0, & otherwise. \end{cases}$$

Condition (iii) of Theorem 2.4 is satisfied with $x_1 = 1$. Condition (iv) of Theorem 2.4 is satisfied with $x_n = T^n x_1 = \frac{1}{3^n}$. Obviously, condition (ii) is satisfied. Let x, y be such that $\alpha(x, y) \geq 1$. Then, $x, y \in [0, 3]$ and $Tx, Ty \in [0, 3]$. Moreover, $\alpha(y, Ty) = \alpha(x, Tx) = 1$ and $\alpha(Tx, T^2x) = 1$. Thus, T is triangular α-orbital admissible and (ii) is satisfied. Finally, to prove that (i) is satisfied. If $0 \leq x, y \leq 3$, then $\alpha(x, y) = 1$ and

$$\begin{aligned}
&\beta(\phi(M_T(x,y)))\phi(M_T(x,y)) - \alpha(x,y)\phi(p(Tx,Ty)) \\
&\quad = \beta(\phi(M_T(x,y)))\phi(M_T(x,y)) - \phi(p(Tx,Ty)) \\
&\quad = \left(\frac{1}{1+\frac{y}{2}}\right)\left(\frac{y}{2}\right) - \left(\frac{1}{3}\right)\left(\frac{y}{2}\right) \\
&\quad = \frac{y(4-y)}{6(2+y)} \\
&\quad \geq 0.
\end{aligned}$$

Therefore, $\alpha(x,y)\phi(p(Tx,Ty)) \leq \beta(\phi(p(x,y)))\phi(p(x,y))$ for $0 \leq x, y \leq 3$. If $x, y > 3$, $x \in [0,3]$, $y > 3$ or vice versa then, obviously, we have

$$\alpha(x,y)\phi(p(Tx,Ty)) \leq \beta(\phi(p(x,y)))\phi(p(x,y)),$$

since $\alpha(x,y) = 0$. Consequently, all assumptions of Theorems 2.4 are satisfied, and T has a unique fixed point $x^* = 0$.

We also notice that Theorem 1.1 in [8] is not satisfied. In fact, for $x = 0, y = 3$ and $d(x,y) = |x-y|$, we have

$$d(T0, T3) = 1 > \frac{3}{4} = \beta(d(0,3))d(0,3).$$

Conclusion

The fixed point results obtained in a S-complete Hausdorff uniform space equipped with a E-distance for generalized α-ϕ-Geraghty contractive type operators, apart from being a generalization and extension of some related works in the literature, paves way for more investigations to unravel conditions for the existence and uniqueness of fixed points in other abstract spaces.

Acknowledgement

The authors are grateful to the anonymous referees for their constructive and valuable remarks, comments and suggestions.

References

[1] M. Aamri and D. El Moutawakil, Common fixed point theorems for E-contractive or E-expansive maps in uniform spaces, *Acta Mathematica Academiae Paedagogicae Nyi Regyhaziensis (New Series)* **20** *(2004), 83–89.*

[2] A. Amini-Harandi and H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations, *Nonlinear Analysis* **72** *(2010), 2238–2242.*

[3] A. H. Ansari, A. Razani and N. Hussain, Fixed and coincidence points for hybrid rational Geraghty contractive mappings in ordered b-metric spaces, *International Journal of Nonlinear Analysis* **8(1)** *(2017), 315–329.*

[4] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fundamenta Mathematicae* **3** *(1922), 133–181.*

[5] N. Bourbaki, Éléments de mathématique Fasc. II Livre III: Topologie générale, Chapitre 1: Structures Topologiques, Chapitre 2: Structures uniformes. Quatriéme édition *Actualités Scientifiques et Industrielles, No. 1142, Hermann, Paris, (1965).*

[6] S. H. Cho, J. S. Bae and E. Karapínar, Fixed point theorems for α-Geraghty contraction type mappings in metric spaces, *Fixed Point Theory and Applications* **2013** *(2013), Article ID 329.*

[7] L. B. Ciric, A generalization of Banach's contraction principle, *Proceedings of the American Mathematical Society* **45** *(1974), 267–273.*

[8] M. Geraghty, On contractive mappings, *Proceedings of the American Mathematical Society* **40** *(1973), 604–608.*

[9] E. Karapínar, α-ψ-Geraghty contraction type mappings and some related fixed point results, *Filomat* **28** *(2014), 37–48.*

[10] E. Karapínar, A Discussion on α-ψ-Geraghty contraction type mappings, *Filomat* **28** *(2014), 761–766.*

[11] E. Karapínar, P. Kumam and P. Salimi, On α-ψ-Meir-Keeler contractive mappings, *Fixed Point Theory and Applications* **2013** *(2013), Article ID 94.*

[12] E. Karapínar and B. Samet, Generalized α-ϕ-contractive type mappings and related fixed point theorems with applications, *Abstract and Applied Analysis* **2012** *(2012), Article ID 793486.*

[13] J. O. Olaleru, A comparison of Picard and Mann iteration for quasi-contraction mappings, *Fixed point theory and Applications* **8** *(2007), 87–95.*

[14] J. O. Olaleru, Common fixed points of three self-mappings in cone metric spaces, *Applied Mathematics E-Note* **11** *(2009), 41–49.*

[15] V. O. Olisama, J. O. Olaleru and H. Akewe, Best proximity point results for some contractive mappings in uniform spaces, *Abstract and Applied Analysis* **2017** *(2017), Article ID 6173468, 8 pages.*

[16] O. Popescu, Some new fixed point theorems for α-Geraghty contraction type mappings in metric spaces, *Fixed Point Theory and Applications* **2014** *(2014), Article ID 190.*

[17] A. Razani, A contraction theorem in fuzzy metric spaces, *Fixed Point Theory and Applications* **3** *(2005), 257–265.*

[18] A. Razani, A fixed point theorem in Menger probabilistic metric spaces, *New Zealand Journal of Mathematics* **35** *(2006), 109–114.*

[19] A. Razani, Existence of fixed point for the nonexpansive mapping of intuitionistic fuzzy metric spaces, *Chaos, Solitons and Fractals* **30** *(2006), 367–373.*

[20] A. Razani, Results in fixed point theory, *Andisheh Zarin publisher, Qazvin, August (2010).*

[21] A. Razani and R. Moradi, Fixed point theory in modular space, *Saieh Ghostar publisher, Qazvin, April (2006).*

[22] A. Razani, Fixed points for total asymptotically nonexpansive mappings in a new version of bead space, *International Journal of Industrial Mathematics* **6(4)** *(2014), 329–332.*

[23] J. Rodríguez-Montes and J. A. Charris, Fixed points for W-contractive or W-expansive mappings in uniform spaces: toward a unified approach, *Southwest Journal of Pure and Applied Mathematics* **1** *(electronic, 2001), 93–101.*

[24] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-contractive mappings, *Nonlinear Analysis* **75** *(2012), 2154–2165.*

[25] R. J. Shahkoohi and A. Razani, Some fixed point theorems for rational Geraghty contractive mappings in ordered b-metric spaces, *Journal of Inequalities and Applications* **373** *(2014), 23 pages.*

[26] J. C. Umudu, J. O. Olaleru and A. A. Mogbademu, Fixed point results for Geraghty quasi-contraction type mappings in dislocated quasi-metric spaces, *Fixed Point Theory and Applications* **2020** *Article ID 16 (2020), doi.org/10.1186/s13663-020-00683-z.*

[27] J. C. Umudu, J. O. Olaleru and A. A. Mogbademu, Best proximity point results for Geraghty p-proximal cyclic quasi-contraction in uniform spaces, *Divulgaciones Matematicas* **21(1-2)** *(2020), 21–31.*

[28] A. Weil, Surles Espaces a Structure Uniforme et sur la Topologie Generale, *Actualit´es Scientifiques et Industrielles, Hermann, Paris, France,* **551** *(1937).*

[29] F. Zabihi and A. Razani, Fixed point theorems for hybrid rational Geraghty contractive mappings in ordered b-metric spaces, *Journal of Applied Mathematics* **2014** *Article ID 929821 (2014), 9 pages.*