Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **10**(2)(2021), 300-308

Spirallike functions in terms of Convolution

E. Amini [1]
[1] Department of Mathematics, Payme Noor University, P. O. Box 19395-4697 Tehran, IRAN.

Abstract. In this paper the author used Salagean differential operator to define a certain subclass of spirallike functions and obtain some convolution results and some upper bounds on the coefficients.

Keywords: Univalent functions, Hadamard product, (n, λ)-spirallike functions of order β, Salagean differential operator.

2000 Mathematics subject classification: 30C45, 30C55.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C}; |z| < 1\}$. The class of functions in $\mathcal{A}$, which are also univalent in $\mathcal{U}$, is denoted(as usual) by $\mathcal{S}$. Authors in [1, 2] investigated the familiar subclass of univalent functions in D, such as starlike, convex and spirallike functions.

For $0 \leq \beta < 1$, $\lambda \in \mathbb{R}$ with $|\lambda| < \pi/2$, by $SP(\lambda, \beta)$ denote the well known subclass of $\mathcal{A}$ consisting of λ-spirallike functions of order β. As well known

$$SP(\lambda, \beta) = \Big\{ f \in \mathcal{A} : Re\big\{ e^{i\lambda} \frac{zf'(z)}{f(z)} \big\} > \beta \cos\lambda, \forall z \in \mathcal{U} \Big\}.$$

[1]Corresponding author: eb.amini.s@pnu.ac.ir
Received: 25 April 2020
Accepted: 11 February 2021

The function $f(z) \in \mathcal{A}$ is convex λ-spriallike of order β in $\mathcal{U}$ if and only if $zf'(z)$ is λ-spriallike of order β in $\mathcal{U}$ (for more details see [3, 14]).

Lemma 1.1. [3] *Let* $0 \leq \beta < 1$, $\lambda \in \mathbb{R}$ *with* $|\lambda| < \pi/2$. *If*

$$\Big|\frac{zf'(z)}{f(z)} - 1\Big| < (1-\beta)\cos\lambda,$$

then $f \in SP(\lambda, \beta)$.

Let D^n be the Salagean differential operator ([6]) $D^n : \mathcal{A} \longrightarrow \mathcal{A}$, $n \in \mathbb{N} = \{1, 2, ...\}$, defined as:

$$D^0 f(z) = f(z), \quad D^1 f(z) = zf'(z), \qquad D^n f(z) = D(D^{n-1} f(z)).$$

If $f \in \mathcal{A}$ is given by (1.1), then

$$D^n f(z) = \sum_{k=1}^{\infty} k^n a_k z^k,$$

and $z(D^n f(z))' = D^{n+1} f(z)$.

Let I^n be the Salagean integral operator ([6]) $I^n : \mathcal{A} \longrightarrow \mathcal{A}$, $n \in \mathbb{N}$, defined as:

$$I^0 f(z) = f(z), \quad I^1 f(z) = If(z) = \int_0^z \frac{f(t)}{t} dt, \quad I^n f(z) = I(I^{n-1} f(z)).$$

If $f \in \mathcal{A}$ is given by (1.1), then

$$I^n f(z) = \sum_{k=1}^{\infty} k^{-n} a_k z^k.$$

For f and g in $\mathcal{A}$, with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution (Hadamard product) of f and g, denoted by $f * g$, is a function, also in $\mathcal{A}$, given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

The basic reference for this theory is the book by Ruscheweyh [5] where many of the results area first were published. Silverman et al in [8] characterized for convex, starlike and spirallike functions in terms of convolutions.

Theorem 1.2. [8] *The function* f *is convex of order* α *in* $|z| < R < 1$ *if and only if*

$$\frac{1}{z}\left[f * \frac{z + \frac{x+\alpha}{1-\alpha} z^2}{(1-z)^3}\right] \neq 0 \ (|z| < R, |x| < 1).$$

Theorem 1.3. [8] *For $|z| < R \leq 1$, λ real with $|\lambda < \pi/2$ and $|x| = 1$, we have*

$$Re\left\{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0$$

if and only if

$$\frac{1}{z}\left[f * \frac{z + \frac{2x+1-e^{-2i\lambda}}{1+e^{-2i\lambda}}z^2}{(1-z)^2}\right] \neq 0.$$

In geometric function theory of function, a verity of sufficient conditions for spirallikeness have been considered. We refer to the monographs [3], [9], [11], [12] for details. In the present work we define a subclass of spirallike functions and we give upper bounds on the coefficients in the classes. Furthermore using the idea of Silverman et al in [8] and we give characterizations for subclass of spirallike functions in terms of convolutions.

Let $f(z)$ and $g(z)$ be analytic in D. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1(z \in D)$ such that $f(z) = g(w(z))$ for $z \in D$. If $g(z)$ is univalent in D, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D(\subset g(D))$. In [4] Miller and Mocanu study of the concept of subordination in the complex plan. In [10], [13] authors studied subordination for spirallike functions.

Lemma 1.4. [10] *The function $f(z) \in \mathcal{A}$ is λ-spirallike of order β $(0 \leq \beta < 1)$, $|\lambda| < \pi/2$, if and only if there exists $w(z)$ analytic function satisfying $w(0) = 0$, $|w(z)| < 1$ such that*

$$e^{i\lambda}\frac{zf'(z)}{f(z)} = \beta\cos\lambda + (1-\beta)\cos\lambda\frac{1-w(z)}{1+w(z)} + i\sin\lambda.$$

2. Convolution Condition

Definition 2.1. For $0 \leq \beta < 1$, λ real with $|\lambda| < \pi/2$ and $n \in \mathbb{N}$, we define the class $SP_n(\lambda, \beta)$ of (n, λ)-spriallike functions of order β by

$$SP_n(\lambda,\beta) = \left\{f \in \mathcal{A} : Re\left\{e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \beta\cos\lambda,\ \forall z \in \mathcal{U}\right\}.$$

Note that the class $SP_0(\lambda, \beta) = SP(\lambda, \beta)$, the class of λ-spriallike of order β, $SP_1(\lambda, \beta)$, the class of convex λ-spriallike functions of order β. In particular, $f \in SP_n(\lambda, \beta)$ if and only if $D^n f$ is λ-spriallike of order β for $n = 1, 2, 3, ...$, and $f \in SP_n(\lambda, \beta)$ if and only if $D^{n-1}f$ is convex λ-spriallike of order β for $n = 2, 3, ...$

Theorem 2.2. *For* $|z| < R \leq 1$, $n \in \mathbb{N}$, λ *real with* $|\lambda| < \pi/2$, $0 \leq \beta < 1$ *and* $|x| = 1$ *we have*

$$Re\left\{e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \beta\cos\lambda \tag{2.1}$$

if and only if

$$\frac{1}{z}\Big(f * \frac{z + (\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda} - 1)z^2}{(1-z)^{n+2}}\Big) \neq 0. \tag{2.2}$$

Proof. The function $D^{n-1}f(z)$ is convex λ- spriallike of order β in $|z| < R \leq 1$ from Lemma 1.4, we have

$$\frac{e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)} - \beta\cos\lambda - i\sin\lambda}{(1-\beta)\cos\lambda} = \frac{1-w(z)}{1+w(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{2.3}$$

Since $\frac{D^{n+1}f(z)}{D^n f(z)} = 1$ at $z = 0$, (2.3) is equivalent to

$$\frac{e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)} - \beta\cos\lambda - i\sin\lambda}{(1-\beta)\cos\lambda} \neq \frac{x-1}{x+1},$$

where $|x| = 1$. which simplifies to

$$(x+1)e^{i\lambda}D^{n+1}f(z) + D^{n+1}f(z)\Big(\cos\lambda(1-2\beta-x) - i(x+1)\sin\lambda\Big) \neq 0. \tag{2.4}$$

Setting f by (1.1), we have

$$\begin{aligned} D^{n+1}f(z) &= \frac{z}{(1-z)^2} * D^n f(z) = k(z) * D^n f(z), \\ D^n f(z) &= D^n f(z) * \frac{z}{1-z}. \end{aligned}$$

So that the left hand side of (2.4) may be expressed as

$$\frac{D^n f(z)}{z} * \Big(\frac{(x+1)e^{i\lambda}}{(1-z)^2} + \frac{(1-2\beta-x)\cos\lambda - i(x+1)\sin\lambda}{1-z}\Big) \neq 0.$$

Thus the above inequality is equivalent to

$$\frac{1}{z}\left(D^n f(z) * \frac{z + \frac{(x+2\beta-1)\cos\lambda + i(x+1)\sin\lambda}{2(1-\beta)\cos\lambda}}{(1-z)^2}\right) \neq 0.$$

Since $D^n(f * g)(z) = D^n f(z) * g(z) = f(z) * D^n g(z)$, we obtain that

$$\frac{1}{z}\left(f * D^n\Big(\frac{z}{(1-z)^2}\Big) + \frac{(x+2\beta-1)\cos\lambda + i(x+1)\sin\lambda}{2(1-\beta)\cos\lambda}D^n\Big(\frac{z^2}{(1-z)^2}\Big)\right) \neq 0.$$

Since

$$D^n\Big(\frac{z}{(1-z)^2}\Big) = \sum_{k=1}^{\infty} k^{n+1}z^k,$$
$$D^n\Big(\frac{z^2}{(1-z)^2}\Big) = \sum_{k=2}^{\infty} k^{n+1}z^k - \sum_{k=2}^{\infty} k^n z^n,$$

we get

$$\frac{1}{z}\Bigg(f*\Big(z+\sum_{k=2}^{\infty}k^{n+1}z^k+\frac{(x+2\beta-1)\cos\lambda+i(x+1)\sin\lambda}{2(1-\beta)\cos\lambda}\big(\sum_{k=2}^{\infty}k^{n+1}z^k-\sum_{k=2}^{\infty}k^n z^k\big)\Big)\Bigg)\neq 0.$$

which simplifies to

$$\frac{1}{z}\Bigg(f*\Big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda}\sum_{k=1}^{\infty}k^n z^k+\frac{-(x+1)\cos\lambda+2(1-\beta)\cos\lambda}{2(1-\beta)\cos\lambda}\sum_{k=1}^{\infty}k^n z^k\Big)\Bigg)\neq 0.$$

Since $\sum_{k=1}^{\infty}k^{n+1}z^k=\frac{z}{(1-z)^{n+2}}$ and $\sum_{k=1}^{\infty}k^n z^k=\frac{z}{(1-z)^{n+1}}$, we have

$$\frac{1}{z}\Bigg(f*\Big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda}\frac{z}{(1-z)^{n+2}}+\big(\frac{-(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda}+1\big)\frac{z}{(1-z)^{n+1}}\Big)\Bigg)\neq 0. \tag{2.5}$$

Thus the inequality (2.5) equivalent to (2.2), and this completes the proof of the theorem. □

In Theorems 1.2 and 1.3 charaterized for convex and spirallike function in terms of convolution. In the following, we give characterizations for λ-spirallike of order β and convex λ-spirallike of order β in terms of convolution.

Putting $n=0$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *For $|z|<R\leq 1$, $n\in\mathbb{N}$, λ real with $|\lambda|<\pi/2$, $0\leq\beta<1$ and $|x|=1$, the function f is λ-spriallike functions of order β if and only if*

$$\frac{1}{z}\Big(f*\frac{z+\big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda}-1\big)z^2}{(1-z)^2}\Big)\neq 0.$$

Putting $n=1$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.4. *For $|z|<R\leq 1$, $n\in\mathbb{N}$, λ real with $|\lambda|<\pi/2$, $0\leq\beta<1$ and $|x|=1$, the function f is convex λ-spriallike functions of order β if and only if*

$$\frac{1}{z}\Big(f*\frac{z+\big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda}-1\big)z^2}{(1-z)^3}\Big)\neq 0.$$

Theorem 2.5. *The function* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *is in* $SP_n(\lambda, \beta)$ *if and only if*

$$1 + \sum_{k=2}^{\infty} \left(k^{n+1} + \Big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda} - 1 \Big)(k-1)^{n+1} \right) a_k z^k \neq 0,$$

for all $z \in \mathcal{U}$ *and* $|x| = 1$.

Proof. By applying definition of convolution and inequality (2.2), the proof is complete. □

Remark 2.6. Since

$$\left| 1 + \sum_{k=2}^{\infty} \left(k^{n+1} + \Big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda} - 1 \Big)(k-1)^{n+1} \right) a_k z^{k-1} \right|$$
$$\geq 1 - \sum_{k=2}^{\infty} \left| k^{n+1} + \Big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda} - 1 \Big)(k-1)^{n+1} \right| |a_k||z^{n-1}|.$$

a sufficient condition for f to be in $SP_n(\lambda, \beta)$ is that

$$\sum_{k=2}^{\infty} \left| k^{n+1} + \Big(\frac{(x+1)e^{i\lambda}}{2(1-\beta)\cos\lambda} - 1 \Big)(k-1)^{n+1} \right| |a_k| \leq 1.$$

3. Coefficient Result

Theorem 3.1. *Let* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. *If*

$$\sum_{k=2}^{\infty} k^n \Big(\frac{k-1}{(1-\beta)\cos\lambda} + 1 \Big) |a_k| \leq 1. \tag{3.1}$$

Then $f \in SP_n(\lambda, \beta)$.

Proof. By Lemma 1.1 , it is suffices to show that

$$\Big| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \Big| < (1-\rho)\cos\lambda.$$

We have

$$\begin{aligned} \Big| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \Big| &= \left| \frac{\sum_{k=2}^{\infty} k^n (k-1) a_k z^k}{z + \sum_{k=2}^{\infty} k^n a_k z^k} \right| \\ &\leq \frac{\sum_{k=2}^{\infty} k^n (k-1) |a_k||z|^{k-1}}{1 - \sum_{k=2}^{\infty} k^n |a_k||z|^{k-1}} \\ &\leq \frac{\sum_{k=2}^{\infty} k^n (k-1) |a_k|}{1 - \sum_{k=2}^{\infty} k^n |a_k|}. \end{aligned}$$

Thus last expression is bounded above by $(1-\beta)\cos\lambda$, if

$$\sum_{k=2}^{\infty} k^n(k-1)|a_k| \le (1-\beta)\cos\lambda\Big(1-\sum_{k=2}^{\infty} k^n|a_k|\Big).$$

which is equivalent to (3.1). □

Theorem 3.2. *Let $n \in \{0,1,2,...\}$, $0 \le \beta < 1$ and λ is real with $|\lambda| < \pi/2$. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^z$ is in $SP_n(\lambda,\beta)$ and $|a_2| = a$, then*

$$|a_k| \le \frac{1+2^n a}{1+2\cos\lambda - 2\beta\cos\lambda}\frac{\prod_{j=2}^{k}(j-2+2\cos\lambda-2\beta\cos\lambda)}{(k-1)!k^n}, \qquad j = 3,4,...$$

Proof. We choose

$$p(z) = e^{i\lambda}\frac{D^{n+1}f(z)}{D^n f(z)} = (\cos\lambda + i\sin\lambda)\frac{z+\sum_{k=2}^{\infty} k^{n+1}a_k z^k}{z+\sum_{k=2}^{\infty} k^n a_k z^k}. \tag{3.2}$$

and we denote

$$q(z) = \frac{p(z)-e^{i\lambda}}{\cos\lambda - \beta\cos\lambda} = q_1 z + q_2 z^2 + ... \tag{3.3}$$

We remark that $Rep(z) > \beta\cos\lambda$, $z \in \mathcal{U}$, because $f \in SP_n(\lambda,\beta)$ and, consequently, $Req(z) > -1$, $z \in \mathcal{U}$. Hence we have the subordination

$$q(z) \prec s(z) = 2z + 2z^2 + ... + 2z^j + ... \tag{3.4}$$

Since s is a convex univalent function, we have

$$|q_k| \le 2, \qquad k = 1,2,3,...$$

From (3.2) and (3.3), we obtain

$$e^{i\lambda} + (1-\beta)\cos\lambda(q_1 z + q_2 z^2 + ...) = (\cos\lambda + i\sin\lambda)\frac{z+\sum_{k=2}^{\infty} k^{n+1}a_k z^k}{z+\sum_{k=2}^{\infty} k^n a_k z^k}. \tag{3.5}$$

From (3.2) and (3.5) by equating coefficients, we deduce

$$(\cos\lambda + i\sin\lambda)k^{n+1}a_k = (1-\beta)\cos\lambda\Big(q_{k-1} + 2^n a_2 q_{k-2} + ... + q_1(k-1)^n a_{k-1}\Big) + (\cos\lambda + i\sin\lambda)k^n a_k. \tag{3.6}$$

Hence the equation (3.6) is equivalent to

$$(\cos\lambda + i\sin\lambda)k^{n+1}(k-1)a_k = (1-\beta)\cos\lambda\Big(q_{k-1} + 2^n a_2 q_{k-2} + ... + q_1(k-1)^n a_{k-1}\Big).$$

Thus, by using (3.4), we finally obtain

$$k^{n+1}(k-1)|a_k| \le 2(1-\beta)\cos\lambda\Big(1 + 2^n|a_2| + ... + (k-1)^n|a_{k-1}|\Big),$$

for $k = 3, 4, ...$ We assume that

$$|a_k| \leq \frac{1 + 2^n a}{1 + 2\cos\lambda - 2\beta\cos\lambda} \frac{\prod_{j=2}^{i}(j - 2 + 2\cos\lambda - 2\beta\cos\lambda)}{(i-1)! i^n}.$$

for $k = 3, 4, 5, ...$ Then using the similar method to that of Silverman and Silva [7] , we have

$$\begin{aligned}
(k-1)k^n|a_k| &\leq 2(1-\beta)\cos\lambda\left(1 + 2^n a + \sum_{i=3}^{k-1} \frac{\prod_{j=2}^{i}(j - 2 + 2\cos\lambda - 2\beta\cos\lambda)}{(i-1)!}\right) \\
&= 2(1-\beta)\cos\lambda(1 + 2^n a)\left(1 + \frac{1}{1 + 2\cos\lambda - 2\beta\cos\lambda}\right. \\
&\left(\frac{\prod_{j=3}^{k}(j - 2 + 2\cos\lambda - 2\beta\cos\lambda)}{(k-2)!} - 1 - 2\cos\lambda + 2\beta\cos\lambda\right)\Bigg) \\
&= \frac{1 + 2^n a}{1 + 2\cos\lambda - 2\beta\cos\lambda} \frac{\prod_{j=2}^{k}(j - 2 - 2\cos\lambda + 2\beta\cos\lambda)}{(k-2)!}.
\end{aligned}$$

and the proof is complete. □

Theorem 3.3. *Let $n \in \{0, 1, 2, ...\}$, $0 \leq \beta < 1$ and λ is real with $|\lambda| < \pi/2$. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^z$ is in $SP_n(\lambda, \beta)$, then*

$$|a_k| \leq \frac{\prod_{j=2}^{k}(j - 2 + 2\cos\lambda - 2\beta\cos\lambda)}{(k-1)! k^n}, \qquad k = 3, 4, ..., \tag{3.7}$$

and this result is sharp.

Proof. By Applying Theorem 3.2 for $k = 2$, we deduce

$$|a_2| \leq \frac{2(1-\beta)\cos\lambda}{2^n}. \tag{3.8}$$

Now, by using Theorem 3.2 and (3.8), we obtain (3.7).

This result is sharp for $f_{n,\beta,\lambda}$. Indeed

$$\frac{z}{(1 - z^{2(1-\beta)\cos\lambda})} = z + \sum_{k=2}^{\infty} \frac{\prod_{j=2}^{k}(j - 2 + 2\cos\lambda - 2\beta\cos\lambda)}{(k-1)!} z^k$$

and

$$f_{n,\beta,\lambda} = I^n \frac{z}{(1-z)^{2(1-\beta)\cos\lambda}} = z + \sum_{k=2}^{\infty} \frac{\prod_{j=2}^{k}(j - 2 + 2\cos\lambda - 2\beta\cos\lambda)}{(k-1)! k^n} z^k$$

□

References

[1] P.L. Duren, Univalent Functions, *Grundelheren der Mathematischen Wissenchaften 259, Springer-Verlag, New York (1983).*

[2] I. Graham, G. Kohr, *Geometric Function Theory in one and Higher Dimensions, Marcel Dekker, Inc, NewYork, (2003).*

[3] R. J. Libera, Univalent α-spiral functions, *Canad. J. Math.* **19** *(1967), 449-456.*

[4] S. Miller and P. T. Mocanu, *Differential Subordinations, Monographs and Texbooks in Pure and Appl. Math., 225, Dekker, New York, (2000).*

[5] St. Ruscheweyh, *Convolutions in Geometric Function Theory, Sem. Math. Sup. 83, Presses Univ. Montreal, (1982).*

[6] G. S. Sălăgean, On some classes of univalent functions, Seminar of geometric function theory, Cluj - Napoca, 1983.

[7] H. Silverman, E. M. Silvia, The influence of the second coefficient on prestarlike functions, Rocky Mountain J., bf 10(3), (1980), 469-474.

[8] H. Silverman, E. M. Silvia and D. Telage, Convolution conditions for convexity starlikeness and spiral-likenes, *Math. Z,* **162** *(1978)125-130.*

[9] E. M. Silvia, On A subclass of spiral-like functions ,*Amer. Math. Soc,***44***(1974), 411-420.*

[10] O. Sang, Kwon, and S. OWA, The subordination theorem for λ spirallike functions of order β, International Journal of Applied Mathematics, (2002), 19-24.

[11] R. Singh, A note on spiral-like functions, *J. Indian Math. Soc.* **33** *(1969), 49-55.*

[12] S. Singh and R. Singh, On a subcllas of α-spiral-like functions, *Indian. J. pure. appl. Math.***11(2)***(1980), 160-164.*

[13] R. Singh, A subordination theorem for spirallike functions, Intenat. J. Math. Math. Sci. **24(7)**(2000), 433-435.

[14] N. Xu, D. Yang, On β uniformly convex α-Spiral functions, Soochow Journal of Mathematics, **31**(2005), 561-571.

Authors' addresses: *Department of Mathematics, Payme Noor University, P. O. Box 19395-4697, Tehran, IRAN. e-mail:* `ab.amini.s@gmail.com` .