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Sequential Henstock Integrals For Interval Valued Functions

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Abstract. In this paper, we introduce the concept of Sequential Henstock integrals for interval valued functions and discuss some of their properties.

Keywords: Sequential Henstock integral, Interval valued functions, guages, right and left endpoints.

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1. Introduction

Henstock integral of a function (also known as Henstock-Kursweil integral) introduced in the mid-1950s is a useful generalisation of the Riemann integral, which can handled nowhere-continuous functions, extreme oscillation functions and gives a simpler and more satisfactory version of the fundamental theorem of calculus that link the concept of differentiation of a function with the concept of integration of that function. While the standard definition of the Henstock integral uses the $\varepsilon - \delta$ definition, then the Sequential Henstock integral was introduced, by employing sequences of guage functions. Many authors have done a lot of work on application of the Henstock integral to functions taking real values and have published interesting results on a number of its' properties, see [1-15]. An interesting area which has not been given

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much attention is the functions taking interval values in real spaces. Interval analysis helps to reduce the uncertainty and error bounds yields in real values computations and measurements as well as find guaranteed solutions to differential equations and optimization problems.
In 2000, Wu and Gong[14] introduced the notion of the Henstock (H) integral of interval valued functions and Fuzzy number-valued functions and obtained a number of properties. In 2016, Yoon[15] introduced the concept of the Henstock Stieltjes (HS) integral of interval valued functions on time scale and investigated some of its properties. In the same year, Hamid and Elmuiz[5] established the concept of the Henstock Stieltjes (HS) integrals of interval valued functions and Fuzzy number-valued functions and obtained some number of properties of these integrals.
In this paper, we introduce the Sequential Henstock (SH)integrals of interval valued functions and discuss some of its properties.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $F(X)$ as an interval valued function, F^-, the left endpoint, F^+ as right endpoint, $\{\delta_n(x)\}_{n=1}^{\infty}$, as set of gauge functions, P_n, as set of partitions of subintervals of a compact interval $[a,b]$, X, as non empty interval in $\mathbb{R}$ and $d(X) = X^+ - X^-$, as width of the interval X and $\ll$ as much more smaller.

Definition 2.1[9,11] A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.

Definition 2.2[9,11] A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n}\, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} <, ..., < u_{m_n} = b$.

Definition 2.3 [11] A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable to α on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that if $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that for $\delta(x)$-fine tagged partitions $P = \{(u_{i-1}\, u_i), t_i\}_{i=1}^{n}$, we have

$$|\sum_{i=1}^{n} f(t_i)[u_i - u_{(i-1)}] - \alpha| < \varepsilon.$$

where the number α is the Henstock integral of f on $[a,b]$. The family of all Henstock integrals function on $[a,b]$ is denoted by $H[a,b]$ with $\alpha = (H)\int_{[a,b]} f(x)dx$ and $f \in H[a,b]$.

Definition 2.4 [11] A function $f : [a, b] \to \mathbb{R}$ is Sequential Henstock integrable to $\alpha \in \mathbb{R}$ on $[a, b]$ if for any $\varepsilon > 0$ there exists a sequence of gauge functions $\delta_\mu(x) = \{\delta_n(x)\}_{n=1}^{\infty}$ such that for any $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon,$$

where the sum $\sum$ is over P_n, we write $\alpha = (SH)\int_{[a,b]} f(x)dx$ and $f \in SH[a, b]$.

Lemma 2.5[5] Let f, k be Sequential Henstock (SH)integrable functions on $[a, b]$, if $f \leq k$ is almost everywhere on $[a, b]$, then

$$\int_a^b f \leq \int_a^b k.$$

Definition 2.6 [10 and 14]
Let $I_{\mathbb{R}} = \{I = [I^-, I^+]$: I is a closed bounded interval on the real line $\mathbb{R}\}$.
For $X, Y \in I_{\mathbb{R}}$, we define
i. $X \leq Y$ if and only if $Y^- \leq X^-$ and $X^+ \leq Y^+$,
ii. $X + Y = Z$ if and only if $Z^- = X^- + Y^-$ and $Z^+ = X^+ + Y^+$,
iii. $X.Y = \{x.y : x \in X, y \in Y\}$, where

$$(X.Y)^- = \min\{X^-.Y^-, X^-.Y^+, X^+.Y^-, X^+.Y^+\}$$

and

$$(X.Y)^+ = \max\{X^-.Y^-, X^-.Y^+, X^+.Y^-, X^+.Y^+\}.$$

Define $d(X, Y) = \max(|X^- - Y^-|, |X^+ - Y^+|)$ as the distance between intervals X and Y.

Definition 2.7 [5]
An interval valued function $F : [a, b] \to I_{\mathbb{R}}$ is Henstock integrable($IH[a, b]$) to $I_0 \in I_{\mathbb{R}}$ on $[a, b]$ if for every $\varepsilon > 0$, there exists a positive gauge function $\delta(x) > 0$ on $[a, b]$ such that for every $\delta(x) - fine$ tagged partitions $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$, we have

$$d(\sum_{i=1}^{n \in \mathbb{N}} F(t_i)(u_i - u_{i-1}), I_o) < \varepsilon$$

We say that α is the Henstock integral of F on $[a, b]$ with $(IH)\int_{[a,b]} F = \alpha$ and $F \in IH[a, b]$.

Now, we will define the Sequential Henstock integral of interval valued function and then discuss some of the properties of the integral.

Definition 2.8
An interval valued function $F : [a,b] \to I_{\mathbb{R}}$ is Sequential Henstock integrable($ISH[a,b]$) to $I_0 \in I_{\mathbb{R}}$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for every $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_o) < \varepsilon.$$

We say that α is the Sequential Henstock integral of F on $[a,b]$ with $(IH)\int_{[a,b]} F = \alpha$ and $F \in ISH[a,b]$.

3. Main Results

In this section, we discuss some of the basic properties of the interval valued Sequential Henstock integrals.

Theorem 3.1
If $F \in ISH[a,b]$, then there exists a unique integral value.

Proof. Suppose the integral value are not unique. Let $\alpha_1 = (ISH)\int_{[a,b]} F$ and $\alpha_2 = (ISH)\int_{[a,b]} F$ with $\alpha_i \neq \alpha_2$. Let $\varepsilon > 0$ then there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$ and $\{\delta_n^2(x)\}_{n=1}^{\infty}$ such that for each $\delta_n^1(x)$-fine tagged partitions P_n^1 of $[a,b]$ and for each $\delta_n^2(x)$-fine tagged partitions P_n^2 of $[a,b]$, we have

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \alpha_1) < \frac{\varepsilon}{2},$$

and

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \alpha_2) < \frac{\varepsilon}{2}.$$

respectively.
Define a positive function $\delta_n(x)$ on $[a,b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a,b]$. Then by triangular

inequality, we have

$$d(\alpha_1,\alpha_2)=d(\sum_{i=1}^{m_n\in\mathbb{N}}F(t_{i_n})(u_{i_n}-u_{(i-1)_n},\alpha_1)+\sum_{i=1}^{m_n\in\mathbb{N}}F(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\alpha_2)$$

$$\leq d(\sum_{i=1}^{m_n\in\mathbb{N}}F(t_{i_n})(u_{i_n}-u_{(i-1)_n},\alpha_1))+d(\sum_{i=1}^{m_n\in\mathbb{N}}F(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\alpha_2)$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

which is a contradiction. Thus $\alpha_1=\alpha_2$. This completes the proof.

Theorem 3.2
An interval valued function $F\in ISH[a,b]$ if and only if $F^-,F^+\in SH[a,b]$ and

$$(ISH)\int_{[a,b]}F=[(SH)\int_{[a,b]}F^-,(SH)\int_{[a,b]}F^+] \tag{1.1}$$

Proof. Let $F\in ISH[a,b]$, from Definition 3.3 there is a unique interval number $I_o=[I_0^-,I_0^+]$ in the property, then for any $\varepsilon>0$, there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$, $n\geq\mu$ on $[a,b]\in\mathbb{R}$ such that for any $\delta_n(x)$-fine tagged partition P_n, we have

$$d(\sum_{i=1}^{m_n\in\mathbb{N}}F(t_{i_n})(u_{i_n}-u_{(i-1)_n}),\alpha)<\varepsilon.$$

Observe that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}}F(t_{i_n})(u_{i_n}-u_{(i-1)_n}),I_0) = \max(|\sum_{i=1}^{m_n\in\mathbb{N}}F^-(t_{i_n})(u_{i_n}-u_{(i-1)_n})-I_0^-|,$$

$$|\sum_{i=1}^{m_n\in\mathbb{N}}F^+(t_{i_n})(u_{i_n}-u_{(i-1)_n}-I_0^+)|).$$

Since $u_{i_n}-u_{(i-1)_n}\geq 0$ for $1\leq i_n\leq m_n$, hen it follows that

$$|\sum_{i=1}^{m_n\in\mathbb{N}}F^-(t_{i_n})(u_{i_n}-u_{(i-1)_n})-I_0^-|<\varepsilon,|\sum_{i=1}^{m_n\in\mathbb{N}}F^-(t_{i_n})(u_{i_n}-u_{(i-1)_n}-I_0^+)|<\varepsilon.$$

for every $\delta_n(x)$-tagged partition $P_n=\{(u_{(i-1)_n},u_{i_n}),t_{i_n}\}_{i=1}^{m_n}$. Thus, by Definition 2.8, we obtain $F^+,F^-\in SH[a,b]$ and

$$I_o^-=(SH)\int_{[a,b]}F^-(x)dx$$

and

$$I_o^+ = (SH)\int_{[a,b]} F^+(x)dx.$$

Conversely, Let $F^- \in SH_{[a,b]}$. Then there exist a unique $\beta_1 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$, such that for any $\delta_n^1(x)$-fine tagged partitions P_n^1 we have

$$|\sum_{i=1}^{m_n\in\mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \beta_1| < \varepsilon.$$

Similarly,
Let $F^+ \in SH[a,b]$. Then there exist a unique $\beta_2 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^2(x)\}_{n=1}^{\infty}$, such that for any $\delta_n^2(x)$-fine tagged partitions P_n^2 we have

$$|\sum_{i=1}^{m_n\in\mathbb{N}} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n} - \beta_2)| < \varepsilon.$$

Let $\beta = [\beta_1, \beta_2]$. If $F^- \leq F^+$, then $\beta_1 \leq \beta_2$. We define $\delta_n(x) = \min(\delta_n^1(x), \delta_n^2(x))$, then for any $\delta_n(x) - fine$ tagged partitions P_n we have

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), \beta) < \varepsilon.$$

Hence, $F : [a,b] \to I_{\mathbb{R}}$ is Sequential Henstock integrable on $[a,b]$.
This completes the proof.

Theorem 3.3
Let $F, K \in ISH[a,b]$ with $F = [F^-, F^+]$ and $H = [K^-, K^+]$ and $\gamma, \xi \in \mathbb{R}$. Then $\gamma F, \xi K \in ISH[a,b]$ and

$$(ISH)\int_{[a,b]} (\gamma F + \xi K)dx = \gamma(ISH)\int_{[a,b]} F dx + \xi(ISH)\int_{[a,b]} K dx$$

Proof. (i) If $F, K \in ISH[a,b]$, then $[F^-, F^+], K = [K^-, K^+] \in SH[a,b]$ by Theorem 3.2. Hence, $\gamma F^- + \xi K^-, \gamma F^- + \xi K^+, \gamma F^+ + \xi K^-, \gamma F^+ + \xi K^+ \in SH[a,b]$.
1) If $\gamma > 0$ and $\xi > 0$, then

$$\begin{aligned}(SH)\int_{[a,b]} (\gamma F + \xi K)^- dx &= (SH)\int_{[a,b]} (\gamma F^- + \xi K^-)dx \\ &= \gamma(SH)\int_{[a,b]} F^- dx + \xi(SH)\int_{[a,b]} K^- dx\end{aligned}$$

$$\begin{aligned} &= \gamma((ISH)\int_{[a,b]} Fdx)^- + \xi((ISH)\int_{[a,b]} Kdx)^- \\ &= (\gamma(ISH)\int_{[a,b]} Fdx + \xi(ISH)\int_{[a,b]} Kdx)^-. \end{aligned}$$

2) If $\gamma < 0$ and $\xi > 0$, then

$$\begin{aligned} (SH)\int_{[a,b]}(\gamma F+\xi K)^- dx &= (SH)\int_{[a,b]}(\gamma F^+ + \xi K^+)dx \\ &= \gamma(SH)\int_{[a,b]} F^+dx + \xi(SH)\int_{[a,b]} K^+dx \\ &= \gamma((ISH)\int_{[a,b]} Fdx)^+ + \xi((ISH)\int_{[a,b]} Kdx)^+ \\ &= (\gamma(ISH)\int_{[a,b]} Fdx + \xi(ISH)\int_{[a,b]} Kdx)^-. \end{aligned}$$

3) If $\gamma > 0$ and $\xi < 0$ (or $\gamma < 0$ and $\xi > 0$), then

$$\begin{aligned} (ISH)\int_{[a,b]}(\gamma F+\xi K)^- dx &= (SH)\int_{[a,b]}(\gamma F^- + \xi K^+)dx \\ &= \gamma(SH)\int_{[a,b]} F^-dx + \xi(SH)\int_{[a,b]} K^+dx \\ &= \gamma((ISH)\int_{[a,b]} Fdx)^- + \xi((ISH)\int_{[a,b]} Kdx)^+ \\ &= (\gamma(ISH)\int_{[a,b]} Fdx + \xi(ISH)\int_{[a,b]} Kdx)^-. \end{aligned}$$

Similarly, for three cases above, we have

$$(ISH)\int_{[a,b]}(\gamma F+\xi K)^+ dx = (\gamma(ISH)\int_{[a,b]} Fdx + \xi(ISH)\int_{[a,b]} Kdx)^+$$

Hence, by Theorem 3.2, $\gamma F, \xi K \in ISH[a,b]$ and

$$(ISH)\int_{[a,b]}(\gamma F+\xi K)dx = \gamma(ISH)\int_{[a,b]} Fdx + \xi(ISH)\int_{[a,b]} Kdx.$$

This completes the proof.

Theorem 3.4

Let $F, K \in ISH[a,b]$ and $F(x) \le K(x)$ nearly everywhere on $[a,b]$, then

$$(ISH)\int_{[a,b]} F(x)dx \le (ISH)\int_{[a,b]} Kdx$$

Proof. If $F(x) \leq K(x)$ nearly everywhere on $[a, b]$ and $F, K \in ISH[a, b]$, then $F^-, F^+, K^-, K^+ \in SH[a, b]$ and $F^- \leq F^+, K^- \leq K^+$ nearly everywhere on $[a, b]$. By Lemma 2.5

$$(SH)\int_{[a,b]} F^-(x)dx \leq (SH)\int_{[a,b]} K^- dx$$

and

$$(ISH)\int_{[a,b]} F^+(x)dx \leq (ISH)\int_{[a,b]} K^+ dx.$$

Hence by Theorem 3.2, we have

$$(ISH)\int_{[a,b]} F(x)dx \leq (ISH)\int_{[a,b]} K dx.$$

This completes the proof.

Theorem 3.5
Let $F, K \in ISH[a, b]$ and $d(F, K)$ is Sequential Lebesgue (SL) integrable on $[a, b]$, then

$$d((ISH)\int_{[a,b]} F dx, (ISH)\int_{[a,b]} K dx) \leq (SL)\int_{[a,b]} d(F, K)dx.$$

Proof. By metric definition,

$$d((ISH)\int_{[a,b]} F dx, (ISH)\int_{[a,b]} K dx)$$

$$\begin{aligned}
&= \max(|((SH)\int_{[a,b]} F dx)^- - ((SH)\int_{[a,b]} K dx)^-|, |((SH)\int_{[a,b]} F dx)^+ - ((SH)\int_{[a,b]} K dx)^+|) \\
&= \max(|(SH)\int_{[a,b]} (F^- - K^-)dx|, |(SH)\int_{[a,b]} (F^+ - K^+)dx|) \\
&\leq \max((SL)\int_{[a,b]} |(F^- - K^-)|dx, (SL)\int_{[a,b]} |(F^+ - K^+)|dx) \\
&\leq (SL)\int_{[a,b]} \max(|(F^- - K^-)|, |(F^+ - K^+)|)dx \\
&= (SL)\int_{[a,b]} d(F, K)dx.
\end{aligned}$$

This completes the proof.

Theorem 3.6

If $F \in ISH[a,c]$ and $F \in ISH[c,b]$, then $F \in ISH[a,b]$ and

$$(ISH)\int_a^b F = (ISH)\int_a^c F + (ISH)\int_c^b F.$$

Proof.
If $F \in ISH[a,c])$, then by Theorem 3.2, $F^- \in SH[a,c])$ and $F^- \in SH[c,b])$. Hence, $F^- \in SH[a,b])$ and

$$\begin{aligned}(SH)\int_a^b F^- &= (SH)\int_a^c F^- + (SH)\int_c^b F^- \\ &= ((ISH)\int_a^c F + (ISH)\int_c^b F)^-.\end{aligned}$$

Similarly,
Since $F \in ISH[a,c])$, then by Theorem 3.2, $F^+ \in SH[a,c])$ and $F^+ \in SH[c,b])$. Hence, $F^+ \in SH[a,b])$ and

$$\begin{aligned}(SH)\int_a^b F^+ &= (SH)\int_a^c F^+ + (SH)\int_c^b F^+ \\ &= ((ISH)\int_a^c F + (ISH)\int_c^b F)^+.\end{aligned}$$

Hence by Theorem 3.2, $F \in ISH[a,b]$ and

$$(ISH)\int_a^b F = (ISH)\int_a^c F + (ISH)\int_c^b F.$$

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