Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **10**(1)(2021), 93-103

Existence and uniqueness of solutions for neutral periodic integro-differential equations with infinite delay on time scale

Haitham A. Makhzoum [1] , Abdelhamid S. Elmabrok [2] and Rafik. A. Elmansouri [3]
[1,2] Department of Mathematic, University of Benghazi, Libya,
[3] College of Electrical and Electronic Technology, Benghazi, Libya.

Abstract. In this article, we will shed the light on the following nonlinear neutral dynamic equation with infinite delay

$$\begin{aligned} x(t)^{\Delta} = \quad & G(t,\ x(t), x(t-\tau(t))) + Q(t, x(t-\tau(t)))^{\Delta} \\ & + \int_{-\infty}^{t} \left(\sum_{i=1}^{p} D_i(t,s) \right) f(x(s)) \Delta s, \end{aligned}$$

where $\mathbb{T}$ is a periodic time scale. Using the fixed-point method by Krasnoselskii, we will show that equation has a periodic solution. In addition, we will prove this solution is unique by using the contraction mapping principle.

Keywords: Fixed point, infinite delay, time scales, periodic solution.

2010 Mathematics subject classification: Primary 34K13, 34N05 ; Secondary 34L30, 45J05.

1. Introduction

Over the past years, Researchers have used different methods to show the existence of solutions of numerous types for nonlinear differential equations one of these methods is fixed-point theory. Accordingly, many

[1] Corresponding author: haitham.makhzoum@uob.edu.ly
Received: 20 September 2020
Revised: 16 December 2020
Accepted: 10 January 2021

articles has been published in this field, and we refer readers to [11,14,15]. On the other hand, Fixed point has been revealed as a very strong and important method for the study of nonlinear neutral dynamic equation. Recently, in [17] by Yankson, the existence and uniqueness of solutions for the neutral periodic integro- differential equation with infinite delay given by

$$\begin{aligned}\frac{d}{dt}x\left(t\right) = \quad & G(t, x\left(t\right), x\left(t-\tau\left(t\right)\right)) + \frac{d}{dt}\mathrm{Q}\left(t, x\left(t-\tau\left(t\right)\right)\right) \\ & + \int_{-\infty}^{t}\left(\sum_{i=1}^{p} D_i\left(t,s\right)\right) f\left(x\left(s\right)\right) ds.\end{aligned} \tag{1.1}$$

was established by using the fixed-point method by Krasnoselskiiand and the contraction mapping principle respectively.

In this paper, we will present the following neutral periodic integro-differential equations with infinite delay

$$\begin{aligned}x(t)^{\Delta} = \quad & G(t, x\left(t\right), x\left(t-g\left(t\right)\right)) + Q(t, x(t-g\left(t\right)))^{\Delta} \\ & + \int_{-\infty}^{t}\left(\sum_{i=1}^{p} D_i\left(t,s\right)\right) f\left(x\left(s\right)\right)\Delta s,\end{aligned} \tag{1.2}$$

by assuming that $G : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous real-valued function, taking into consideration $Q : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $D_i : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $f : \mathbb{R}\to\mathbb{R}$ are continuous functions, and to ensure periodicity the following assumption has been made $g\left(t\right)$, $D_i\left(t,x\right)$ and $Q\left(t,x\right)$ are periodic functions.

We are interested to study the existence of periodic solutions of Eq (1.2) on the Time scale space $\mathbb{T}$. Time scale is a relatively new subject it has been presented by the following definition a time scale $\mathbb{T}$ is a closed non-empty subset of $\mathbb{R}$. The main point of this space is unifying the theory of difference with differential equations.

Let $0\in\mathbb{T}$, $g : \mathbb{T}\to\mathbb{R}$ and $id-g : \mathbb{T}\to\mathbb{T}$ is strictly increasing, this leads that $x\left(t-g\left(t\right)\right)$ is well-defined over $\mathbb{T}$. The work is inspired and motivated by the works done by Ardjouni and Djoudi [2], and for more details on this subject, we refer the reader to [3]- [9] and [12]. To achieve the intended result we have to follow the requirements of Krasnoselskii's fixed point where the theory asks for $z = Az + Bz$ yields $z\in M$, where M is a convex set, Az is continuous and compact, Bz is a contraction. The methodology used in this paper is transformed Eq(1.2) into an integral equation that allows us to create two mappings and it is the condition of the fixed point theorem of Krasnoselskii and this done in Lemma 3.5. Afterwards, we proved that Az is continuous and compact, Bz is a contraction. It helped us to implement Krasnoselskii's theorem and to grant us to prove the existence of periodic solutions. In the end, we show the uniqueness of the periodic solution by the use of the contraction mapping principle.

This paper is structured as follows. In Section 2, we present outlines some preliminary background material to be used in the upcoming sections. Also, some facts will provide about the exponential function on a time scale well. The main result has been presented in Section 3.

2. Preliminaries

This section focus to provide the significant notations which related to concepts concerning the calculus on time scales for dynamic equations mostly all definitions, lemmas and theorems can be found in Bohner and Peterson books [8, 9].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ and the backward jump operator ρ, respectively, are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \textit{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

These operators allow elements in the time scale to be classified as follows. We say t is

(1) *right scattered* if $\sigma(t) > t$,
(2) *right dense* if $\sigma(t) = t$,
(3) *left scattered* if $\rho(t) < t$,
(4) *left dense* if $\rho(t) = t$.

The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

(1) If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$.
(2) If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted by $f^{\Delta}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \epsilon \left| \sigma(t) - s \right|,$$

for all $s \in U$. For example,

(1) If $\mathbb{T} = \mathbb{R}$ then $f^{\Delta}(t) = \grave{f}(t)$ is the usual derivative.
(2) If $\mathbb{T} = \mathbb{Z}$ then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now able to state some properties of the delta-derivative of f. Note that $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1. [8] *Assume that, $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(1) $\{(f+g)\}^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$,
(2) $\{(\alpha f)\}^{\Delta}(t) = \alpha f^{\Delta}(t)$,
(3) $\{(fg)\}^{\Delta}(t) = f^{\Delta}(t)\, g(t) + f^{\sigma}(t)\, g^{\Delta}(t)$,
(4) $\left\{\left(\frac{f}{g}\right)\right\}^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}$, *with* $g(t)g^{\sigma}(t) \neq 0$.

The next two theorems deal with the composition of two functions. The first one is the chain rule on time scales [8, Theorem 1.93].

Theorem 2.2. *(Chain Rule). Assume, $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$ and $w^{\grave{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then $\{(w \circ v)\}^{\Delta} = \left(\{w\}^{\acute{\Delta}} \circ v\right) v^{\Delta}$.*

In the sequel, we will need to differentiate and integrate functions of the form $f(t - g(t)) = f(v(t))$, where $f(v(t)) = t - g(t)$. The second theorem is the substitution rule [8, Theorem 1.98].

Theorem 2.3. *(Substitution). Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\, v^{\Delta}(t)\, \Delta t = \int_{v(a)}^{v(b)} f \circ v^{-1}(s) \widetilde{\Delta} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by

$$\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)\, f(t) > 0 \quad \texttt{for all} \quad t \in \mathbb{T}\}.$$

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \left[\frac{1}{\mu(z)} \log(1 + \mu(z)\, p(z))\right] \Delta z\right). \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,\ s)$ is the solution to the initial value problem $y^{\Delta} = p(t)\, y,\ y(s) = 1$. Other properties of the exponential function are given in the following Lemma [8, Theorem 2.36].

Lemma 2.4. *Let* $p,\ q \in \mathcal{R}$. *Then*

(1) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,
(2) $e_p(\sigma(t), s) = (1 + \mu(t)\, p(t)) e_p(t,s)$,
(3) $\frac{1}{e_p(t,s)} = e_{\ominus}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,
(4) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,
(5) $e_p(t,s)\, e_p(s,r) = e_p(t,r)$,
(6) $\left\{\left(\frac{1}{e_p(\cdot\ ,s)}\right)\right\}^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$.

The notion of periodic time scales and the next two definitions are quoted from [7] and [13].

Definition 2.5. We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$, such that, $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p with this property is called the period of the time scale.

Example 2.6. The following time scales are periodic.

(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$ has period $p = 2h$,
(2) $\mathbb{T} = hZ$ has period $p = h$,
(3) $\mathbb{T} = \mathbb{R}$,
(4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z},\ m \in \mathbb{N}_0\}$, where $0 < q < 1$ has period $p = 1$.

Remark 2.7. [13] All periodic time scales are unbounded above and below.

Definition 2.8. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = npf(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.

Let $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$, if T is the smallest positive number such that $f(t \pm T) = f(t)$, for all $t \in \mathbb{T}$.

Remark 2.9. [13] If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies

$$\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$$

and so, is a periodic function with period p.

3. Existence of periodic solutions

In this section we will present the main result. The following conditions should be assumed. Let $C(\mathbb{T}, \mathbb{R})$ be the space of all real valued continuous functions on $\mathbb{T}$. Define

$$\mathcal{H}_T = \{\varphi \in C(\mathbb{T}, \mathbb{R}) : \varphi(t+T) = \varphi(t)\}, \ where\, T > 0,\ T \in \mathbb{T},$$

then $\mathcal{H}_T$ is a Banach space with the supremum norm

$$\|x\| = sup\,|x(t)|,\ t \in [0, t].$$

If $\mathbb{T} \neq \mathbb{R}$ and $T = np$ for some, $n \in \mathbb{N}$. By the notation $[a, b]$ we mean $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$, unless otherwise specified. The intervals $[a, b)$, $(a, b]$, and (a, b) are defined similarly. for all $t \in \mathbb{T}$, let $a(t) > 0$ and $a \in \mathcal{R}^+$, where $a(t)$ is a continuous, and

$$a(t + T) = a(t), \tag{3.1}$$

$$g(t+T) = g(t), \tag{3.2}$$

$$D_i(t+T, u+T) = D_i(t, u), \tag{3.3}$$

$$(id - g)(t+T) = (id - g)(t), \tag{3.4}$$

where, id is the identity function on $\mathbb{T}$. We also assume that $Q(t, x)$ and $f(x)$ are continuous and periodic in t and Lipschitz continuous in x. That is,

$$Q(t+T, x) = Q(t, x) \tag{3.5}$$

and there are positive constants E_1, E_2, E_3, E_4 and E_5 such that

$$\sum_{i=1}^{p} |Q_i(t, x) - Q_i(t, y)| \leq E_1\|x - y\|, \tag{3.6}$$

$$|G(t, x, y) - G(t, w, z)| \leq E_2\|x - w\| + E_3\|y - z\|, \tag{3.7}$$

$$|f(x) - f(y)| \leq E_4\|x - y\|, \tag{3.8}$$

and

$$\int_{-\infty}^{t} |D(t, u)|\Delta u \leq E_5, \tag{3.9}$$

Lemma 3.1. [13] *Let $x \in \mathcal{H}_T$. Then $\|x^\sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

The following lemma allow to convert Eq(1.2) to an equivalent integral equation.

Lemma 3.2. *Suppose,* (3.1) *and* (3.5) *hold, if $x \in \mathcal{H}_T$, then x is a solution of Eq* (1.2) *if and only if*

$$\begin{aligned} x(t) &= Q(t, x(t - g(t))) \\ &\quad + (1 - e_{\ominus a}(t, t-T))^{-1} \int_{t-T}^{t} [-a(u)\, Q^\sigma(u, x(u - g(u))) \\ &\quad + a(u)x(u)^\sigma + \int_{-\infty}^{u} \left(\left(\sum_{i=1}^{p} D_i(u, s) \right) f(x(s)) \right) \Delta s \\ &\quad + G(u, x(u), x(u - \tau(u)))] e_{\ominus a}(t, u)\Delta u. \end{aligned} \tag{3.10}$$

We will introduce the state of Krasnoselskii's fixed point theorem and apply this theorem to prove the existence of a periodic solution

Theorem 3.3. *(Krasnoselskii) Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(B, \|.\|)$. Suppose that A and B map $\mathbb{M}$ into B such that*

(1) *$x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,*
(2) *A is compact and continuous,*
(3) *B is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

For its proof, we refer the reader to [16]. As structures hypothesis of Theorem 3.3 states there are two mappings, one is a contraction and the other is compact. Therefore we will define the operator $P : \mathcal{H}_T \to \mathcal{H}_T$ by

$$\begin{aligned}(P\varphi)(x) =& Q(t, \varphi(t-g(t))) + a(u)x(u)^{\sigma} \\ &+ (1-e_{\ominus a}(t, t-T))^{-1} \int_{t-T}^{t} [-a(u)Q^{\sigma}(u, x(u-g(u))) \\ &+ \int_{-\infty}^{u} \left(\sum_{i=1}^{p} D_i(u,s) f(\varphi(s)) \right) \Delta s \\ &+ G(u, \varphi(u), \varphi(u-\tau(u)))] \;\; e_{\ominus a}(t,u)\Delta u \end{aligned} \tag{3.11}$$

By using the same stpes in [1], we can proof that, $(P\varphi)(x)$ is periodic in t of period T. Now by expressing equation (3.11) as

$$(P\varphi)(t) = (B\varphi)(t) + (A\varphi)(t),$$

where, A and B are given by

$$(B\varphi)(t) = Q(t, \varphi(t-g(t))). \tag{3.12}$$

And

$$\begin{aligned}(A\varphi)(t) = & (1-e_{\ominus a}(t, t-T))^{-1} \int_{t-T}^{t} [-a(u)Q^{\sigma}(u, \varphi(u-g(u))) \\ &+ a(u)\varphi(u)^{\sigma} + \int_{-\infty}^{u} \left(\sum_{i=1}^{p} D_i(u,s) f(\varphi(s)) \right) \Delta s \\ &+ G(u, \varphi(u), \varphi(u-\tau(u)))] \;\; e_{\ominus a}(t,u)\Delta u] \;. \end{aligned} \tag{3.13}$$

We are trying to achieve that $(B\varphi)(t)$ is contraction and $(A\varphi)(t)$ is compact this can be done by providing these two lemmas. Before introduce the lemmas we define the following constants

$$\tau = \max_{t \in [0,T]} \left| (1-e_{\theta a}(t, t_T))^{-1} \right|, \tag{3.14}$$

$$\rho = \max_{t\in[0,T]} |a(t)|, \tag{3.15}$$

$$\nu = \left| \max_{u\in[t-T,t]} e_{\theta a}(t,u) \right|. \tag{3.16}$$

Lemma 3.4. *IfA is defined by Eq(3.13), then A is continuous and the image of A is contained in a compact set.*

Proof. We will start by proving A is continuous we define A as in Eq(3.13). Let $\varphi, \psi \in \mathcal{H}_T$, for a given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{\mathrm{N}}$ with

$$N = \tau\nu T\left[\rho E_1 + \rho + E4E_5 + E_2 + E_3\right].$$

Now for $\|\varphi - \psi\| < \delta$, using (3.6) into (3.9) by utilizing the same steps in [17] , we get

$$\begin{aligned} \|A_\varphi - A_\psi\| &\leq \tau\nu T\left[\rho E_1 + \rho + E_4E_5 + E_2 + E_3\right]\|\varphi - \psi\| \\ &\leq \mathrm{N}\|\varphi - \psi\| \\ &\leq \mathrm{N}\delta \leq \varepsilon. \end{aligned}$$

This is show that A is continuous. The second step is showing A is a compact set using Ascoli-Arzela's theorem [10], which states that for $A \subset X$, A is compact if and only if A is bounded, and equicontinuous. Let $\Omega = \{\varphi \in \mathcal{H}_T : \|\varphi\| \leq \Upsilon\}$, where Υ is any fixed positive constant. From (3.6) and (3.8) we have,

$$\begin{aligned} |Q(t,x)| &= |Q(t,x) - Q(t,0) + Q(t,0)|, \\ &\leq |Q(t,x) - Q(t,0)| + |Q(t,0)|, \\ &\leq E_1|x| + \alpha_1, \end{aligned}$$

where, $\alpha_1 = \sup_{t\in[0,T]} |Q(t,0)|$.

In the same way,

$$\begin{aligned} |f(x)| &= |f(x) - f(0) + f(0)|, \\ &\leq |f(x) - f(0)| + |f(0)|, \\ &\leq E_4\|x\|. \end{aligned}$$

and

$$\begin{aligned} |G(t,x,y)| &= |G(t,x,y) - G(t,0,0) + G(t,0,0)|, \\ &\leq |G(t,x,y) - G(t,0,0)| + |G(t,0,0)|, \\ &\leq E_2\|x\| + E_3\|y\|, \end{aligned}$$

where, $\alpha_2 = \sup_{t\in[0,T]} |G(t,0,0)|$.

Taking into consideration, $f(0) = 0$ and $G(t, 0, 0) = 0$. Let $\varphi_n \in \Omega$ where n is a positive integer with

$$L = \tau\nu T\left[\rho\left(E_1\Upsilon + \alpha_1\right) + \rho\Upsilon + \Upsilon E_4E_5 + \Upsilon E_2 + \Upsilon E_3 + \alpha_2\right]$$

, where $L > 0$. Therefore,

$$\begin{aligned}
\|A_{\varphi_n}\| &= |(1 - e_{\ominus a}(t, t - T))^{-1}\int_{t-T}^{t}[-a(u)\,Q^\sigma(u, \varphi_n(u - g(t))) \\
&\quad + a(u)\,\varphi_n(u)^\sigma + \int_{-\infty}^{u}\left(\sum_{i=1}^{p} D_i(u, s)f(\varphi_n(s))\right)\Delta s \\
&\quad + G(u, \varphi_n(u), \varphi_n(u - \tau(u)))]e_{\ominus a}(t, u)\,\Delta u| \\
&\le \max_{t\in[0,T]}|(1 - e_{\ominus a}(t, t - T))^{-1}\int_{t-T}^{t}[-a(u)\,Q^\sigma(u, \varphi_n(u - g(t))) \\
&\quad + a(u)\,\varphi_n(u)^\sigma + \int_{-\infty}^{u}\left(\sum_{i=1}^{p} D_i(u, s)f(\varphi_n(s))\right)\Delta s \\
&\quad + G(u, \varphi_n(u), \varphi_n(u - \tau(u)))]e_{\ominus a}(t, u)\,\Delta u \\
&\le \tau\nu T\left[\rho\left(E_1\Upsilon + \alpha_1\right) + \rho\Upsilon + \Upsilon E_4E_5 + \Upsilon E_2 + \Upsilon E_3 + \alpha_2\right] \\
&\le L. \qquad (3.17)
\end{aligned}$$

This is showing that A is bounded. To prove A is equicontinuous we need to find $(A\varphi_n)^\Delta(t)$ and prove that it is uniformly bounded. Therefore, after derivative Eq(3.13) with using (3.6) - (3.9) we get,

$$\begin{aligned}
(A\varphi_n)^\Delta(t) \quad &= -a(t)\,A(\varphi_n)^\sigma(t) - a(t)\,Q^\sigma(t, \varphi_n(t - g(t))) \\
&\quad + a(t)\,\varphi_n(t)^\sigma + \int_{-\infty}^{u}\left(\sum_{i=1}^{p} D_i(u, s)\,f(\varphi(s))\right)\Delta s \\
&\quad + G(u, \varphi(u), \varphi(u - \tau(u))).
\end{aligned}$$

The above expression yields $|(A\varphi_n)^\Delta| \le Z$, where Z is some positive constant. Hence, by Ascoli-Arzela's Theorem $A\varphi$ is compact. □

Lemma 3.5. *If B is given by Eq(3.12) with $E_1 < 1$, and (3.6) hold, then B is a contraction.*

Proof. Let B be defined by Eq(3.12). Then for $\varphi, \psi \in \mathcal{H}_T$ we have

$$\begin{aligned}
\|(B\varphi)(t) - (B\psi)(t)\| \quad &= \sup_{t\in[0,T]}|(B\varphi)(t) - (B\psi)(t)| \\
&= \sup_{t\in[0,T]}|Q(t, \varphi(t - g(t))) - Q(t, \psi(t - g(t)))|.
\end{aligned}$$

By using (3.6), then we get

$$\|(B\varphi)(t) - (B\psi)(t)\| \le E_1\sup_{t\in[0,T]}\|\varphi(t - g(t)) - \psi(t - g(t))\|.$$

As $E_1 < 1$, therefore B defines a contraction. □

Theorem 3.6. *Suppose* (3.1)-(3.9) *hold. Let*

$$\alpha_1 = \sup_{t\in[0,T]} |Q(t,0)|, \qquad \alpha_2 = \sup_{t\in[0,T]} |G(t,0,0)|$$

and there exist that a positive constant $\mathcal{K}$ *satisfying the inequality*

$$\tau\nu T\left[\rho\left(E_1\mathcal{K}+\alpha_1\right)+\left(\rho+E_4E_5+E_2+E_3\right)\mathcal{K}+\alpha_2\right]+E_1\mathcal{K}+\alpha_1 \le \mathcal{K}.$$

Let $\mathcal{M} = \{\varphi \in \mathcal{H}_T : \|\varphi\| \le \mathcal{K}\}$. *Then Eq* (1.2) *has a solution in* $\mathcal{M}$.

Proof. First, we will define $\mathcal{M} = \{\varphi \in \mathcal{H}_T : \|\varphi\| \le \mathcal{K}\}$. By knowing that A is continuous and $A\mathcal{M}$ is contained in a compact set from Lemma 3.4. Also, the mapping B is a contraction from Lemma 3.5. It is clear that $A, B : \mathcal{H}_T \to \mathcal{H}_T$. Our goal is to show that, $\|A_\varphi + B_\psi\| \le \mathcal{K}$. Let $\varphi, \psi \in \mathcal{M}$, with $\|\varphi\|, \|\psi\| \le \mathcal{K}$. Then,

$$\|A_\varphi + B_\psi\| \le \|A_\varphi\| + \|B_\psi\|.$$

Lemma 3.4 says that,

$$\|A_{\varphi_n}\| \le \tau\nu T\left[\rho\left(E_1\mathcal{K}+\alpha_1\right)+\left(\rho+E_4E_5+E_2+E_3\right)\mathcal{K}+\alpha_2\right]$$

Therefore,

$$\begin{aligned}\|A\|+\|B\| \quad &\le \tau\nu T\left[\rho\left(E_1\mathcal{K}+\alpha_1\right)+\left(\rho+E_4E_5+E_2+E_3\right)\mathcal{K}+\alpha_2\right]+E_1\|\psi\|+\alpha_1\\ &\le \tau\nu T\left[\rho\left(E_1\mathcal{K}+\alpha_1\right)+\left(\rho+E_4E_5+E_2+E_3\right)\mathcal{K}+\alpha_2\right]+E_1\mathcal{K}+\alpha_1\\ &\le \mathcal{K}.\end{aligned}$$

Hence, all conditions of Theorem 3.3 are proven. Thus, there exists a fixed point z in $\mathcal{M}$. By Lemma 3.2, this fixed point is a solution of Eq (1.2). Therefore, Eq (1.2) has a T-periodic solution. □

Theorem 3.7. *Suppose* (3.1)-(3.9) *hold. If*

$$E_1 + \tau\nu T\left[\rho E_1 + \rho + E_5E_4 + E_2 + E_3\right] < 1, \tag{3.18}$$

then Eq (1.2) *has a unique* T*-periodic solution.*

Proof. Let $\varphi, \psi \in \mathcal{H}_T$. Define P as Eq (3.11), we have,

$$\|P_\varphi - P_\psi\| < E_1 + \tau\nu T\left[\rho E_1 + \rho + E_5E_4 + E_2 + E_3\right]\|\varphi - \psi\|.$$

This completes the proof of Theorem 3.7. □

Acknowledgement The authors would like to express their appreciation to referees for reading the manuscript, correcting errors, and valuable hints.

References

[1] S. Althubiti, H. Makhzoum, and Y. N. Raffoul, Periodic solution and stability in nonlinear neutral system with infinite delay, *Appl. Math. Sci.* **7, 136***(2013), 6749-6764.*

[2] A. Ardjouni and A. Djoudi, Periodicity and Stability in Nonlinear Neutral Dynamic Equations with Infinite Delay on a Time Scale, *Khayyam Journal of Mathematics.* **2, 1***(2016), 51-62.*

[3] A. Ardjouni and A. Djoudi, Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale, *Commun. Nonlinear Sci. Numer. Simul.* **17***(2012), 3061-3069.*

[4] A. Ardjouni and A. Djoudi, Existence of positive periodic solutions for a nonlinear neutral differential equation with variable delay, *Appl. Math. E-Notes,* **12***(2012), 94-101.*

[5] A. Ardjouni and A. Djoudi, Periodic solutions for impulsive neutral dynamic equations with infinite delay on time scales, *Kragujevac Journal of Mathematics,* **42***, No 1, (2018), 69-82.*

[6] A. Ardjouni and A. Djoudi, Periodic solutions in totally nonlinear dynamic equations withfunctional delay on a time scale, *Rend. Semin. Mat. Univ. Politec. Torino,* **68, 4***(2010), 349-359.*

[7] F.M. Atici, G.S. Guseinov and B. Kaymakcalan, Stability criteria for dynamic equationson time scales with periodic coefficients, *Proceedings of the International Conference onDynamic Systems and Applications III,* **74***(1999), 43-48.*

[8] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, An introduction withApplications, Birkhäuser,Boston, 2001.

[9] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser,Boston, 2003.

[10] E. DiBenedetto and E. Debenedetto, Real analysis, Birkhäuser, Boston,, 2002.

[11] A.S. Elmabrok and S.R Alkaleeli, Generalization of b_2-metric spaces and some fixed point theorems, *Libyan Journal of Science and Technology.* **8, 1***(2018), 51-55.*

[12] E.R. Kaufmann and Y.N. Raffoul, Stability in neutral nonlinear dynamic equations on atime scale with functional delay, *Dynam. systems Appl.* **16***(2007), 561-570.*

[13] E.R. Kaufmann and Y.N. Raffoul, Periodic solutions for a neutral nonlinear dynamicalequation on a time scale, *J. Math. Anal. Appl.* **319, 1***(2006), 315-325.*

[14] Y. Khalili, and M. Yadollahzadeh, Existence Results for a New Class of Nonlinear Langevin Equations of Fractional Orders, Iranian Journal of Science and Technology, *Transactions A: Science.* **43, 5***(2019), 2335-2342.*

[15] N. S.Rao, K. Kalyani, and B. Mitiku, Fixed point theorems for nonlinear contractive mappings in ordered b-metric space with auxiliary function, *BMC Research Notes.* **13, no.1***(2020), 1-8.*

[16] D.R. Smart, Fixed Point Theorems, Cambridge Univ. Press, Cambridge, 1980.

[17] E .Yankson, Existence and uniqueness of solutions for neutral periodic integro-differential equations with infinite delay, *Caspian Journal of Mathematical Sciences.* **5, 1***(2016), 37-46.*