

Some geometrical properties of Berger spheres

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ABSTRACT. Our aim in this paper is to investigate some geometrical properties of Berger spheres i.e., homogeneous Ricci solitons and harmonicity properties of invariant vector fields. We determine all vector fields, which are critical points for the energy functional restricted to vector fields of the same length. We also see that do not exist any vector fields defining harmonic map, and the energy of critical points is explicitly calculated.

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1. INTRODUCTION

In Riemannian geometry, a Berger sphere is a standard 3-sphere with Riemannian metric from a one-parameter family, which can be obtained from the standard metric by shrinking along fibers of a Hopf fibration. These spaces, found by M. Berger [6] in his classification of all simply connected normal homogeneous Riemannian manifolds of positive sectional curvature, have not constant curvature, and their metrics are obtained from the round metric on S^3 by deforming it along the fibers of the Hopf fibration $S^3 \rightarrow S^2$ by ϵ . In [11], the homogeneous Riemannian structures on the 3-dimensional Berger spheres, their corresponding

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reductive decompositions, and the associated groups of isometries have been obtained. These spaces are of great interest in Riemannian geometry and provide excellent examples; for instance, they served as counterexamples to a conjecture of Klingenberg about closed geodesics and to conjectures on the first eigenvalue of the Laplacian on spheres ([5]). The Berger spheres S_ϵ^3 are homogeneous Riemannian spaces diffeomorphic to the 3-dimensional sphere. The metrics so constructed are known as Berger metrics; they consist of a 1-parameter variation g_ϵ for $\epsilon > 0$. We will also consider the Lorentzian Berger metrics, i.e., when $\epsilon < 0$. Up to our knowledge, no geometrical properties such as homogeneous Ricci solitons have been obtained yet for Berger spheres.

A natural generalization of an Einstein manifold is Ricci soliton, i.e., a pseudo-Riemannian metric g on a smooth manifold M , such that the equation

$$\mathcal{L}_X g = \varsigma g - \varrho, \quad (1.1)$$

holds for some $\varsigma \in R$ and some smooth vector field X on M , where ϱ denotes the Ricci tensor of (M, g) and \mathcal{L}_X is the usual Lie derivative. According to whether $\varsigma > 0$, $\varsigma = 0$, or $\varsigma < 0$, a Ricci soliton g is said to be shrinking, steady, or expanding, respectively. A homogeneous Ricci soliton on a homogeneous space $M = G/H$ is a G -invariant metric g for which the equation (1.1) holds, and an invariant Ricci soliton is a homogeneous space, such that equation (1.1) is satisfied by an invariant vector field. Indeed, the study of Ricci solitons homogeneous spaces is an interesting area of research in pseudo-Riemannian geometry. For example, algebraic solitons and the Alekseevskii Conjecture properties [14], conformally flat Lorentzian gradient Ricci solitons [7], algebraic Ricci solitons [4]. In case (G, g) be a simply-connected completely solvable Lie group equipped with a left-invariant metric, and $(\mathfrak{g}, \langle, \rangle)$ be the corresponding metric Lie algebra, then (G, g) is a Ricci soliton if and only if $(\mathfrak{g}, \langle, \rangle)$ is an algebraic Ricci soliton [15].

On the other hand, investigating critical points of the energy associated with vector fields is an interesting purpose under different points of view. As an example, by the Reeb vector field ξ of a contact metric manifold, somebody can see how the criticality of such a vector field is related to the geometry of the manifold ([16],[17]). Recently, it has been [12] proved that critical points of $E : \mathfrak{X}(M) \rightarrow R$, that is, the energy functional restricted to vector fields, are again parallel vector fields. Moreover, the same paper, also has been determined the tension field associated with a unit vector field V , and investigated the problem of determining when V defines a harmonic map.

A Riemannian manifold admitting a parallel vector field is locally reducible. The same is true for a pseudo-Riemannian manifold admitting

either space-like or time-like parallel vector field. This leads us to consider different situations, where some interesting types of non-parallel vector fields can be characterized in terms of harmonicity properties. We may refer to the recent references [2] and [3] for an overview of harmonic vector fields.

This paper is organized in the following way. We devote Section 2 to recall the definitions and state the results we will need in the sequel. In Section 3, we investigate the required conditions for Berger spheres Ricci solitons. Harmonicity properties of vector fields on Berger spheres will be determined in Sections 4. Finally, the energy of all these vector fields is explicitly calculated in Section 5.

2. PRELIMINARIES

Let $M = G/H$ be a homogeneous manifold (with H connected), \mathfrak{g} the Lie algebra of G , and \mathfrak{h} the isotropy subalgebra. Consider $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ the factor space, which identifies with a subspace of \mathfrak{g} complementary to \mathfrak{h} . The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropy representation

$$\psi : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{m}), \quad \psi(x)(y) = [x, y]_{\mathfrak{m}},$$

for all $x \in \mathfrak{g}, y \in \mathfrak{m}$. Suppose that $\{e_1, \dots, e_r, u_1, \dots, u_n\}$ be a basis of \mathfrak{g} , where $\{e_j\}$ and $\{u_i\}$ are bases of \mathfrak{h} and \mathfrak{m} respectively, then with respect to $\{u_i\}$, H_j would be the isotropy representation for e_j . With regard to curvature properties, an invariant nondegenerate symmetric bilinear form g on \mathfrak{m} uniquely defines its invariant linear Levi-Civita connection as the corresponding homomorphism of \mathfrak{h} -modules $\Lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$. In other word

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \quad (2.1)$$

for all $x, y \in \mathfrak{g}$, where $v : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{m}$ is the \mathfrak{h} -invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}),$$

for all $x, y, z \in \mathfrak{g}$. Then the curvature tensor can be determined by

$$R : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{gl}(\mathfrak{m}), \quad R(x, y) = [\Lambda(x), \Lambda(y)] - \Lambda([x, y]), \quad (2.2)$$

and with respect to u_i , the Ricci tensor ρ of g is given by

$$\rho(u_i, u_j) = \sum_{k=1}^4 g(R(u_k, u_i)u_j, u_k), \quad i, j = 1, \dots, 4. \quad (2.3)$$

Let (M, g) be a compact Riemannian manifold and g_s be the Sasaki metric on the tangent bundle TM , then the energy of a smooth vector field $V : (M, g) \longrightarrow (TM, g^s)$ on M is;

$$E(V) = \frac{n}{2} \text{vol}(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2 dv, \quad (2.4)$$

(assuming M compact; in the non-compact case, one works over relatively compact domains see [8]). If $V : (M, g) \rightarrow (TM, g^s)$ be a critical point for the energy functional, then V is said to define a harmonic map. The Euler-Lagrange equations characterize vector fields V defining harmonic maps as the ones whose tension field $\theta(V) = \text{tr}(\nabla^2 V)$ vanishes. Consequently, V defines a harmonic map from (M, g) to (TM, g^s) if and only if

$$\text{tr}[R(\nabla \cdot V, V)] = 0, \quad \nabla^* \nabla V = 0, \quad (2.5)$$

where with respect to an orthonormal local frame $\{e_1, \dots, e_n\}$ on (M, g) , with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i , one has

$$\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V).$$

A smooth vector field V is said to be a harmonic section if and only if it is a critical point of $E^v(V) = (1/2) \int_M \|\nabla V\|^2 dv$ where E^v is the vertical energy. The corresponding Euler-Lagrange equations are given by

$$\nabla^* \nabla V = 0, \quad (2.6)$$

Let $\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : \|V\|^2 = \rho^2\}$ and $\rho \neq 0$. Then, one can consider vector fields $V \in \mathfrak{X}(M)$, which are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$, restricted to vector fields of the same constant length. The Euler-Lagrange equations of this variational condition are given by

$$\nabla^* \nabla V \text{ is collinear to } V. \quad (2.7)$$

As usual, condition (2.7) is taken as a definition of critical points for the energy functional restricted to vector fields of the same length in the non-compact case.

Following [11], the sphere \mathbb{S}^3 and the Lie group $SU(2)$ have been identified by a map that sends $(z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2$ to

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in SU(2).$$

We consider the basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ given by

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (2.8)$$

Then, the Lie brackets are determined by

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2. \quad (2.9)$$

The one-parameter family $\{g_\epsilon : \epsilon > 0\}$ of left-invariant Riemannian metrics on $\mathbb{S}^3 = SU(2)$ given at the identity, with respect to the basis

of left-invariant vector fields X_1, X_2, X_3 by

$$g_\epsilon = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.10)$$

are called the Berger metrics on \mathbb{S}^3 ; if $\epsilon = 1$ we have the canonical (bi-invariant) metric and for $\epsilon < 0$ the one-parameter family $\{g_\epsilon : \epsilon < 0\}$ are left-invariant Lorentzian metrics. The Berger spheres are the simply connected complete Riemannian manifolds $\mathbb{S}_\epsilon^3 = (\mathbb{S}^3, g_\epsilon)$, $\epsilon > 0$. We will use the name "Lorentzian Berger spheres" for case $\epsilon < 0$.

3. HOMOGENEOUS RICCI SOLITONS ON BERGER SPHERES

Setting $\Lambda_i = \nabla_{e_i}$, the components of the Levi-Civita connection are calculated using the well known *Koszul* formula and are

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon - 2 \\ 0 & 2 - \epsilon & 1 \end{pmatrix}, \quad (3.1)$$

$$\Lambda_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\epsilon & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & -1 & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using (2.2) we can determine the non-zero curvature components;

$$\begin{aligned} R(X_1, X_2)X_1 &= \epsilon^2 X_2, & R(X_1, X_2)X_2 &= -\epsilon X_1, \\ R(X_1, X_3)X_3 &= -\epsilon X_1, & R(X_1, X_3)X_1 &= \epsilon^2 X_3, \\ R(X_2, X_3)X_2 &= (4 - 3\epsilon)X_3, & R(X_2, X_3)X_3 &= (3\epsilon - 4)X_2. \end{aligned}$$

Since $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ we have;

$$\begin{aligned} R(X_1, X_2, X_1, X_2) &= R(X_1, X_3, X_1, X_3) = \epsilon^2, \\ R(X_2, X_3, X_2, X_3) &= (4 - 3\epsilon). \end{aligned}$$

Applying the Ricci tensor formula (2.3), we get;

$$(\rho)_{ij} = \begin{pmatrix} 2\epsilon^2 & 0 & 0 \\ 0 & 4 - 2\epsilon & 0 \\ 0 & 0 & 4 - 2\epsilon \end{pmatrix}, \quad (3.2)$$

which is diagonal with eigenvalues $r_1 = 2\epsilon^2$ and $r_2 = r_3 = 4 - 2\epsilon$.

For an arbitrary left-invariant vector field $X = aX_1 + bX_2 + cX_3 \in \mathfrak{su}(2)$ we have;

$$\begin{aligned} \nabla_{X_1} X &= (\epsilon - 2)(cX_2 - bX_3), \\ \nabla_{X_2} X &= cX_1 - a\epsilon X_3, \\ \nabla_{X_3} X &= -bX_1 + a\epsilon X_2. \end{aligned}$$

Using the relation $(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$ we have;

$$L_X g = \begin{pmatrix} 0 & 2(1-\epsilon)c & 2(\epsilon-1)b \\ 2(1-\epsilon)c & 0 & 0 \\ 2(\epsilon-1)b & 0 & 0 \end{pmatrix}. \quad (3.3)$$

By the Ricci soliton formula (1.1), we get the following system of differential equations;

$$\begin{aligned} 2(1-\epsilon)c &= 0, \\ 2(\epsilon-1)b &= 0, \\ 2\epsilon^2 - \lambda\epsilon &= 0, \\ 4 - 2\epsilon - \lambda &= 0. \end{aligned} \quad (3.4)$$

If $\epsilon = 1$, then from (3.2) we can see $\rho_{ij} = \lambda g_{ij}$ for all indices i, j and therefore g_1 is an Einstein metric on \mathbb{S}^3 . So, we suppose that $\epsilon \neq 1$. From the first and the second equations in (3.4), we get $b = c = 0$ and the third and the last equations in (3.4) give us $\epsilon = 0$ and $\lambda = 4$. Therefore the only solution occurs when $\epsilon = 0$, and for an arbitrary $\epsilon \neq 0, 1$ the system of differential equations (3.4) is incompatible. Thus, we have the following propositions.

Proposition 3.1. *Let $(\mathbb{S}^3, g_\epsilon)$ be a Berger sphere ($\epsilon > 0$). Then $(\mathbb{S}^3, g_\epsilon)$ can not be a homogeneous Ricci soliton manifold.*

Proposition 3.2. *Let $(\mathbb{S}^3, g_\epsilon)$ be a Lorentzian Berger sphere ($\epsilon < 0$). Then $(\mathbb{S}^3, g_\epsilon)$ can not be a homogeneous Ricci soliton manifold.*

Remark 1. *Proposition (3.2) confirms the classification result in [7], while Proposition (4.3) emphasizes that there are no left-invariant Ricci solitons on three-dimensional Riemannian Lie groups [9].*

A D' Atri space is defined as a Riemannian manifold (M, g) whose local geodesic symmetries are volume-preserving. Let us recall that the property of being a D' Atri space is equivalent to the infinite number of curvature identities called the odd Ledger conditions L_{2k+1} , $k \geq 1$. In particular, the two first non-trivial Ledger conditions are:

$$\begin{aligned} L_3 : (\nabla_X \rho)(X, X) &= 0, \\ L_5 : \sum_{a,b=1}^n R(X, E_a, X, E_b)(\nabla_X R)(X, E_a, X, E_b) &= 0, \end{aligned} \quad (3.5)$$

where X is any tangent vector at any point $m \in M$ and $\{E_1, \dots, E_n\}$ is an orthonormal basis of $T_m M$. Here R denotes the curvature tensor and ρ the Ricci tensor of (M, g) , respectively, and $n = \dim M$.

Thus, it is natural to start with the investigation of all homogeneous Riemannian Berger spheres satisfying the simplest Ledger condition L_3 , which is the first approximation of the D' Atri property. This condition is called in [18] "the class \mathcal{A} condition." Equivalently Ledger condition L_3

holds if and only if the Ricci tensor is cyclic-parallel, i.e. $(\nabla_X \rho)(Y, Z) + (\nabla_Y \rho)(Z, X) + (\nabla_Z \rho)(X, Y) = 0$. For more detail, see [8].

Proposition 3.3. *Let $(\mathbb{S}^3, g_\epsilon)$ be a Berger sphere. Then $(\mathbb{S}^3, g_\epsilon)$ is a D' Atri space, which its first approximation holds.*

Proof. In Ledger condition L_3 ,

$$\nabla_i \rho_{jk} = - \sum_t (\varepsilon_j B_{ijt} \rho_{tk} + \varepsilon_k B_{ikt} \rho_{tj}),$$

where B_{ijk} components can be obtained by the relation $\nabla_{e_i} e_j = \sum_k \varepsilon_j B_{ijk} e_k$ with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i . Hence $\nabla_1 \rho_{11} = \nabla_2 \rho_{22} = \nabla_3 \rho_{33} = 0$, as desired. \square

A pseudo-Riemannian manifold which admits a parallel degenerate distribution is called a *Walker* manifold. Walker spaces were introduced by Arthur Geoffrey Walker in 1949. The existence of such structures causes many interesting properties for the manifold with no Riemannian counterpart. Walker also determined standard local coordinates for these kinds of manifolds [19].

Proposition 3.4. *Let $(\mathbb{S}^3, g_\epsilon)$ be a Lorentzian Berger sphere ($\epsilon < 0$). Then $(\mathbb{S}^3, g_\epsilon)$ can not be a Walker manifold.*

Proof. Set $X = aX_1 + bX_2 + cX_3 \in \mathfrak{su}$ and suppose that $\mathcal{D} = \text{span}(X)$ is an invariant null parallel line field. Then, the following equations must satisfy for some parameters $\omega_1, \dots, \omega_3$

$$\nabla_{X_1} X = \omega_1 X, \quad \nabla_{X_2} X = \omega_2 X, \quad \nabla_{X_3} X = \omega_3 X.$$

By straight forward calculations we conclude that the following equations must satisfy

$$\begin{array}{lll} \omega_1 a = 0, & \omega_1 b + c(2 - \epsilon) = 0, & \omega_1 c + b(\epsilon - 2) = 0, \\ \omega_2 b = 0, & \omega_2 a - c = 0, & -\omega_2 c + a\epsilon = 0, \\ \omega_3 c = 0, & \omega_3 a + b = 0, & \omega_3 b - a = 0. \end{array}$$

X is null, hence X must satisfy $g(X, X) = a^2 \epsilon + b^2 + c^2 = 0$ described in (2.10). For solving the above system of equations, as we can see, since $\epsilon \neq 0$, a non-trivial solution can not occur. \square

4. HARMONICITY OF VECTOR FIELDS ON BERGER SPHERES

In this section, we investigate the harmonicity of invariant vector fields on a Berger sphere $(\mathbb{S}^3, g_\epsilon)$ in both Riemannian ($\epsilon > 0$) and Lorentzian ($\epsilon < 0$). We treat separately the case when g is Lorentzian and the Riemannian case.

4.1. **Lorentzian case.** In Lorentzian case ($\epsilon < 0$) we can construct an orthonormal frame field $\{e_1, e_2, e_3\}$ with respect to g_ϵ ;

$$e_1 = \frac{1}{\sqrt{-\epsilon}}X_1, \quad e_2 = X_2, \quad e_3 = X_3, \quad (4.1)$$

and we get;

$$[e_1, e_2] = \frac{2}{\sqrt{-\epsilon}}e_3, \quad [e_2, e_3] = 2\sqrt{-\epsilon}e_1, \quad [e_3, e_1] = \frac{2}{\sqrt{-\epsilon}}e_2. \quad (4.2)$$

Considering formula (2.1) the connection components are;

$$\begin{aligned} \nabla_{e_1}e_2 &= \frac{2-\epsilon}{\sqrt{-\epsilon}}e_3, & \nabla_{e_1}e_3 &= \frac{\epsilon-2}{\sqrt{-\epsilon}}e_2, & \nabla_{e_2}e_1 &= \sqrt{-\epsilon}e_3, \\ \nabla_{e_2}e_3 &= \sqrt{-\epsilon}e_1, & \nabla_{e_3}e_1 &= -\sqrt{-\epsilon}e_2, & \nabla_{e_3}e_2 &= -\sqrt{-\epsilon}e_1, \end{aligned} \quad (4.3)$$

while $\nabla_{e_i}e_j = 0$ in the remaining cases.

For an arbitrary left-invariant vector field $V = ae_1 + be_2 + ce_3 \in \mathfrak{su}(2)$ we can now use (4.3) to calculate $\nabla_{e_i}V$ for all indices i . We get

$$\begin{aligned} \nabla_{e_1}V &= \frac{2-\epsilon}{\epsilon}ce_2 + \frac{\epsilon-2}{\epsilon}be_3, & \nabla_{e_2}V &= \sqrt{-\epsilon}ce_1 + ae_3, \\ \nabla_{e_3}V &= -\sqrt{-\epsilon}be_1 - ae_2. \end{aligned} \quad (4.4)$$

From (4.10) it follows at once that there are no parallel vector fields $V \neq 0$ belonging to $\mathfrak{su}(2)$. We can now calculate $\nabla_{e_i}\nabla_{e_i}V$ and $\nabla_{\nabla_{e_i}e_i}V$ for all indices i . We obtain

$$\begin{aligned} \nabla_{e_1}\nabla_{e_1}V &= -\frac{(\epsilon-2)^2}{\epsilon^2}(be_2 + ce_3), & \nabla_{e_2}\nabla_{e_2}V &= \sqrt{-\epsilon}ae_1 - \epsilon ce_3, \\ \nabla_{e_3}\nabla_{e_3}V &= \sqrt{-\epsilon}ae_1 - \epsilon be_2, & \nabla_{\nabla_{e_1}e_1}V &= 0, \\ \nabla_{\nabla_{e_2}e_2}V &= 0, & \nabla_{\nabla_{e_3}e_3}V &= 0. \end{aligned} \quad (4.5)$$

Thus, we find

$$\begin{aligned} \nabla^*\nabla V &= \sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V) \\ &= -2\epsilon ae_1 - \left(\frac{\epsilon^3 - \epsilon^2 + 4\epsilon - 4}{\epsilon^2}\right)(be_2 + ce_3). \end{aligned} \quad (4.6)$$

Since $\nabla^*\nabla V = -\frac{\epsilon^3 - \epsilon^2 + 4\epsilon - 4}{\epsilon^2}V + \frac{-\epsilon^3 - \epsilon^2 + 4\epsilon - 4}{\epsilon^2}ae_1$, condition (2.7) results that $a = 0$. In the other direction, let $V = -\frac{\epsilon^3 - \epsilon^2 + 4\epsilon - 4}{\epsilon^2}(be_2 + ce_3)$. A direct calculation yields that $\nabla^*\nabla V = -\frac{\epsilon^3 - \epsilon^2 + 4\epsilon - 4}{\epsilon^2}V$.

On the other hand, from (4.12), since $\nabla^*\nabla V = -2\epsilon V + \frac{\epsilon^3 + \epsilon^2 - 4\epsilon + 4}{\epsilon^2}(be_2 + ce_3)$, then (2.7) results that $b = c = 0$. Vice versa, let $V = ae_1$, by standard calculations, we obtain $\nabla^*\nabla V = -2\epsilon V$. Thus, we have the following.

Theorem 4.1. *Let $(\mathbb{S}^3, g_\epsilon)$ be a Lorentzian Berger sphere and $V = ae_1 + be_2 + ce_3 \in \mathfrak{su}(2)$ be a left-invariant vector field on the Berger sphere for some real constants a, b, c .*

- (a) V is a critical point for the energy functional restricted to vector fields of the same length if and only if $V = -2\left(\frac{\epsilon^2-2\epsilon+2}{\epsilon}\right)(be_2 + ce_3)$. However, none of these vector fields is harmonic (in particular, defines a harmonic map).
- (b) V is a critical point for the energy functional restricted to vector fields of the same length if and only if $V = -2\epsilon ae_1$. However, none of these vector fields is harmonic (in particular, defines a harmonic map).

4.2. Riemannian case. Consider now a Riemannian Berger sphere $(\mathbb{S}^3, g_\epsilon)$, $\epsilon > 0$. $\mathfrak{su}(2)$ admits a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, where

$$e_1 = \frac{1}{\sqrt{\epsilon}}X_1, \quad e_2 = X_2, \quad e_3 = X_3, \quad (4.7)$$

and we get;

$$[e_1, e_2] = \frac{2}{\sqrt{\epsilon}}e_3, \quad [e_2, e_3] = 2\sqrt{\epsilon}e_1, \quad [e_3, e_1] = \frac{2}{\sqrt{\epsilon}}e_2. \quad (4.8)$$

Considering formula (2.1) the connection components are;

$$\begin{aligned} \nabla_{e_1}e_2 &= \frac{2-\epsilon}{\sqrt{\epsilon}}e_3, & \nabla_{e_1}e_3 &= \frac{\epsilon-2}{\sqrt{\epsilon}}e_2, & \nabla_{e_2}e_1 &= \sqrt{\epsilon}e_3, \\ \nabla_{e_2}e_3 &= \sqrt{\epsilon}e_1, & \nabla_{e_3}e_1 &= -\sqrt{\epsilon}e_2, & \nabla_{e_3}e_2 &= -\sqrt{\epsilon}e_1, \end{aligned} \quad (4.9)$$

while $\nabla_{e_i}e_j = 0$ in the remaining cases.

For an arbitrary left-invariant vector field $V = ae_1 + be_2 + ce_3 \in \mathfrak{su}(2)$ we can now use (4.9) to calculate $\nabla_{e_i}V$ for all indices i . We get

$$\begin{aligned} \nabla_{e_1}V &= -\frac{2-\epsilon}{\epsilon}ce_2 - \frac{\epsilon-2}{\epsilon}be_3, & \nabla_{e_2}V &= \sqrt{\epsilon}ce_1 - ae_3, \\ \nabla_{e_3}V &= -\sqrt{\epsilon}be_1 + ae_2. \end{aligned} \quad (4.10)$$

From (4.10) it follows at once that there are no parallel vector fields $V \neq 0$ belonging to $\mathfrak{su}(2)$. We can now calculate $\nabla_{e_i}\nabla_{e_i}V$ and $\nabla_{\nabla_{e_i}e_i}V$ for all indices i . We obtain

$$\begin{aligned} \nabla_{e_1}\nabla_{e_1}V &= -\frac{(\epsilon-2)^2}{\epsilon^2}(be_2 + ce_3), & \nabla_{e_2}\nabla_{e_2}V &= -\sqrt{\epsilon}ae_1 - \epsilon ce_3, \\ \nabla_{e_3}\nabla_{e_3}V &= -\sqrt{\epsilon}ae_1 - \epsilon be_2, & \nabla_{\nabla_{e_1}e_1}V &= 0, \\ \nabla_{\nabla_{e_2}e_2}V &= 0, & \nabla_{\nabla_{e_3}e_3}V &= 0. \end{aligned} \quad (4.11)$$

Thus, we find

$$\begin{aligned} \nabla^*\nabla V &= \sum_i \varepsilon_i (\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V) \\ &= -2\epsilon ae_1 - \left(\frac{\epsilon^3+\epsilon^2-4\epsilon+4}{\epsilon^2}\right)(be_2 + ce_3). \end{aligned} \quad (4.12)$$

Since $\nabla^*\nabla V = -\frac{\epsilon^3+\epsilon^2-4\epsilon+4}{\epsilon^2}V + \frac{-\epsilon^3+\epsilon^2-4\epsilon+4}{\epsilon^2}ae_1$, condition (2.7) results that $a = 0$. In the other direction, let $V = -\frac{\epsilon^3+\epsilon^2-4\epsilon+4}{\epsilon^2}(be_2 + ce_3)$. A direct calculation yields that $\nabla^*\nabla V = -\frac{\epsilon^3+\epsilon^2-4\epsilon+4}{\epsilon^2}V$.

On the other hand, from (4.12) since $\nabla^*\nabla V = -2\epsilon V + \frac{\epsilon^3-\epsilon^2+4\epsilon-4}{\epsilon^2}(be_2 +$

ce_3), then (2.7) results that $b = c = 0$. Vice versa, let $V = ae_1$, by standard calculations we obtain $\nabla^*\nabla V = -2\epsilon V$. So, we proved the following.

Theorem 4.2. *Each vector field $V = ae_1$ and $V = be_2 + ce_3 \in \mathfrak{su}(2)$ on a Riemannian Berger sphere $(\mathbb{S}^3, g_\epsilon)$, $\epsilon > 0$ are a critical point for the energy functional restricted to vector fields of the same length. However, none of these vector fields is harmonic (in particular, defines a harmonic map).*

Therefore by theorems 4.1 and 4.2 there is no harmonic vector field nor harmonic map on Berger spheres.

4.3. Geodesic and Killing vector fields. A vector field V is a geodesic if $\nabla_V V = 0$, and is Killing if $\mathcal{L}_V g = 0$, where \mathcal{L} denotes the Lie derivative i.e. X is Killing if and only if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathfrak{X}(M)$ (the equation above is called the Killing equation). Parallel vector fields are both geodesic and Killing, and vector fields with these special geometric features often have particular harmonicity properties [1, 13]. By standard calculations, we obtain the following result.

Proposition 4.3. *Let $(\mathbb{S}^3, g_\epsilon)$ be the Berger sphere and $V \in \mathfrak{su}(2)$ be a left-invariant vector field on the Berger sphere, then V is geodesic if and only if $V = be_2 + ce_3$ or $V = ae_1$. Moreover, by (3.3), V is Killing if and only if $V = ae_1$.*

In particular, from Theorems 4.1, 4.2, and Proposition 4.3, a straightforward calculation proves the following main classification result.

Theorem 4.4. *For a vector field $V = ae_1 + be_2 + ce_3 \in \mathfrak{su}$ on the $(\mathbb{S}^3, g_\epsilon)$, the following conditions are equivalent:*

- (1) V is geodesic;
- (2) V is a critical point for the energy functional restricted to vector fields of the same length;
- (3) $V = be_2 + ce_3$ or $V = ae_1$.
- (4) V is Killing if and only if $b = c = 0$.

5. THE ENERGY OF VECTOR FIELDS ON BERGER SPHERES

We calculate explicitly the energy of a vector field $V \in \mathfrak{su}(2)$ of a Berger sphere. This gives us the opportunity to determine some critical values of the energy functional on Berger spheres. We shall first discuss the geometric properties of the map V defined by a vector field $V \in \mathfrak{su}(2)$.

Proposition 5.1. *Let $(\mathbb{S}^3, g_\epsilon)$ be a Berger sphere, $V = ae_1 + be_2 + ce_3 \in \mathfrak{su}(2)$ be a left-invariant vector field on the Berger sphere for some real constants a, b, c . Denote by $E(V)$ the energy of V .*

(a) *In the Lorentzian case ($\epsilon < 0$) the energy of V is*

$$E(V) = (2 + \frac{\epsilon^3 - (\epsilon - 2)^2}{2\epsilon^2} \|V\|^2 - \frac{\epsilon^3 - (\epsilon - 2)^2 - 2\epsilon^2}{2\epsilon^2} a^2) \text{vol}(\mathbb{S}^3, g_\epsilon).$$

(b) *In the Riemannian case ($\epsilon > 0$) the energy of V is*

$$E(V) = (2 + \frac{\epsilon^3 + (\epsilon - 2)^2}{2\epsilon^2} \|V\|^2 - \frac{\epsilon^3 + (\epsilon - 2)^2 - 2\epsilon^2}{2\epsilon^2} a^2) \text{vol}(\mathbb{S}^3, g_\epsilon).$$

Proof. Let $(\mathbb{S}^3, g_\epsilon)$ be the Berger sphere. For case (a) consider the local orthonormal basis $\{e_1, e_2, e_3\}$ of vector fields described in (4.1). Then, locally,

$$\|\nabla V\|^2 = \sum_{i=1}^n \varepsilon_i g(\nabla_{e_i} V, \nabla_{e_i} V).$$

Let $V \in \mathfrak{su}(2)$ be a left-invariant vector field on the Berger sphere, then (4.10) easily yields that

$$\|\nabla V\|^2 = \frac{\epsilon^3 - (\epsilon - 2)^2}{\epsilon^2} \|V\|^2 - \frac{\epsilon^3 - (\epsilon - 2)^2 - 2\epsilon^2}{\epsilon^2} a^2.$$

In the Riemannian case, consider the local orthonormal basis $\{e_1, e_2, e_3\}$ of vector fields, where e_i is the base described in (4.7). After a similar and straightforward calculation, we find that

$$\|\nabla V\|^2 = \frac{\epsilon^3 + (\epsilon - 2)^2}{\epsilon^2} \|V\|^2 - \frac{\epsilon^3 + (\epsilon - 2)^2 - 2\epsilon^2}{\epsilon^2} a^2.$$

□

We already know from Theorems 4.1 and 4.2 which vector fields in $\mathfrak{su}(2)$ of Berger spheres are critical points for the energy functional. Taking into account the Proposition (5.1), we then have the following.

Theorem 5.2. *Let $(\mathbb{S}^3, g_\epsilon)$ be the Berger sphere (in both Riemannian and Lorentzian cases).*

- (a) *If $\epsilon > 0$, $(2 + (\frac{\epsilon^3 + (\epsilon - 2)^2}{2\epsilon^2})\rho^2) \text{vol}(\mathbb{S}^3, g_\epsilon)$ is the minimum value of the energy functional E restricted to vector fields of constant length ρ . Such a minimum is attained by all vector fields $V = be_2 + ce_3 \in \mathfrak{su}(2)$ of length $\|V\| = \rho = \sqrt{b^2 + c^2}$.*
- (b) *If $\epsilon < 0$, $(2 + (\frac{\epsilon^3 - (\epsilon - 2)^2}{2\epsilon^2})\rho^2) \text{vol}(\mathbb{S}^3, g_\epsilon)$ is the minimum value of the energy functional E restricted to vector fields of constant length ρ . Such a minimum is attained by all vector fields $V = be_2 + ce_3 \in \mathfrak{su}(2)$ of length $\|V\| = \rho = \sqrt{b^2 + c^2}$.*
- (c) *$(2 + \epsilon\rho^2) \text{vol}(\mathbb{S}^3, g_\epsilon)$ is the minimum value of the energy functional E restricted to vector fields of constant length ρ . Such a minimum is attained by all vector fields $V = ae_1 \in \mathfrak{su}(2)$ of length $\|V\| = \rho = \sqrt{a^2}$.*

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