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Some inequalities on the order of the higher multiplier of groups

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Abstract. The Schur multiplier $\mathcal{M}(G)$ of a group G was introduced by Schur in 1904 during his works on projective representations of groups. Ellis extended the theory of the Schur multiplier for a pair of groups. Several authors generalized the concept of the Schur multiplier of a pair of groups to the c-nilpotent multiplier of a pair of groups. This was a motivation to define the Baer-invariant of the pair (N, G) with respect to a variety of groups. In this paper, we prove some inequalities for the order of the c-nilpotent multiplier of a pair of groups.

Keywords: Pair of groups, c-nilpotent multiplier, p-groups.

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1. Introduction and Preliminaries

Let $1 \to R \to F \to G \to 1$ be a free presentation of a group G. Then, the c-nilpotent multiplier of G is defined as

$$\mathcal{M}^{(c)}(G) = \frac{\gamma_{c+1}(F) \cap R}{\gamma_{c+1}(R, F)},$$

in which $\gamma_{c+1}(F)$ is the $(c+1)$-th term of the lower central series of F and $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$, inductively. If $c = 1$,

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then $\mathcal{M}^{(c)}(G) = \mathcal{M}(G)$ is celled the Schur multiplier of G (see [3, 8] for more information).

Let (N, G) be a pair of grouos, in which N is a normal subgroup of G. Ellis defined the Schur multiplier of a pair (N, G) to be the abelian group $\mathcal{M}(N, G)$ appearing in the following exact sequence

$$H_3(G) \to H_3(\frac{G}{N}) \to \mathcal{M}(N, G) \to \mathcal{M}(G) \to \mathcal{M}(\frac{G}{N}) \\ \to \frac{N}{[N, G]} \to (G)^{ab} \to (\frac{G}{N})^{ab} \to 1,$$

in which $H_3(G)$ is the third homology of G with integer coefficients. Let $1 \to R \to F \to G \to 1$ be a free presentation of G and S be a normal subgroup of F with $N \cong S/R$. If N admits a complement in G then

$$\mathcal{M}(N, G) \cong \frac{R \cap [S, F]}{[R, F]}.$$

If $N = G$, then $\mathcal{M}(G, G) = \mathcal{M}(G)$ is the usual Schur multiplier of G.

Let G and N be two groups with an action of G on N. Then, the G-commutator subgroup and G-center subgroup of N are defined, as follows:

$$[N, G] = \langle [n, g] = n^g n^{-1} | n \in N, g \in G \rangle,$$
$$Z(N, G) = \{n \in N | n^g = n, \ \forall \ g \in G\}.$$

Let (N, G) be a pair of groups, and S be a normal subgroup of F with $N \cong S/R$. If N admits a complement in G, then the c-nilpotent multiplier of the pair (N, G) is defined as

$$\mathcal{M}^{(c)}(N, G) = \frac{R \cap [S,_c F]}{[R,_c F]}$$

where, $[X,_c Y] = [X, \underbrace{Y, \cdots, Y}_{c\text{-times}}]$. One can check that $\mathcal{M}^{(c)}(N, G)$ is abelian and independent of the free presentation of G. In particular, if $N = G$, then $\mathcal{M}^{(c)}(G, G) = \mathcal{M}^{(c)}(G)$ is the c-nilpotent multiplier of G. (See [1, 2, 4, 7, 9, 10, 11] for more information). Let (N, G) be a pair of groups, in which N is a normal subgroup of G. We define the lower central series of normal subgroups of N as follows

$$N = [N,_0 G] \supseteq [N, G] \supseteq [N, G, G] \supseteq \cdots \supseteq [N,_c G] \supseteq \cdots,$$

where $[N,_c G] = \gamma_{c+1}(N, G) = [N, \underbrace{G, \ldots, G}_{c\text{-times}}]$, $c > 0$. Similarly, we define the upper central series of N in G as follows

$$0 = Z_0(N, G) \subseteq Z_1(N, G) \subseteq \cdots \subseteq Z_c(N, G) \subseteq \cdots,$$

where, $Z_c(N,G) = \{n \in N \big| [n, g_1, \ldots, g_c] = 1 \text{ for all } g_1, \ldots, g_c \in G\}$.
Let (N,G) be a pair of groups. A relative c-central extension of the pair (N,G) is a homomorphism $\sigma : M \to G$ together with an action of G on M such that

(i) $\sigma(M) = N$
(ii) $\sigma(m^g) = g^{-1}\sigma(m)g$, for all $g \in G$, $m \in M$,
(iii) $m'^{\sigma(m)} = m^{-1}m'm$, for all $m, m' \in M$,
(iv) $\ker\sigma \subseteq Z_c(M,G)$.

In addition, the relative c-central extension $\sigma : M \to G$ is said to be a c-cover of (N,G) if there exists a subgroup A of M such that

(i) $A \subseteq Z_c(M,G) \cap [M,_c G]$,
(ii) $A \cong \mathcal{M}^{(c)}(N,G)$,
(iii) $N \cong M/A$.

In this paper, we prove some inequalities for the order of the c-nilpotent multiplier of a pair of groups.

2. Main results

Let X and Y be two groups, we recall $X \otimes^c Y = X \otimes \underbrace{Y \otimes \ldots \otimes Y}_{c\text{-times}}$ is the abelian tensor product. Also, the exterior product $N \wedge G$ is $N \otimes G/\langle n \otimes n |\ n \in N\rangle$. So $N \wedge^c G = N \wedge \underbrace{G \wedge \ldots \wedge G}_{c\text{-times}}$, (see [5] for more information).
Let H be a group and $|H/Z(H)| = p^n$, then Wiegold [12] proved that $|H'| \leq p^{1/2n(n+1)}$. If $H/Z(H) \cong G$ then Gaschutz et al. [6] showed that

$$|H' \cap Z(H)| \leq |\mathcal{M}(\frac{G}{G'})||G'|^{d(\frac{G}{Z(G)})-1},$$

where $d(X)$ is the minimal generator of group X. In particular,

$$|H'| \leq |\mathcal{M}(\frac{G}{G'})||G'|^{d(\frac{G}{Z(G)})-1}.$$

In [9] Moghaddam et. al. generalized the works of Wiegold and Gaschutz et al. to a pair of groups. In this section, we extend the above results to the c-nilpotent multiplier of a pair of groups. The following Lemmas are useful for the proof of the next results.

Lemma 2.1. *Let G and K be two groups with central subgroups N and M, respectively. If $\theta : G \to K$ is an epimorphism with $\theta(N) = M$, then*

$$|\mathcal{M}^{(c)}(M,K)| \leq |\mathcal{M}^{(c)}(N,G)|.$$

Proof. One can check that θ induces the following epimorphism

$$\psi : N \otimes^c G^{ab} \to M \otimes^c K^{ab}$$
$$\psi\big(n \otimes (g_1G') \otimes \cdots \otimes (g_cG')\big) = \theta(n) \otimes \big(\theta(g_1)K'\big) \otimes \cdots \otimes \big(\theta(g_c)K'\big),$$

where, $n \in N$ and $g_1, \dots, g_c \in G$. Thus, we can see that there exists an epimorphism from $\mathcal{M}^{(c)}(N,G)$ on to $\mathcal{M}^{(c)}(M,K)$. Hence,

$$|\mathcal{M}^{(c)}(M,K)| \leq |\mathcal{M}^{(c)}(N,G)|.$$

□

Lemma 2.2. *Let (N,G) be a pair of finite groups and M be a normal subgroup of G such that $M \subseteq Z(N,G)$. Then*

$$|M \cap [N,_c G]| \leq \big|\mathcal{M}^{(c)}\big(\frac{N}{M}, \frac{G}{M}\big)\big|.$$

Proof. Define

$$\sigma : N \wedge^c G \to G$$
$$\sigma\big(n \wedge (g_1 \wedge \cdots \wedge g_c)\big) = [n, g_1, \cdots, g_c].$$

Thus,

$$\mathrm{Im}(\sigma) = [N,_c G] \quad \text{and} \quad \ker(\sigma) \cong \mathcal{M}^{(c)}(N,G).$$

So, there exists an epimprphism

$$\varphi : N \wedge^c G \to \frac{N}{M} \wedge^c \frac{G}{M}$$
$$\varphi\big(n \wedge (g_1 \wedge \cdots \wedge g_c)\big) = (nM') \wedge (g_1M' \wedge \cdots \wedge g_cM'),$$

for $g_1, \dots, g_c \in G$ and $n \in N$. Thus, we have an epimorphism $\delta : N/M \wedge^c G/M \to [N,_c G]$ such that $\delta\varphi = \sigma$. Therefore,

$$|[N,_c G]| \leq \big|\frac{N}{M} \wedge^c \frac{G}{M}\big|,$$

and so, we have

$$\big|\frac{N}{M} \wedge^c \frac{G}{M}\big||M \cap [N,_c G]| = \big|\mathcal{M}^{(c)}\big(\frac{N}{M}, \frac{G}{M}\big)\big||[N,_c G]|.$$

Thus, the proof is completes. □

Theorem 2.3. *Let (M,K) be a pair of finite p-groups. If (N,G) is a pair of finite groups such that $\frac{G}{Z_c(N,G)} \cong K$ and $\frac{N}{Z_c(N,G)} \cong M$. Then*

$$|[N,_c G]| \leq \big|\mathcal{M}^{(c)}\big(\frac{M}{[M,_c K]}, \frac{K}{[M,_c K]}\big)\big| \cdot |[M,_c K]|^{d\left(\frac{K}{Z_c(M,K)}\right)},$$

where $d(X)$ is the minimal number of generators of a group X.

Proof. we prove the result by using induction on the order of $[M,_c K]$. If $|[M,_c K]| = 1$, then by using Lemma 2.2, we obtain the result. Now, let $|[M,_c K]| = n > 1$ and the result holds for any pair (M', K') of finite p-groups with $|[M',_c K']| < n$. Let $Z_{c+1}(N,G)$ be the pre-image in the normal subgroup N of $Z\big(\frac{N}{Z_c(N,G)}, \frac{G}{Z_c(N,G)}\big)$. We have

$$Z_c(N,G) \lneqq Z_{c+1}(N,G) \cap \big([N,_c G] Z_c(N,G)\big),$$

thus there exists $x \in \big(Z_{c+1}(N,G) \cap ([N,_c G] Z_c(N,G)\big) - Z_c(N,G)$.

Hence, the following mapping is a well defined epimorphism

$$\delta : \frac{G}{Z_{c+1}(N,G)} \to [x,_c G]$$
$$\delta\big(gZ_{c+1}(N,G)\big) = [x, \underbrace{g, \cdots, g}_{c\text{-times}}].$$

Put $T = [x,_c G]$. So, we obtain $|T| \leq p^{d\left(\frac{K}{Z_c(M,K)}\right)}$. Put

$(N^*, G^*) = \big(\frac{N}{T}, \frac{G}{T}\big)$ and $(M^*, K^*) = \big(\frac{N^*}{Z_c(N^*,G^*)}, \frac{G^*}{Z_c(N^*,G^*)}\big)$.

As $xT \in Z_c(N^*, G^*) - \frac{Z_c(N,G)}{T}$. Thus,

$$\frac{Z_c(N,G)}{T} \lneqq Z_c(N^*, G^*).$$

Also, the following map is an epimorphism with $\theta(M) = M^*$ and $\ker\theta \neq 1$

$$\theta : K \cong \frac{G}{Z_c(N,G)} \to K^*$$
$$\theta\big(gZ_c(N,G)\big) = (gT)Z_c(N^*, G^*).$$

Now, by Lemma 2.1 we have

$$\big|\mathcal{M}^{(c)}\big(\frac{M^*}{Z_c(M^*,K^*)}, \frac{K^*}{Z_c(M^*,K^*)}\big)\big| \leq \big|\mathcal{M}^{(c)}\big(\frac{M}{Z_c(M,K)}, \frac{K}{Z_c(M,K)}\big)\big|.$$

Also,

$$d\big(\frac{K^*}{Z_c(M^*,K^*)}\big) \leq d\big(\frac{K}{Z_c(M,K)}\big) \textit{ and } |[M^*,_c K^*]| \leq |[M,_c K]|$$

Hence, we have

$$\begin{aligned}|[N^*,_c G^*]| &\leq \big|\mathcal{M}^{(c)}\big(\frac{M^*}{Z_c(M^*,K^*)}, \frac{K^*}{Z_c(M^*,K^*)}\big)\big| \cdot |[M^*,_c K^*]|^{d\left(\frac{K^*}{Z_c(M^*,K^*)}\right)} \\ &\leq \big|\mathcal{M}^{(c)}\big(\frac{M}{Z_c(M,K)}, \frac{K}{Z_c(M,K)}\big)\big| \frac{[M,_c K]}{p}\big|^{d\left(\frac{K}{Z_c(M,K)}\right)}.\end{aligned}$$

On the other hand,

$$|[N,_c G]| = |[N^*,_c G^*]\big||T|.$$

So, we have

$$\begin{aligned}|[N,_c G]| &\leq \left|\mathcal{M}^{(c)}\Big(\frac{M}{Z_c(M,K)}, \frac{K}{Z_c(M,K)}\Big)\right| \left|\frac{[M,_c K]}{p}\right|^{d\left(\frac{K}{Z_c(M,K)}\right)} \cdot |T| \\ &\leq \left|\mathcal{M}^{(c)}\Big(\frac{M}{Z_c(M,K)}, \frac{K}{Z_c(M,K)}\Big)\right| \left|[M,_c K]\right|^{d\left(\frac{K}{Z_c(M,K)}\right)}.\end{aligned}$$

□

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Let (M,K) be a pair of finite p-groups. Then for each pair (N,G) of finite groups with $\frac{G}{Z_c(N,G)} \cong K$ and $\frac{N}{Z_c(N,G)} \cong M$,*

$$|[N,_c G] \cap Z_c(N,G)| \leq \left|\mathcal{M}^{(c)}\Big(\frac{M}{[M,_c K]}, \frac{K}{[M,_c K]}\Big)\right| \cdot \left|[M,_c K]\right|^{d\left(\frac{K}{Z_c(M,K)}\right)-1}.$$

Now by Lemma 2.1 and Corollary 2.4 we prove the last result.

Corollary 2.5. *Let (N,G) be a pair of finite p-groups such that N has a complement in G. Then*

$$|\mathcal{M}^{(c)}(N,G)| \leq \left|\mathcal{M}^{(c)}\Big(\frac{N}{[N,_c G]}, \frac{G}{[N,_c G]}\Big)\right| \left|[N,_c G]\right|^{d\left(\frac{G}{Z_c(N,G)}\right)-1}.$$

Proof. If $\sigma : M \to G$ is a c-cover of the pair (N,G), then there exists a group H such that $M \subseteq H$, and

$$\mathcal{M}^{(c)}(N,G) \cong \ker\sigma \subseteq [M,_c H] \cap Z_c(M,H).$$

And,

$$(N,G) \cong \Big(\frac{M}{\ker\sigma}, \frac{H}{\ker\sigma}\Big).$$

Put

$$(P,K) = \Big(\frac{M}{[M,_c H] \cap Z_c(M,H)}, \frac{H}{[M,_c H] \cap Z_c(M,H)}\Big).$$

Thus, by Lemma 2.1 and Corollary 2.4 we obtain

$$\begin{aligned}|\mathcal{M}^{(c)}(N,G)| &\leq |[M,_c H] \cap Z_c(M,H)| \\ &\leq \left|\mathcal{M}^{(c)}\Big(\frac{P}{[P,_c K]}, \frac{K}{[P,_c K]}\Big)\right| \left|[P,_c K]\right|^{d\left(\frac{K}{Z_c(P,K)}\right)-1} \\ &\leq \left|\mathcal{M}^{(c)}\Big(\frac{N}{[N,_c G]}, \frac{G}{[N,_c G]}\Big)\right| \left|[N,_c G]\right|^{d\left(\frac{G}{Z_c(N,G)}\right)-1}.\end{aligned}$$

□

In the following, we present some examples which satisfy in our results.

Example 2.6. Suppose that D and Q denote the dihedral and the quaternion group of order 8, also E_1 and E_2 denote the extra special p-groups of order p^3 of odd exponent p and p^2, respectively. Also, E_4 denotes the unique central product of a cyclic group of order p^2 and a non-abelian group of order p^3, and $Z_n^{(m)}$ denotes the direct product of m copies of Z_n. Then the following groups satisfy in our results;

(i) $G \cong N \times K$ where $N \cong Z_{p^2}$ and $K = 1$.
(ii) $G \cong N \times K$ where $N \cong E_4$ and $K = Z_p$.
(iii) $G \cong N \times K$ where $N \cong E_1 \times Z_p^{(3)}$ and $K = Z_p$.
(iv) $G \cong N \times K$ where $N \cong Q$ and $K = Z_p^{(2)}$.
(v) $G \cong N \times K$ where $N \cong D \times Z_2$ and $K = Z_p^{(2)}$.
(vi) $G \cong N \times K$ where $N \cong E_2$ and $K = Z_p^{(2)}$
(vii) $G \cong N \times K$ where $N \cong D \times Z_2$ and $K = Z_p^{(2)}$

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