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Hermite-Hadamard type integral inequalities for uniformly convex functions by using Riemann-Liouville fractional integral transformations with respect to geodesic in Hadamard space

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Abstract. In this paper, we investigate the Hermite-Hadamard inequality for the Riemann-Liouville integral transformations for uniformly convex functions from a geodesic perspective in the Hadamard spaces. This inequality is widely used in some fractional integral approximations.

Keywords: Hermite-Hadamard inequalities, Hadamard space, special means, uniformly convex.

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1. Introduction

The theory of inequalities is one of the important and practical aspects of mathematics. Most inequalities are closely related to the concept of convexity. Convexity plays a vital role in mathematics, chemistry, biology, physics, and other sciences. One of the most important inequalities

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discovered by Hermite-Hadamard for convex functions is the Hermite-Hadamard inequality, which is as follows:
Let $g : I \to \mathbb{R}$ be a convex function, and let $a, b \in I$ with $a < b$. Then the following inequality holds:

$$g(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b g(x)dx \leq \frac{g(a)+g(b)}{2},$$

where I is an interval (finite or infinite) in $\mathbb{R}$.
This inequality is widely used in some integral approximations and means approximations, so many mathematicians have researched this inequality and improved its boundaries for different categories of functions such as convex, quasi-convex, uniformly convex, and so on (see [?], [?],[?],[?],[?],[?],[?],[?],[?],[?]). Recently, some researchers used this inequality in different spaces such as Hadamard as well as for different integral transformations (see[?], [?]). In this paper, we investigate the Hermite-Hadamard inequality for the Riemann-Liouville integral transformations for uniformly convex functions from a geodesic perspective in the Hadamard spaces.

2. Preliminaries

In this section, we consider the basic concepts and results, which are needed to obtain our main results.

Definition 2.1 ([?, ?])**.** Let $g \in L[a, b]$, in this case the left and right Riemann-Liouville fractional integrals denoted by $J^{\alpha}_{a^+}g$ and $J^{\alpha}_{b^-}g$ of order $\alpha > 0$ with $b \geq a \geq 0$, respectively, are defined as follows

$$J^{\alpha}_{a^+}g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt \text{ with } x > a,$$

$$J^{\alpha}_{b^-}g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t)dt \text{ with } x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and its definition is

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt.$$

It is to be noted that $J^0_{a^+}g(x) = J^0_{b^-}g(x) = g(x)$.

We need the following lemma which has been proved in [?].

Lemma 2.2 ([?])**.** *Let $g : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $g' \in L[a, b]$ the following equality for fractional integrals* *holds:*

$$\frac{g(a)+g(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\Big(J_{a^-}g(b) + J_{b^-}g(a)\Big)$$
$$= \frac{b-a}{2}\int_0^1 \Big((1-t)^{\alpha} - t^{\alpha}\Big)g'\Big(ta+(1-t)b\Big)dt.$$

Definition 2.3. A geodesic is a rectifiable curve $\gamma : [0,1] \to N$ such that the length of $\gamma|_{[t_1,t_2]}$ is $d(\gamma(t_1),\gamma(t_2))$ for all $0 \le t_1 \le t_2 \le 1$.

Definition 2.4. A complete metric space (N,d) is called a global NPC space(or Hadamard space) if for $x_1, x_2 \in N$ there exists a point $z \in N$ such that for each $x \in N$, we have

$$d(x,z)^2 \le \frac{1}{2}d(x,x_1)^2 + \frac{1}{2}d(x,x_2)^2 - \frac{1}{4}d(x_1,x_2)^2.$$

Remark 2.5. The point z occurring in the preceding definition plays the role of a midpoint between x_1 and x_2.

Remark 2.6. If $(N;d)$ is a global NPC and α, β are two geodesic arcs starting at $x \in X$, then the distance map $t \to d(\alpha(t),\beta(t))$ is a convex function.

Remark 2.7. The categories of complete Riemannian manifolds with non-positive cross-sectional curvature, Hilbert spaces, Bruhat Tits buildings, especially metric trees are NPC.

Definition 2.8. A subset $C \subseteq B$ is called convex if for each geodesic $\gamma:[0,1]\to B$ joining two arbitrary points in C holds that $\gamma([0,1]) \subset C$.

3. Main results

Let X be a vector space. For any two distinct points a and b in X, we define the line segment connecting a to b by

$$\gamma(\lambda) := \{(1-\lambda)a + \lambda b : \lambda \in [0,1]\}.$$

If it is necessary to emphasize a and b, it will be written $\gamma(\lambda) = \alpha_{a,b}(\lambda)$. The set of all $(1-\lambda)a+\lambda b$ with $\lambda \in [0,1]$ is shown to be $\alpha_{(a,b)}$.

Here we refer to more restrictive versions of strict convexity introduced in reference [**?**].

Definition 3.1. Let $g : H \longrightarrow (-\infty,+\infty]$ be proper and H be a Hilbert space. Then g is uniformly convex with modulus $\phi : [0,+\infty) \to [0,+\infty]$ if ϕ is increasing, ϕ vanishes only at zero and

$$g(tr+(1-t)s) + t(1-t)\phi(|r-s|) \le tg(r) + (1-t)g(s).$$

Using the above definition, we introduce the definition of uniformly convex with geodesic for a function which defined from a subset of a global NPC space to $\mathbb{R}$.

Definition 3.2. Suppose that (N, d) is a global NPC space, $C \subset N$ be a convex set. A function $g : C \subseteq N \to \mathbb{R}$ is called uniformly convex with modulus $\phi : [0, +\infty) \to [0, +\infty]$ with respect to γ (or simply γ-uniformly convex) if ϕ is increasing, ϕ vanishes only at 0, and also, the function $go\gamma : [0, 1] \to \mathbb{R}$ is uniformly convex function (in sense of Definition**??**), defined as

$$g(\gamma(tr + (1-t)s)) + t(1-t)\phi(|r-s|) \leq (1-t)g(\gamma(s)) + tg\big(\gamma(r)\big),$$

for each $t \in [0, 1]$. Where $\gamma : [0, 1] \to C$ is geodesic and $0 \leq r, s \leq 1$

Theorem 3.3. *Let $\alpha > 0$, g be a γ-uniformly convex function and γ be a geodesic, then the following inequality is established;*

$$\begin{aligned}
&g(\gamma(\frac{1}{2})) + \frac{\Gamma(\alpha+1)}{2^{\alpha+2}} J^{\alpha}_{0^+}\phi(1) + \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha)_k J^{k+1}_{0^+}\phi(1)}{2^{k+3}} \\
&\leq \frac{1}{2}\Gamma(\alpha+1) J^{\alpha}_{1^-} g(\gamma(0)) + \frac{1}{2}\Gamma(\alpha+1) J^{\alpha}_{0^+} g(\gamma(1)) \\
&\leq \frac{g(\gamma(1)) + g(\gamma(0))}{2} - \frac{\alpha\phi(1)}{(\alpha+1)(\alpha+2)} \\
&= \alpha_{g(\gamma(0)), g(\gamma(1))}(\frac{1}{2}) - \frac{\alpha\phi(1)}{(\alpha+1)(\alpha+2)},
\end{aligned}$$

where $(\alpha)_k := \alpha(\alpha-1) \times \cdots \times (\alpha-k+1), (\alpha)_0 := 1,\ \alpha \geq 0, k \in \mathbb{N}$

Proof. we have

$$g(\gamma(\frac{x+y}{2})) + \frac{1}{4}\phi(|x-y|) \leq \frac{g(\gamma(x)) + g(\gamma(y))}{2}.$$

Now, set $y = 1 - x$ then

$$g(\gamma(\frac{1}{2})) + \frac{1}{4}\phi(|2x-1|) \leq \frac{g(\gamma(x)) + g(\gamma(1-x))}{2}.$$

Multiplying two side the above relation in $x^{\alpha-1}$ and integrating on $[0, 1]$, we have

$$\frac{g(\gamma(\frac{1}{2}))}{\alpha} + \frac{1}{4}\int_0^1 x^{\alpha-1}\phi(|2x-1|)dx \leq \frac{1}{2}(\int_0^1 (x^{\alpha-1})g(\gamma(x))) + x^{\alpha-1}g(\gamma(x-1))dx, \quad (3.1)$$

also

$$\int_0^1 x^{\alpha-1}\phi(|2x-1|)dx = \frac{1}{2}\int_{-1}^{1}(\frac{t+1}{2})^{\alpha-1}\phi(|t|)dt$$
$$= \frac{1}{2^\alpha}\int_{-1}^{0}(t+1)^{\alpha+1}\phi(|t|)dt + \frac{1}{2}\int_0^1(1+\frac{t-1}{2})^{\alpha-1}\phi(t)dt$$
$$= \frac{1}{2^\alpha}\int_0^1(1-t)^{\alpha-1}\phi(t)dt + \frac{1}{2}\int_0^1(\sum_{k=0}^{\infty}\binom{\alpha-1}{k}(\frac{t-1}{2})^k\phi(t))dt$$
$$= \frac{\Gamma(\alpha)}{2^\alpha}J_{0^+}^{\alpha}\phi(1) + \sum_{k=0}^{\infty}\frac{\binom{\alpha-1}{k}}{2^{k+1}}\int_0^1(t-1)^k\phi(t)dt$$
$$= \frac{\Gamma(\alpha)}{2^\alpha}J_{0^+}^{\alpha}\phi(1) + \sum_{k=0}^{\infty}\frac{(\alpha-1)_k}{2^{k+1}k!}(-1)^k\int_0^1(1-t)^k\phi(t)dt$$
$$= \frac{\Gamma(\alpha)}{2^\alpha}J_{0^+}^{\alpha}\phi(1) + \sum_{k=0}^{\infty}\frac{(\alpha-1)_k}{2^{k+1}k!}(-1)^k\Gamma(k+1)J_{0^+}^{k+1}\phi(1)$$
$$= \frac{\Gamma(\alpha)}{2^\alpha}J_{0^+}^{\alpha}\phi(1) + \sum_{k=0}^{\infty}\frac{(\alpha-1)_k}{2^{k+1}}(-1)^k J_{0^+}^{k+1}\phi(1), \tag{3.2}$$

where

$$\binom{\alpha}{0} := 1,$$

and

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} = \frac{(\alpha)_k}{k!}$$

Also, we have

$$\int_0^1 x^{\alpha-1}g(\gamma(x))dx = \Gamma(\alpha)J_{1^-}^{\alpha}g(\gamma(0))$$

$$\int_0^1 x^{\alpha-1}g(\gamma(1-x))dx = \int_0^1(1-x)^{\alpha-1}g(\gamma(x))dx = \Gamma(\alpha)J_{0^+}^{\alpha}g(\gamma(1)). \tag{3.3}$$

By using the relation (**??**), (**??**) and (**??**) the left inequality is proved. To prove the right side of inequality, for each $0 \le x \le 1$, we obtain

$$g(\gamma(x)) \le xg(\gamma(x)) + (1-x)g(\gamma(0)) - x(1-x)\phi(1)$$
$$g(\gamma(1-x)) \le (1-x)g(\gamma(1)) + xg(\gamma(0)) - x(1-x)\phi(1).$$

By adding sides of the two above equations with each other, we have

$$g(\gamma(x)) + g(\gamma(1-x)) \le g(\gamma(0)) + g(\gamma(1)) - 2x(1-x)\phi(1). \tag{3.4}$$

Multiply the sides of Equation (**??**) in $x^{\alpha-1}$ and integrating from $x=0$ to $x=1$, we obtain

$$\Gamma(\alpha)J^{\alpha}_{1-}g(\gamma(0))+\Gamma(\alpha)J^{\alpha}_{0+}g(\gamma(1))\leq\frac{g(\gamma(0))+g(\gamma(1))}{\alpha}-\frac{2\phi(1)}{(\alpha+1)(\alpha+2)},$$

consequently, Equations (2), (3), and (4) require that the right-hand side of inequality be true. □

Theorem 3.4. *Let $g:C\to\mathbb{R}$ be a function and $\gamma:[0,1]\to C$ be a geodesic and also $|(go\gamma)'|$ be a convex function, then*

$$|\frac{g(\gamma(1))+g(\gamma(0))}{2}-\frac{\Gamma(\alpha+1)}{2}(J^{\alpha}_{0+}g(\gamma(1))+J^{\alpha}_{1-}g(\gamma(0))| \leq\frac{1}{2(\alpha+1)}(1-\frac{1}{2^{\alpha}})(|(go\gamma)'(1)|+|(go\gamma)'(0)|). \tag{3.5}$$

Proof. By using Lemma **??** and the convexity of $|(go\gamma)'|$, we have

$$\begin{aligned}
&|\frac{g(\gamma(1))+g(\gamma(0))}{2}-\frac{\Gamma(\alpha+1)}{2}(J^{\alpha}_{0+}g(\gamma(1))+J^{\alpha}_{1-}g(\gamma(0))|\\
&\leq\frac{1}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}||(go\gamma)'(t\gamma(0)+(1-t)\gamma(1)|dt\\
&\leq\frac{1}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left(t|(go\gamma)'(\gamma(0)|+(1-t)|(go\gamma)'(\gamma(1))|\right)dt\\
&\leq\frac{1}{2}\left\{\int_0^{\frac{1}{2}}|(1-t)^{\alpha}-t^{\alpha}|\left(t|(go\gamma)'(\gamma(0)|+(1-t)|(go\gamma)'(\gamma(1))|\right)dt\right.\\
&\left.+\int_{\frac{1}{2}}^1|(1-t)^{\alpha}-t^{\alpha}|\left(t|(go\gamma)'(\gamma(0)|+(1-t)|(go\gamma)'(\gamma(1))|\right)dt\right\}\\
&=\frac{1}{2}(K_1+K_2).
\end{aligned}\tag{3.6}$$

Calculating K_1 and K_2, we have

$$\begin{aligned}
K_1&=|(go\gamma)'(0)|\left[\int_0^{\frac{1}{2}}t(1-t)^{\alpha}dt-\int_0^{\frac{1}{2}}t^{\alpha+1}dt\right]+|(go\gamma)'(1)|\left[\int_0^{\frac{1}{2}}(1-t)^{\alpha+1}dt-\int_0^{\frac{1}{2}}(1-t)t^{\alpha}dt\right]\\
&=|(go\gamma)'(0)|\left[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\right]+|(go\gamma)'(1)|\left[\frac{1}{\alpha+2}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\right]
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
K_2&=|(go\gamma)'(0)|\left[\int_{\frac{1}{2}}^1t^{\alpha+1}dt-\int_{\frac{1}{2}}^1t(1-t)^{\alpha}dt\right]+|(go\gamma)'(1)|\left[\int_{\frac{1}{2}}^1(1-t)t^{\alpha}dt-\int_{\frac{1}{2}}^1(1-t)^{\alpha+1}dt\right]\\
&=|(go\gamma)'(0)|\left[\frac{1}{\alpha+2}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\right]+|(go\gamma)'(1)|\left[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}\right].
\end{aligned}\tag{3.8}$$

So if we use **??** and **??** in **??**, we get the inequality **??** that completes the proof. □

Lemma 3.5. *Let $\gamma : [0,1] \to C$, $g : C \to \mathbb{R}$, $go\gamma$ be differentiable and $(go\gamma)' \in L[0,1]$, then*

$$\frac{g(\gamma(0)) + g(\gamma(1))}{2} - \frac{\Gamma(\alpha+1)}{2}[(J^{\alpha}_{0+}g(\gamma(1)) + J^{\alpha}_{1-}g(\gamma(0)))] \\ = \frac{1}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}](go\gamma)'(1-t)dt. \tag{3.9}$$

Proof. As proof of Lemma 2 in [**?**], if we divide the right integral of Equation **??** into two parts, we have:

$$\frac{1}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha}](go\gamma)'(1-t)dt \\ = \frac{1}{2}\int_0^1 (1-t)^{\alpha}(go\gamma)'(1-t)dt - \frac{1}{2}\int_0^1 t^{\alpha}(go\gamma)'(1-t)dt \tag{3.10}$$

Integrating by parts

$$\frac{1}{2}\int_0^1 (1-t)^{\alpha}(go\gamma)'(1-t)dt = \frac{1}{2}\Big(g(\gamma(1) - \Gamma(\alpha+1)J^{\alpha}_{1-}g(\gamma(0))\Big) \tag{3.11}$$

and similarly we get,

$$\frac{1}{2}\int_0^1 t^{\alpha}(go\gamma)'(1-t)dt = \frac{1}{2}\Big(g(\gamma(0) - \Gamma(\alpha+1)(J^{\alpha}_{0+}g(\gamma(1))\Big) \tag{3.12}$$

By adding the sides of Equations **??** and **??** together, we get to Equation **??**. □

Theorem 3.6. *Let (N,d) be a global NPC space, $C \subset N$ be a convex set. If $\gamma : [0,1] \to C$, $g : C \to \mathbb{R}$ are two functions and also $go\gamma$ is an invertible function such that $|(go\gamma)'|$ is uniformly convex with module ϕ, then*

$$|\frac{g(\gamma(0)) + g(\gamma(1))}{2} - \frac{\Gamma(\alpha+1)}{2}[J^{\alpha}_{0+}g(\gamma(1)) + J^{\alpha}_{1-}g(\gamma(0))]| \\ \leq \frac{1}{2(\alpha+1)}(1 - \frac{1}{2^{\alpha}})[|(go\gamma)'(1)| + |(go\gamma)'(0)|] - \frac{2^{\alpha+1} - \alpha - 1}{2^{\alpha+2}(\alpha+2)(\alpha+3)}\phi(1).$$

Proof. Using Lemma **??** and because the $|(go\gamma)'|$ is uniformly convex, so we have

$$
\begin{aligned}
&|\frac{g(\gamma(1))+g(\gamma(0))}{2}-\frac{\Gamma(\alpha+1)}{2}[J^{\alpha}_{0+}(\gamma(1))+J^{\alpha}_{1-}(\gamma(0))]| \\
&=\frac{1}{2}|\int_0^1[(1-t)^{\alpha}-t^{\alpha}]|(go\gamma)'(1-t)dt| \\
&\leq \frac{1}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}||(go\gamma)'(1-t)|dt \\
&\leq \frac{1}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|[t|(go\gamma)'(0)|+(1-t)|(go\gamma)'(1)|-t(1-t)\phi(1)]dt \\
&=\frac{1}{2}\{\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}][t|(go\gamma)'(0)|+(1-t)|(go\gamma)'(1)|]\}dt \\
&+\frac{1}{2}\int_{\frac{1}{2}}^1[t^{\alpha}-(1-t)^{\alpha}][t|(go\gamma)'(0)|+(1-t)|(go\gamma)'(1)|]dt \\
&-\frac{1}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|t(1-t)\phi(1)dt \\
&=\frac{1}{2}(K_1+K_2)-\frac{1}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|t(1-t)\phi(1)dt.
\end{aligned}
$$

Calculating K_1 and K_2, we have

$$
\begin{aligned}
&K_1=|(go\gamma)'(0)|[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}] \\
&+|(go\gamma)'(1)|[\frac{1}{\alpha+2}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}] \\
&K_2=|(go\gamma)'(0)|[\frac{1}{\alpha+2}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}]+|(go\gamma)'(1)|[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}].
\end{aligned}
$$

Also,

$$\int_0^1 |(1-t)^\alpha - t^\alpha| t(1-t)\phi(1)dt$$
$$= \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] t(1-t)\phi(1)dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] t(1-t)\phi(1)dt$$
$$= \phi(1)\int_0^{\frac{1}{2}} [t(1-t)^{1+\alpha} - t^{\alpha+1} + t^{\alpha+2}]dt$$
$$+ \phi(1)\int_{\frac{1}{2}}^1 [t^{\alpha+1} - t^{\alpha+2} - t(1-t)^{1+\alpha}]dt$$
$$= \phi(1)[-\frac{(1-t)^{\alpha+2}}{\alpha+2} + \frac{(1-t)^{\alpha+3}}{\alpha+3} - \frac{t^{\alpha+2}}{\alpha+2} + \frac{t^{\alpha+3}}{\alpha+3}]_0^{\frac{1}{2}}$$
$$+ \phi(1)[\frac{t^{\alpha+2}}{\alpha+2} - \frac{t^{\alpha+3}}{\alpha+3} + \frac{(1-t)^{\alpha+2}}{\alpha+2} - \frac{(1-t)^{\alpha+3}}{\alpha+3}]_{\frac{1}{2}}^1$$
$$= \phi(1)[-\frac{1}{(\alpha+2)2^{\alpha+2}} + \frac{1}{(\alpha+3)2^{\alpha+3}} - \frac{1}{(\alpha+2)2^{\alpha+2}} + \frac{1}{(\alpha+3)2^{\alpha+3}}$$
$$+ \frac{1}{\alpha+2} - \frac{1}{\alpha+3} + \frac{1}{\alpha+2} - \frac{1}{\alpha+3} - \frac{1}{(\alpha+2)2^{\alpha+2}} + \frac{1}{(\alpha+3)2^{\alpha+3}}$$
$$- \frac{1}{(\alpha+2)2^{\alpha+2}} + \frac{1}{(\alpha+3)2^{\alpha+3}}]$$
$$= \phi(1)[\frac{1}{(\alpha+3)2^{\alpha+1}} - \frac{1}{(\alpha+2)2^{\alpha}} + \frac{2}{\alpha+2} - \frac{2}{\alpha+3}]$$
$$= \frac{2^{\alpha+1} - \alpha - 1}{2^{\alpha+1}(\alpha+2)(\alpha+3)}\phi(1),$$

which completes the proof. □

Corollary 3.7. *Suppose (N, d) is a global NPC space and $C \subset N$ is a convex set. If $\gamma : [0,1] \to C$, $g : C \to \mathbb{R}$ are two functions and also $go\gamma$ is an invertible function such that $|(go\gamma)'|$ is uniformly convex with module ϕ also $\alpha = 1$, then*

$$|\frac{g(\gamma(0)) + g(\gamma(1))}{2} - \int_0^1 g(\gamma(t))dt| \leq \frac{1}{8}(|(go\gamma)'(0)| + |(go\gamma)'(1)|) - \frac{1}{48}\phi(1).$$

Proof. This follows from Theorem **??**. □

4. Conclusions

In this paper, we demonstrate Hermite-Hadamard's inequalities in Riemann-Liouville fractional integral transformations for uniformly convex functions from a geodesic perspective and obtain some other integral inequalities for fractional integrals.

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