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Strongly hollow elements in the lattices

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Abstract. Let L be a lattice with the greatest element 1. Following the concept of strongly hollow elements of commutative rings, we define strongly hollow elements of lattices and we will make an intensive investigate the basic properties and possible structures of these elements.

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1. Introduction

Let M be a module over a commutative ring R. An R-submodule N of M is said to be irreducible if N is not the intersection of two submodules of M that properly contain it. An ideal I of R which is irreducible if it is irreducible as a submodule of the R-module R. Heinzer et al. in [7], generalized the concept of irreducible ideals as follows: a proper ideal I of R is said to be strongly irreducible if for ideals J and K of R $J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$. The notion of strongly irreducible submodules was introduced and studied in [4]. A submodule N of an R-module M is said to be strongly irreducible if for submodules N_1 and N_2 of M, the inclusion $N_1 \cap N_2 \subseteq N$ implies that either $N_1 \subseteq N$ or $N_2 \subseteq N$. A non-zero submodule N of M is strongly hollow in M

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if for any submodules N_1 and N_2 of M, if $N \subseteq N_1 + N_2$, then either $N \subseteq N_1$ or $N \subseteq N_2$, also a non-zero submodule N of M is completely hollow in M if for any non-empty family $\{N_i\}_{i\in J}$ of submodules of M, if $N = \sum_{i\in J} N_i$, then there is $j \in J$ such that $N = N_j$. A non-zero ideal I of R is strongly hollow (resp. completely hollow) whenever I is a strongly hollow (resp. completely hollow) submodule of the R-module R. The notion of strongly hollow submodules was introduced and studied in [1] as a dual notion of strongly irreducible submodules. The notion of strongly hollow ideals (resp. strongly hollow elements) was introduced and studied in [8].

Let L be a distributive lattice with 1. In the present paper, we are interested in investigating strongly hollow elements of L to use other notions of strongly hollow, and associate which exist in the literature as laid forth in [8]. Here, we extend several concepts from module theory to lattice theory. With a careful generalization, we can cover some basic corresponding results in the former setting. The main difficulty is figuring out what additional hypotheses the lattice or filter must satisfy to get similar results. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [2, 3, 5, 6]). We shortly summarize the content of the paper. In Section 2, the notion of completely strongly hollow filters (as a generalization of strongly hollow filters) and strongly hollow elements is introduced and some related properties are investigated. Also, we characterize completely strongly hollow filters and strongly hollow elements of distributive lattices.

Let us recall some notions and notations [2]. By a lattice we mean a poset $(L, \leq)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter F of L is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of L is said to be maximal if G is a filter in L with $F \subsetneqq G$, then $G = L$. If F is a filter of a lattice L, then the radical of F, denoted by $\mathrm{rad}(F)$, is the intersection of all maximal subfilters of F. If A is a subset of a lattice L, then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A. A filter F is called finitely generated if there is a finite subset A of F such that $F = T(A)$.

Lemma 1.1. *[5] Let L be a lattice.*

(1) A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.

(2) If $F_1, \cdots, F_n$ are filters of L and $a \in L$, then $\vee_{i=1}^n F_i = \{\vee_{i=1}^n a_i : a_i \in F_i\}$ and $a \vee F_i = \{a \vee a_i : a_i \in F_i\}$ are filters of L and $\vee_{i=1}^n F_i = \bigcap_{i=1}^n F_i$.

(3) If L is distributive, F, G are filters of L, and $x \in L$, then $(G :_L F) = \{x \in L : x \vee F \subseteq G\}$, $(F :_L T(x)) = (F :_L x) = \{a \in L : a \vee x \in F\}$ and $(1 :_L F) = \{x \in L : x \vee F = 1\}$ are filters of L.

(4) If $\{F_i\}_{i \in \Delta}$ is a chain of filters of L, then $\cup_{i \in \Delta} F_i$ is a filter of L.

Lemma 1.2. *[6] Let A be an arbitrary non-empty subset of L. Then $T(A) = \{x \in L : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in A$ $(1 \leq i \leq n)\}$. Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$.*

2. Basic properties of strongly hollow elements

Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. In this section, we collect some basic properties concerning strongly hollow elements of lattices. We begin with the key definitions of this paper.

Definition 2.1. Let $F \neq \{1\}$ be a filter of L.

(1) F is called strongly hollow in L if for any filters F_1 and F_2 of L, if $F \subseteq T(F_1 \cup F_2)$, then either $F \subseteq F_1$ or $F \subseteq F_2$.

(2) F is called completely hollow in L if for any non-empty family $\{F_i\}_{i \in \Lambda}$ of filters of L, if $F = T(\cup_{i \in \Lambda} F_i)$, then there is $j \in \Lambda$ such that $F = F_j$.

(3) F is called completely strongly hollow in L if for any non-empty family $\{F_i\}_{i \in \Lambda}$ of filters of L, if $F \subseteq T(\cup_{i \in \Lambda} F_i)$, then there is $j \in \Lambda$ such that $F \subseteq F_j$.

Lemma 2.2. *Let $F \neq \{1\}$ be a filter of L.*

(1) If F is a completely strongly hollow, then there is $a \in F$ such that $F = T(\{a\})$.

(2) Let F be a finitely generated filter. Then F is completely strongly hollow if and only if it is strongly hollow.

Proof. (1) An inspection will show that $F = T(\cup_{x \in F} T(\{x\}))$. By assumption, there is $a \in F$ such that $F \subseteq T(\{a\})$; hence $F = T(\{a\})$.

(2) By definition, if F is completely strongly hollow, then it is strongly hollow. Conversely, assume that F is a strongly hollow filter and let $F \subseteq T(\cup_{i\in\Lambda} F_i)$. By assumption, there are elements $a_1, \cdots, a_n \in F$ such that $F = T(\{a_1, \cdots, a_n\})$. As $a_1, \cdots, a_n \in T(\cup_{i\in\Lambda} F_i)$, there exist $i_1, \cdots, i_n \in \Lambda$ such that $F \subseteq T(F_{i_1} \cup \cdots \cup F_{i_n})$; hence $F \subseteq F_{i_j}$ for some j, as required.

□

Definition 2.3. An element $1 \neq a$ of a lattice L is said to be strongly hollow in L if the filter $T(\{a\})$ of L is a (completely) strongly hollow filter of L.

Remark 2.4. (1) Let F be a filter of a lattice L with $F \neq L$. Since the filter F is proper, $\sum = \{G : G \text{ is a filter of L with } F \subseteq G, G \neq L\} \neq \emptyset$. Moreover, $(\sum, \subseteq)$ is a partial order. Clearly, $\sum$ is closed under taking unions of chains and so F contained in a maximal filter of L by Zorn's Lemma.

(2) A lattice L is called a chain lattice if all its filters form a chain under inclusion. Assume that F is a finitely generated filter of a chain lattice L with $F \neq \{1\}$ and let $F \subseteq T(G \cup H)$ for some filters G, H of L. Then either $F \subseteq T(G) = G$ or $F \subseteq T(H) = H$; hence F is completely strongly hollow by Lemma 2.2. Moreover, every element $a \neq 1$ of a chain lattice L is strongly hollow.

Proposition 2.5. *Let L be a lattice. The following hold:*

(1) If F is a finitely generated (completely) strongly hollow filter of L, then the set $\sum = \{G : G$ is a filter of L such that $G \subsetneqq F\}$ has exactly one maximal element with respect to the inclusion.

(2) Let L be a lattice with 0. L is a completely strongly hollow filter if and only if L has exactly one maximal filter.

Proof. (1) Since $\{1\} \in \sum$, $\sum \neq \emptyset$. Of course, the relation of inclusion is a partial order on $\sum$. Now *sum* easily seen to be inductive under inclusion, so by Zorn's lemma $\sum$ has a maximal element H with $H \subsetneqq F$. Let H and H' be maximal elements of $\sum$ with $H \neq H'$ which implies that there exists $x \in H \setminus H'$. Then $H' \subsetneqq T(H' \cup T(\{x\})) \subseteq F$; so $F = (H' \cup T(\{x\}))$. By assumption, $F \subseteq H'$ or $F \subseteq T(\{x\}) \subseteq H$ which is impossible. Thus $H = H'$.

(2) Let L be a completely strongly hollow filter of L and set

$$\sum = \{G : G \text{ is a filter of } L \text{ such that } G \subsetneqq L\}.$$

Then by (1), L has exactly one maximal filter. Conversely, assume that P is the unique maximal filter of L and let $\{F_i\}_{i\in\Lambda}$ be a non-empty family of filters of L such that $L \subseteq T(\cup_{i\in\Lambda} F_i)$. Now suppose that for each $i \in \Lambda$, $L \nsubseteq F_i$. As $L = T(\{0\})$, there exist $F_{i_1}, \cdots, F_{i_n}$ such that $L \subseteq T(F_{i_1} \cup \cdots \cup F_{i_n}) \subseteq P$ which is impossible, as required. □

A simple (minimal) filter is a filter that has no filters besides the $\{1\}$ and itself.

Proposition 2.6. *Let L be a lattice. The following hold:*

(1) Let G and H be two completely (strongly) hollow filters of L. Then $T(G \cup H)$ is completely (strongly) hollow if and only if either $G \subseteq H$ or $H \subseteq G$.

(2) If F is a minimal filter of L, then F is a (completely) strongly hollow filter. In this case, every element $a \neq 1$ of F is strongly hollow.

Proof. (1) Let $T(G \cup H)$ be completely (strongly) hollow. Then $T(H \cup G) \subseteq T(G \cup H)$ gives either $G \subseteq T(G \cup H) \subseteq H$ or $H \subseteq T(G \cup H) \subseteq G$. The other implication is clear.

(2) Let $F \subseteq T(G \cup H)$ for some filters G and H of L. We show that either $F \subseteq G$ or $F \subseteq H$. Assume to the contrary, $F \nsubseteq G$ and $F \nsubseteq H$. Since $F \vee G = F \cap G \subseteq F$, $F \nsubseteq G$ and F is minimal, we have $F \vee G = \{1\}$; hence $G \subseteq (1 :_L F)$. Similarly, $H \subseteq (1 :_L F)$. Let $x \in T(G \cup H)$. Then $x = (x \vee g) \wedge (x \vee h)$ for some $g \in G$ and $h \in H$. Since $x \vee g \in G$ and $x \vee h \in H$, we get that $x \in (1 :_L F)$; hence $F \subseteq T(G \cup H) \subseteq (1 :_L F)$ which implies that $F \vee F = F = \{1\}$, a contradiction. Thus F is a (completely) strongly hollow filter.

□

We next give two other characterizations of strongly hollow elements.

Theorem 2.7. *Let $1 \neq a$ be an element of a lattice L. Then the following are equivalent:*

(1) a is strongly hollow;

(2) If $a = b \wedge c$ for some $b, c \in L$, then either $T(\{a\}) \subseteq T(\{b\})$ or $T(\{a\}) \subseteq T(\{c\})$;

(3) If $a = b \wedge c$ for some $b, c \in L$, then either $T(\{b\}) \subseteq T(\{c\})$ or $T(\{c\}) \subseteq T(\{b\})$.

Proof. (1) $\Rightarrow$ (2) If $a = b \wedge c$ for some $b, c \in L$, then $a \in T(T(\{b\}) \cup T(\{c\}))$ which implies that $T(\{a\}) \subseteq T(T(\{b\}) \cup T(\{c\}))$; hence $T(\{a\}) \subseteq T(\{b\})$ or $T(\{a\}) \subseteq T(\{c\})$ by (1).

(2) $\Rightarrow$ (3) Suppose that $a = b \wedge c$ for some $b, c \in L$. Then either $T(\{a\}) \subseteq T(\{b\})$ or $T(\{a\}) \subseteq T(\{c\})$. If $T(\{a\}) \subseteq T(\{b\})$, then $a \in T(\{b\})$; so $a = b \vee t$ for some $t \in L$. It follows that $c = c \vee (b \wedge c) = c \vee (b \vee t) = b \vee (c \vee t) \in T(\{b\})$; hence $T(\{c\}) \subseteq T(\{b\})$. A similar argument works for the case $T(\{b\}) \subseteq T(\{c\})$.

(3) $\Rightarrow$ (1) Assume that $T(\{a\}) \subseteq T(G \cup H)$ for some filters G and H of L. There exist $g \in G$ and $h \in H$ such that $a = a \vee (g \wedge h) = (a \vee g) \wedge (a \vee h)$.

By (3), either $T(\{a \vee g\}) \subseteq T(\{a \vee h\})$ or $T(\{a \vee h\}) \subseteq T(\{a \vee g\})$. Suppose that $T(\{a \vee g\}) \subseteq T(\{a \vee h\})$. Then $a \vee h, a \vee g \in T(\{a \vee h\})$ gives $a \in T(\{a \vee h\})$; hence $T(\{a\}) \subseteq T(\{a \vee h\}) \subseteq H$. A similar argument works for the case $T(\{a \vee h\}) \subseteq T(\{a \vee g\})$. Thus a is a strongly hollow element. □

Proposition 2.8. *Let a be an element of L with $a \neq 1$. If a is a strongly hollow element of L, then either $a \in \mathrm{rad}(L)$ or there exists exactly one maximal filter of L not containing a.*

Proof. If $a \in \mathrm{rad}(L)$, we are done. Suppose that $a \notin \mathrm{rad}(L)$. Then there exists a maximal filter P of L such that $a \notin P$. Let P' be a maximal filter of L such that $P \neq P'$ and $a \notin P'$. Then $T(P \cup P') = L$ gives $a = a \vee (p \wedge p') = (a \vee p) \wedge (a \vee p')$ for some $p \in P$ and $p' \in P'$. By Theorem 2.7, either $T(\{a\}) \subseteq T(\{a \vee p\}) \subseteq P$ or $T(\{a\}) \subseteq T(\{a \vee p'\}) \subseteq P'$ which is impossible. □

Remark 2.9. Let F be a filter of L. Set

$$S_F = \{G : G \text{ is a filter of } L \text{ such that } F \nsubseteq G\}$$

and $\Gamma_F = T(\cup_{G \in S_F} G)$. It is easy to see that $\Gamma_a = \Gamma_{T(\{a\})}$, where $\Gamma_a = \Gamma_{\{a\}}$.

Theorem 2.10. *Let F be a finitely generated filter of L with $F \neq \{1\}$. Then F is completely strongly hollow if and only if there exists the greatest filter of L with respect to not containing F, namely Γ_F.*

Proof. Let F be a completely strongly hollow filter of L. Then $F = T(\{a\})$ for some $a \in F$ by Lemma 2.2. If $F \subseteq \Gamma_F$, then by definition of Γ_F, there exist $F \subsetneqq F_{i_1}, \cdots, F \subsetneqq F_{i_m}$ such that $F \subseteq T(F_{i_1} \cup \cdots \cup F_{i_m})$. By assumption, $F \subseteq F_{i_j}$ for some j which is impossible. Thus $F \nsubseteq \Gamma_F$. Hence by the definition of Γ_F, Γ_F is the greatest filter of L with respect to not containing F. Conversely, assume that there exists the greatest filter of L with respect to not containing F, say H and let $F \subseteq T(\cup_{i \in \Lambda} F_i)$, where $\{F_i\}_{i \in \Lambda}$ is a non-empty family of filters of L. Now Assume that for each $i \in \Lambda$, $F \nsubseteq F_i$. Let $i \in \Lambda$ be fixed. We put

$$S_i = \{G : G \text{ is a filter of } L \text{ such that } F_i \subseteq G \text{ and } F \nsubseteq G\}.$$

Since F is finitely generated, every non-empty chain of the poset $(S_i, \subseteq)$ has an upper bound in S_i; hence S_i has a maximal element by Zorn's lemma, say H_i. Since every filter containing H_i contains F_i, H_i is a filter of L maximal with respect to not containing F; so $H = H_i$. It follows that for each $i \in \Lambda$, $F_i \subseteq H$, and so $F \subseteq H$ which is a contradiction. Thus F is a completely strongly hollow filter of L. □

Definition 2.11. A filter F of L is called completely strongly irreducible if $\{F_i\}_{i\in\Lambda}$ is a non-empty family of filters of L such that $\cap_{i\in\Lambda}F_i \subseteq F$, then there exists $j \in \Lambda$ such that $F_j \subseteq F$.

Proposition 2.12. *Let $a \in L$ with $a \neq 1$. Then a is a strongly hollow element if and only if $a \notin \Gamma_a$. In this case, $\Gamma_a = \{x \in L : a \notin T(\{x\})\}$.*

Proof. Let a be a strongly hollow element of L. Then by Theorem 2.10 and Remark 2.9, $\Gamma_a = \Gamma_{T(\{a\})}$ is the greatest filter of L with respect to not containing $T(\{a\})$; so $a \notin \Gamma_a$. Conversely, assume that $a \notin \Gamma_a$ and $a = b \wedge c$ for some $b, c \in L$. If $T(\{a\}) \nsubseteq T(\{b\})$ and $T(\{a\}) \nsubseteq T(\{c\})$, then $a \notin T(\{b\})$ and $a \notin T(\{c\})$; hence $b, c \in \Gamma_a$ which implies that $a \in \Gamma_a$, a contradiction. Thus either $T(\{a\}) \subseteq T(\{b\})$ or $T(\{a\}) \subseteq T(\{c\})$; hence a is a strongly hollow element of L by Theorem 2.7. □

Lemma 2.13. *Let $a, b \in L$ with $a \neq 1$ and $b \neq 1$. Then the following hold:*

(1) If a is a strongly hollow element of L, then Γ_a is a completely strongly irreducible filter of L.

(2) If a, b are strongly hollow elements, then $T(\{a\}) \subseteq T(\{b\})$ if and only if $\Gamma_a \subseteq \Gamma_b$.

Proof. (1) Let $\{F_i\}_{i\in\Lambda}$ be a non-empty family of filters of L such that $\cap_{i\in\Lambda}F_i \subseteq \Gamma_a$. Therefore, $a \notin \cap_{i\in\Lambda}F_i$ by Proposition 2.12; hence there is an element $j \in \Lambda$ such that $a \notin F_j$ and so $F_j \subseteq \Gamma_a$.

(2) Assume that $T(\{a\}) \subseteq T(\{b\})$ and let $x \in \Gamma_a$. Then $a \notin T(\{x\})$ by Proposition 2.12. If $x \notin \Gamma_b$, then $b \in T(\{x\})$; hence $a \in T(\{a\}) \subseteq T(\{b\}) \subseteq T(\{x\})$ which is impossible. Thus $x \in \Gamma_b$, and so $\Gamma_a \subseteq \Gamma_b$. The other implication is similar. □

Proposition 2.14. *Let a and b be two strongly hollow elements of L. Then the following are equivalent:*

(1) The filter $T(\{a, b\})$ is a completely strongly hollow filter of L;

(2) Either $\Gamma_a \subseteq \Gamma_b$ or $\Gamma_b \subseteq \Gamma_a$;

(3) Either $T(\{a\}) \subseteq T(\{b\})$ or $T(\{b\}) \subseteq T(\{a\})$.

Proof. (1) ⇒ (2) Let $T(\{a, b\})$ be a completely strongly hollow filter of L. Assume to the contrary, $\Gamma_a \nsubseteq \Gamma_b$ and $\Gamma_b \nsubseteq \Gamma_a$. Then there exist $x \in \Gamma_a \setminus \Gamma_b$ (so by Proposition 2.12, $a \notin T(\{x\})$ and $b \in T(\{x\})$) and $y \in \Gamma_b \setminus \Gamma_a$ (so $b \notin T(\{y\})$ and $a \in T(\{y\})$). Then $a \in T(\{y\}) \setminus T(\{x\})$ and $b \in T(\{x\}) \setminus T(\{y\})$. Let $z \in T(\{a, b\})$. Then $z = z \vee (a \wedge b) = (z \vee a) \wedge (z \vee b) \in T(T(\{x\}) \cup T(\{y\}))$; hence $T(\{a, b\}) \subseteq T(T(\{x\}) \cup T(\{y\}))$, but $T(\{a, b\}) \nsubseteq T(\{x\})$ and $T(\{a, b\}) \nsubseteq T(\{y\})$, a contradiction.

(2) ⇒ (3) It follows by Lemma 2.13 (2).

(3) ⇒ (1) Let either $T(\{a\}) \subseteq T(\{b\})$ or $T(\{b\}) \subseteq T(\{a\})$. Without loss of generality, we can assume that $T(\{a\}) \subseteq T(\{b\})$. Let $T(\{a, b\}) \subseteq$

$T(\cup_{i\in\Lambda}F_i)$, where $\{F_i\}_{i\in\Lambda}$ is a non-empty family of filters of L. By assumption, $T(\{b\}) \subseteq T(\{a,b\})$ gives $T(\{b\}) \subseteq F_j$ for some $j \in \Lambda$ which implies that $T(\{a,b\}) \subseteq F_j$, as required.

□

Let a be an element of L. Set $U_a = (\Gamma_a : a) = \{x \in L : x \vee a \in \Gamma_a\}$. Now we consider the behavior of strongly hollow elements under quotient lattice.

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(L, \leq)$, we define a relation on L, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on L, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leq_Q$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. It is straightforward to check that $(\frac{L}{F}, \leq_Q)$ is a poset. The following notation below will be kept in this section: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X = \{a \wedge F, b \wedge F\}$. By definition of $\leq_Q$, $(a \vee b) \wedge F$ is an upper bound for the set X. If $c \wedge F$ is any upper bound of X, then we can easily show that $(a \vee b) \wedge F \leq_Q c \wedge F$. Thus $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$. Thus $(\frac{L}{F}, \leq_Q)$ is a lattice.

Remark 2.15. Let G be a a subfilter of a filter F of L.

(1) If $a \in F$, then $a \wedge F = F$. By the definition of $\leq_Q$, it is easy to see that $1 \wedge F = F$ is the greatest element of $\frac{L}{F}$.

(2) If $a \in F$, then $a \wedge F = b \wedge F$ (for every $b \in L$) if and only if $b \in F$. In particular, $c \wedge F = F$ if and only if $c \in F$. Moreover, if $a \in F$, then $a \wedge F = F = 1 \wedge F$.

(3) By the definition $\leq_Q$, we can easily show that if L is distributive, then $\frac{L}{F}$ is distributive.

(4) $\frac{F}{G} = \{a \wedge G : a \in F\}$ is a filter of $\frac{L}{G}$.

(5) If K is a filter of $\frac{L}{G}$, then $K = \frac{F}{G}$ for some filter F of L.

(6) If H is a filter of L such that $G \subseteq H$ and $\frac{F}{G} = \frac{H}{G}$, then $F = H$.

(7) If H and V are filters of L containing G, then $\frac{F}{G} \cap \frac{H}{G} = \frac{V}{G}$ if and only if $V = H \cap F$.

(8) If H is a filter of L containing G, then $\frac{T(F \cup H)}{G} = T(\frac{H}{G} \cup \frac{F}{G})$.

(9) Let H be a subfilter of F with $G \subseteq H$. H is a maximal subfilter of F if and only if $\frac{H}{G}$ is a maximal subfilter of $\frac{F}{G}$.

Proposition 2.16. *Assume that a is a strongly hollow element of L and let F be a filter of L such that $a \notin F$. Then the following hold:*

(1) $a \wedge F$ is a strongly hollow element of the lattice $\frac{L}{F}$.

(2) $\Gamma_{a\wedge F} = \frac{\Gamma_a}{F}$.
(3) $\frac{T(U_a\cup F)}{F} = U_{a\wedge F}$.

Proof. (1) Since $a \notin F$, $a\wedge F \neq F$. Let $x \in F$. If $a \in T(\{x\})$, then there exists $c \in L$ such that $a = a\vee c \in F$, a contradiction. Thus $a \notin T(\{x\})$; hence $F \subseteq \Gamma_a$. Let $a \wedge F = (b\wedge F) \wedge_Q (c\wedge F) = (b\wedge c)\wedge F$ for some $b, c \in L$. Since $a = a\wedge 1 \in (b\wedge c)\wedge F$, $a = (b\wedge c)\wedge f = b\wedge(c\wedge f)$ for some $f \in F$. By Theorem 2.7, we have either $T(\{a\}) \subseteq T(\{b\})$ or $T(\{a\}) \subseteq T(\{c\wedge f\})$ which implies that either $a = a\vee b$ or $a = a\vee(c\wedge f)$. Thus either $a\wedge F = (a\vee b)\wedge F = (a\wedge F)\vee_Q (b\wedge F) \in T(\{b\wedge F\})$ (so $T(\{a\wedge F\}) \subseteq T(\{b\wedge F\})$) or $a\wedge F = (a\vee(c\wedge f))\wedge F = (a\wedge F)\vee_Q (c\wedge f)\wedge F = (a\wedge F)\vee_Q (c\wedge F) \in T(\{c\wedge F\})$ (so $T(\{a\wedge F\}) \subseteq T(\{c\wedge F\})$) in the lattice $\frac{L}{F}$. Thus $a\wedge F$ is a strongly hollow element of the lattice $\frac{L}{F}$.

(2) Let $x\wedge F \in \Gamma_{a\wedge F}$. If $x \notin \Gamma_a$, then $a \in T(\{x\})$ by Proposition 2.12; so $a = a\vee x$. Then $a\wedge F = (a\vee x)\wedge F = (a\wedge F)\vee_Q (x\wedge F) \in T(\{x\wedge F\})$ which contradicts Proposition 2.12. Thus $x \in \Gamma_a$ and so $\Gamma_{a\wedge F} \subseteq \frac{\Gamma_a}{F}$. For the reverse inclusion, assume that $t\wedge F \in \frac{\Gamma_a}{F}$. Since $a \notin \Gamma_a$, we have $a\wedge F \notin \frac{\Gamma_a}{F}$. If $t\wedge F \notin \Gamma_{a\wedge F}$, then $a\wedge F \in T(\{t\wedge F\})$; so $a\wedge F = (a\wedge F)\vee_Q (t\wedge F) \in \frac{\Gamma_a}{F}$, a contradiction. Hence $\frac{\Gamma_a}{F} \subseteq \Gamma_{a\wedge F}$, and so we have equality.

(3) By (2), we have $U_{a\wedge F} = \{x\wedge F : (a\wedge F)\vee_Q (x\wedge F) \in \Gamma_{a\wedge F}\} =$

$$\{x\wedge F : (a\vee x)\wedge F \in \Gamma_{a\wedge F} = \frac{\Gamma_a}{F}\} =$$

$\{x\wedge F : a\vee x \in \Gamma_a\} = \{x\wedge F : x \in U_a\} = \frac{T(U_a\cup F)}{F}$. □

A filter F of L will be called a L-second filter provided $F \neq 1$ and $(1 :_L F) = (G :_L F)$ for every proper subfilter G of F [5]. We need the following proposition proved in [5 Proposition 2.1].

Proposition 2.17. *Let $F \neq 1$ be a filter of L. Then the following hold:*
(1) F is L-second if and only if for each a in L, either $a\vee F = \{1\}$ or $a\vee F = F$.
(2) F is L- second if and only if it is a minimal filter.

Corollary 2.18. *If F is a L- second filter of L, then F is a strongly hollow filter.*

Proof. Let F be a L- second filter of L. Then F is minimal, by Proposition 2.17. Therefore it is strongly hollow by Proposition 2.6. □

Theorem 2.19. *Let a be a strongly hollow element of L. Then the following hold:*
(1) $T(\{a\wedge\Gamma_a\})$ is a minimal filter of the lattice $\frac{L}{\Gamma_a}$.

(2) $a \wedge \Gamma_a$ is a co-atom in the lattice $\frac{L}{\Gamma_a}$.

Proof. (1) Let a be a strongly hollow element of L. Then $a \notin \Gamma_a$ by Proposition 2.12. Therefore, $a \wedge \Gamma_a$ is a strongly hollow element of the lattice $\frac{L}{\Gamma_a}$ by Proposition 2.16. Let $x \wedge \Gamma_a$ be an element of the lattice $\frac{L}{\Gamma_a}$ such that $T(\{x \wedge \Gamma_a\}) \subsetneqq T(\{a \wedge \Gamma_a\})$. Then $a \notin T(\{x\})$, and hence $x \in \Gamma_a$, and so $T(\{x \wedge \Gamma_a\}) = \{\Gamma_a\} = \{1_{\frac{L}{\Gamma_a}}\}$. It means that $T(\{a \wedge \Gamma_a\})$ is a minimal filter of the lattice $\frac{L}{\Gamma_a}$.

(2) Let $a \wedge \Gamma_a \leq_Q b \wedge \Gamma_a$ for some $b \in L \setminus \Gamma_a$. Then $b \wedge \Gamma_a \in T(\{a \wedge \Gamma_a\})$. By (1), $T(\{a \wedge \Gamma_a\})$ is a minimal filter of the lattice $\frac{L}{\Gamma_a}$. Hence $T(\{a \wedge \Gamma_a\}) = T(\{b \wedge \Gamma_a\})$. Therefore $a \wedge \Gamma_a \in T(\{b \wedge \Gamma_a\})$, and so $b \wedge \Gamma_a \leq_Q a \wedge \Gamma_a$. Therefore $a \wedge \Gamma_a = b \wedge \Gamma_a$. □

Theorem 2.20. *Let a be a strongly hollow element of a lattice L with 0. Then $\Gamma_a = U_a$ is a prime filter of L.*

Proof. We will show that Γ_a is a prime filter. Let $x \vee y \in \Gamma_a$ and $x, y \notin \Gamma_a$, for some $x, y \in \Gamma_a$. By Proposition 2.12, $a \in T(\{x\})$ and $a \in T(\{y\})$. Hence $x \leq a$ and $y \leq a$. Therefore $x \vee y \leq a$ and so $a \in T(\{x \vee y\})$, a contradiction with $x \vee y \in \Gamma_a$. Therefore Γ is prime. Now, we will show $\Gamma_a = U_a$. Since $U_a = (\Gamma_a : a) = \{x \in L : x \vee a \in \Gamma_a\}$, and Γ_a is a filter by Proposition 2.12, $c \vee a \in \Gamma_a$, for each $c \in \Gamma_a$. Thus $\Gamma_a \subseteq U_a$. Let $b \in U_a$. Then $b \vee a \in \Gamma_a$. By Proposition 2.12, $a \notin \Gamma_a$. Since Γ_a is prime, we have $b \in \Gamma_a$. Therefore $U_a \subseteq \Gamma_a$, and so $U_a = \Gamma_a$. □

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