Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/CJMS.2022.23653.1627

Caspian J Math Sci. **12**(1)(2023), 51-67 (Research Article)

Evasion from many Pursuers on a Compact and not Necessarily Convex Set

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Abstract. In a given compact subset of $\mathbb{R}^2$ which is not necessarily convex, we study a pursuit differential game of many pursuers and one evader. The game must be done in this set. The constraints that we used for control are of coordinate-wise integral type. Pursuers want to complete the pursuit and the evader is doing the opposite. Completion of the pursuit means that the distance of evader becomes zero with some pursuers. Some conditions for the completion of pursuit is derived.

Keywords: Differential Game, Integral Constraint, Nonconvexity.

2000 Mathematics subject classification: xxxx, xxxx; Secondary xxxx.

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Received: 08 June 2022
Revised: 06 September 2022
Accepted: 10 September 2022
How to Cite: Bigdeli, Hamid; Mohammadkarimi, Javad; Biranvand, Nader. Evasion from many Pursuers on a Compact and not Necessarily Convex Set, Casp.J. Math. Sci.,**12**(1)(2023), 51-67.

1. Introduction

What does it mean a mathematical game? a mathematical game deals with players, strategies and explaination that why winers win the game. Differential games are using diffrential equations this is the reason for the terminology. Differential games and pursuit-evasion problems are investigated by many researchers. For example see, Isaacs [10], Petrosyan [13] and Pshenichii [15]. Also A large part of literature has investigated two person differential games and fundamental results were made by researchers such as Blaquiere et al. [1], Krasovskii and Subbotin [11], Pontryagin [14]. Important results were obtained by researchers such as Friedman [6], Hajek [7], Nikol'skii [12], Pshenichnyi and Ostapenko [16], and further new methods were developed in many works such as Chikrii [2], Satimov and Rikhsiev [17].

The game of lion and man [3] is a generally known problem. Lion wants to catch the man in a given set. So lion needs the strategy to make this happen and we can interpret that as a pursuit-evasion game. Mathematical catching means after a period of time say, $T < \infty$, the distance between pursuer and evader is zero. Besicovitch proved that evasion can happen in the game of lion. In [5] Flynn studied the game for pursuer by imposing a condition on the speed bound of lion, say, 1 and lion tries to decrease the distance for getting man, with speed by $v \geq 1$.

In [8] Ibragimov and Salimi study a differential game for inertial players. They assume the control resource of the each pursuer is greater than that of evader. Ibragimov et al. in [9] studied an evasion from many pursuers in a differential games by imposing integral constraints. Recently, in [18] Salimi et al. study a differential game that countable objects try to get one evader. All the players must have simple motions and some pursuers have integral constraints and some other pursuers and the evader have geometric constraints. In other line of research in the field of pursuit-evasion game, authors in [19] investigates the problem of spacecraft interception game with incomplete-information and proposes switching strategies based on the differential game theory. In the interception process, the target can switch among multiple strategies to evade the interceptor. This leads to the formulation of switching strategies pursuit problem for the interceptor. Also, authors in [20] considered reach-avoid differential game with two evaders and one pursuer in the plane which is divided into a play region and a goal region by a straight line. Two evaders, starting from the play region, aim at reaching the goal region protected by the faster pursuer who tries to capture the evaders.

In the present paper, we study a differential game of pursuers and one evader with integral constraints. Game must be done in a compact set in $\mathbb{R}^2$ which is not necessarily convex.

2. Construction of the problem

We study a following differential game

$$\frac{dx_i}{dt} = p_i, \quad x_i(0) = x_{i0}, \quad i = 1, ...m, \quad \frac{dy}{dt} = e, \quad y(0) = y_0, \qquad (2.1)$$

where $x_i,\ p_i,\ y,\ e \in \mathbb{R}^2$, p_i is control parameter of the pursuer x_i $i = 1, ..., m$ and e is control parameter of the evader y. In $\mathbb{R}^2$, we are given a nonempty compact set $M \subset \mathbb{R}^2$. According to the rule of the game, all players must move in M. Suppose the nonempty compact convex set N in [4] and the circle N' with radius r for which we have $N' \subset N$ are given. Let n be a center of N'. We assume $\text{dist}(n, \partial N) > r$ (Figure1). For the rest we use assumption 1 for the $\text{dist}(n, \partial N) > r$. Our nonconvex set M is $N - N'$.

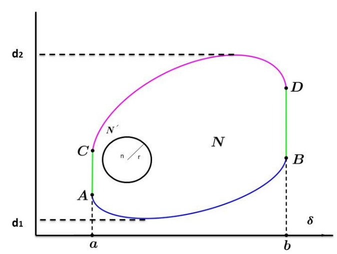


Figure 1. Nonconvex set

Definition 2.1. [4] A measurable function $p_i(t) = (p_{i1}(t), p_{i2}(t))$ $t \geq 0$, is called admissible control of the pursuer x_i if

$$\int_0^\infty |p_{ij}(s)|^2 ds \leq \psi_{ij}^2, \qquad (2.2)$$

where ψ_{ij}, $i = 1, ..., m$, $j = 1, 2$, are given positive numbers.

Definition 2.2. [4] A measurable function $e(t) = (e_1(t), e_2(t))$ $t \geq 0$, is called admissible control of the evader if

$$\int_0^\infty |e_j(s)|^2 ds \leq \varphi_j^2, \qquad (2.3)$$

where φ_j, $j = 1, 2$ are given positive numbers.

Definition 2.3. [4] Let

$$x_i(t) = x_{i0} + \int_0^t p_i(s)ds \in M,$$

$$y(t) = y_{i0} + \int_0^t e(s)ds \in M.$$

We need the quantities $w_i(t)$, $i = 1, \dots, m$, and $k_1(t)$ described by the following equations

$$\frac{dw_{i1}}{dt} = -p_{i1}^2, \quad w_{i1}(0) = \psi_{i1}^2,$$

$$\frac{dk_1}{dt} = -e_1^2, \quad k_1(0) = \varphi_1^2.$$

Clearly,

$$w_{i1}(t) = \psi_{i1}^2 - \int_0^t p_{i1}^2(s)ds,$$

$$k_1(t) = \varphi_1^2 - \int_0^t e_1^2(s)ds,$$

which we call x-energies of the pursuer x_i and evader y, respectively, available at the time t.

Definition 2.4. [4] A measurable function

$$P_i(t, x_i, y, k_1, e) = (P_{i1}(t, x_i, y, k_1, e), P_{i2}(t, x_i, y, q_1, e))$$

,

$$P_i \colon \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

is called a strategy of the pursuer x_i if for any control of the evader $e(t)$, $t \geq 0$, the initial value problem

$$\frac{dx_i}{dt} = P_i(t, x_i, y, k_1, e), \quad x_i(0) = x_{i0},$$

$$\frac{dy}{dt} = e, \quad y(0) = y_0,$$

$$\frac{dk_1}{dt} = -e_1^2, \quad k_1(0) = \varphi_1^2.$$

has a unique solution $(x_i(t), y(t), k_1(t))$ and the inequalities

$$\int_0^\infty P_{i1}^2(s, x_i(s), y(s), k_1(s), e(s))ds \leq \psi_{i1}^2,$$

$$\int_0^\infty P_{i2}^2(s, x_i(s), y(s), k_1(s), e(s))ds \leq \psi_{i2}^2.$$

hold.

Definition 2.5. [4] We are given initial position $\{x_{10}, \dots, x_{m0}, y_0\}$ for the time T in the game (2.1)-(2.3). Pursuit can be completed from the initial position, if there exist strategies P_i, $i = 1, ...m$, of the pursuers such that for any control $e = e(t)$ of the evader the equality $x_i(t) = y(t)$ holds for some $i \in \{1, \dots, m\}$ and $t \in [0, T]$.

Problem 2.6. For the completion of pursuit in the above game the goal is to find a sufficient condition.

3. Main result

The following is the main theorem of the paper.

Theorem 3.1. *If one of the following*

$$\psi_{11}^2 + \psi_{21}^2 + ... + \psi_{m1}^2 > \varphi_1^2, \quad \psi_{12}^2 + \psi_{22}^2 + ... + \psi_{m2}^2 > \varphi_2^2 \tag{3.1}$$

holds, then pursuit can be completed in the above game for a finite time.

We need to provide a simple lemma which says that in our case convexity is not important.

Let $[a, b]$ and $[d_1, d_2]$ be the projection of the given set N in [4] on the x-axis and y-axis, respectively. Define the functions $f(\delta) = \min_{(\kappa,\delta)\in N} \kappa$ and $F(\delta) = \max_{(\kappa,\delta)\in N} \kappa$ for all $\delta \in [a, b]$. Now set $g(\delta) = \min_{(\kappa,\delta)\in M} \kappa$ and $G(\delta) = \max_{(\kappa,\delta)\in M} \kappa$ for all $\delta \in [a, b]$.

Lemma 3.2. *The functions f and g with assumption 1 are equal as a funtion from $[a, b]$ to $[d_1, d_2]$. The same is true for F and G.*

Proof. It is obvious that when we project N and M on the x-axis and y-axis (Figure1), the resulting sets which are closed and bounded, are exactly the same in both cases. □

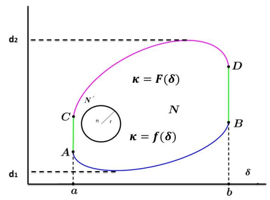
In general, the boundary of $N - N'$ is divided by the points $A = (a, f(a))$, $B = (b, f(b))$, $C = (a, F(a))$, and $D = (b, F(b))$, which lie on the vertical lines $\delta = a$ and $\delta = b$, into 4 arcs: the graphs of the functions $\kappa = f(\delta)$, $\kappa = F(\delta)$, $\delta \in [a, b]$, and the segments AC and BD as shown in (Fig.2A). Note that the segments AC and BD may shrink to points A and B, respectively, as shown in (Fig.2B).

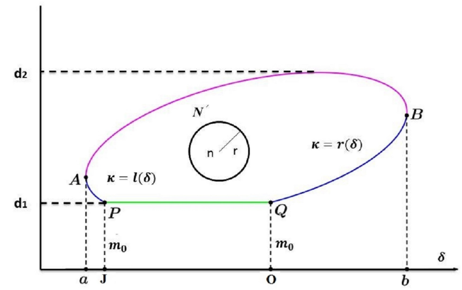
Let the minimum value of the function $f(\delta)$ on $[a, b]$ be m_0 (Figure 2B). Set

$$J = \min_\delta \{f(\delta) = m_0\}, \quad O = \max_\delta \{f(\delta) = m_0\}.$$

Since $f(\delta)$ is convex by Lemma 3.2, we have $f(\delta) = m_0$ for all $\delta \in [J, O]$. Note that if the function $f(\delta)$ attains the value m_0 at a unique point, then $J = O$.

For $a < J$, set $\kappa = l(\delta)$ which is the restriction of the function $\kappa = f(\delta)$, convex and continuous. For $O < b$, set $\kappa = z(\delta)$ which is the





A. $\kappa = f(\delta)$ and $\kappa = F(\delta)$ B. $\kappa = l(\delta)$ and $\kappa = z(\delta)$.

FIGURE 2. Functions $\kappa = f(\delta)$, $\kappa = F(\delta)$, $\kappa = l(\delta)$, and $\kappa = z(\delta)$.

restriction of $\kappa = f(\delta)$, convex and continuous (Fig.2B). Note that in the cases $a = J$ and $O = b$, the graphs of the functions $\kappa = l(\delta)$ and $\kappa = z(\delta)$ will be the points $(a, l(a))$ and $(b, r(b))$, respectively.

Let

$$z'_-(\delta) = \lim_{h\to 0^-} \frac{z(\delta + h) - z(\delta)}{h}, \quad \delta \in (O, b),$$

$$z'_+(\delta) = \lim_{h\to 0^+} \frac{z(\delta + h) - z(\delta)}{h}, \quad \delta \in [O, b),$$

denote the left and right derivatives of the function $z(\delta)$, respectively. Define $z'_-(O) = f'_-(O)$.

The following lists some properties of the function $\kappa = z(\delta)$.

Property 3.3. Let $O < b$. For the function $\kappa = z(\delta)$ we have:

Ω_1. The functions $z'_-(\delta)$ and $z'_+(\delta)$ are increasing on $[O, b)$. Moreover, for any $\delta_1 < \delta_2$ in $[O, b)$

$$z'_-(\delta_1) \le z'_+(\delta_1) \le z'_-(\delta_2) \le z'_+(\delta_2).$$

Ω_2. The set of discontinuities of the $z'_+(\delta)$, $\delta \in (O, b)$, is not of the second kind.

Ω_3. The set of discontinuous points of $z'_+(\delta)$ is at most countable subset of (O, b)

Ω_4. The function $z(\delta)$ increases on $[O, b]$.

Proof. Since for $O < b$, $\kappa = z(\delta)$ is the restriction of $\kappa = f(\delta)$, and by Lemma 3.2, the function f with its projected doamin and codomain is the same as a related function in the convex case, the proof is exactly the same as of [4] Property 1. □

Proof. We prove the theorem. Let $\psi_{11}^2+\psi_{21}^2+...+\psi_{m1}^2>\varphi_1^2$. Similarly we have the second inequality. Denote

$$\psi_1=(\psi_{11}^2+\psi_{21}^2+...+\psi_{m1}^2)^{1/2},$$
$$\varphi_{i1}=\frac{\varphi_1}{\psi_1}\psi_{i1},\ i=1,2,...,m.$$

Clearly, $\varphi_{i1}<\psi_{i1}$. □

3.1. **Construction of pursuers' strategies.** We construct the following strategies of pursuers for $0\le t\le T$, where $T=\frac{L}{\beta}$, $L=\max_{x,y\in M}|x-y|$,

$$\beta=\frac{1}{2L}\min\left\{\psi_{11}^2-\varphi_{11}^2,\psi_{m1}^2-\varphi_{m1}^2,\psi_{12}^2,...,\psi_{m2}^2\right\},0<u_0\le\frac{1}{6L}\min_i\{\psi_{i1}^2-\varphi_{i1}^2\}. \tag{3.2}$$

Without any loss of generality, we assume that $f(a)\le f(b)$. Set

$$p_{i1}(t)=\frac{\beta}{L}(\bar{x}_1-x_{i1}^0),\quad p_{i2}(t)=\frac{\beta}{L}(\bar{x}_2-x_{i2}^0),\quad 0\le t\le T, \tag{3.3}$$

where $(\bar{x}_1,\bar{x}_2)=(a,f(a))$. Then the position of all pursuers will be $(a,f(a))$ at the time T. Indeed,

$$x_{ij}(T)=x_{ij}^0+\int_0^T\frac{\beta}{L}(\bar{x}_j-x_{ij}^0)dt=\bar{x}_j,\quad j=1,2.$$

Pursuers spent the total amount of energies on $[0,T]$ as follows. For controls (3.3), using the

$$\beta\le\frac{1}{2L}\left(\psi_{i1}^2-\varphi_{i1}^2\right),\quad i\in\{1,2,...,m\},$$

following from (3.2), we have

$$\int_0^T p_{i1}^2(t)dt=\int_0^T\frac{\beta^2}{L^2}(\bar{x}_1-x_{i1}^0)^2dt=\frac{\beta}{L}|\bar{x}_1-x_{i1}^0|^2\le L\beta\le\frac{\psi_{i1}^2-\varphi_{i1}^2}{2}. \tag{3.4}$$

Then, obviously, by using (3.4) we have

$$w_{i1}(T)=\psi_{i1}^2-\int_0^T p_{i1}^2(t)dt\ge\frac{\psi_{i1}^2+\varphi_{i1}^2}{2}>\varphi_{i1}^2.$$

This means that φ_{i1}^2 is less than x-energy of pursuer x_i at T. So we have

$$\sum_{i=1}^m w_{i1}(T)>\sum_{i=1}^m\varphi_{i1}^2=\varphi_1^2. \tag{3.5}$$

Similarly, by (3.2) $L\beta \leq \frac{1}{2}\psi_{i2}^2$ and hence

$$\int_0^T p_{i2}^2(t)dt = \int_0^T \frac{\beta^2}{L^2}(\bar{x}_2 - x_{i2}^0)^2 dt = \frac{\beta}{L}|\bar{x}_2 - x_{i2}^0|^2 \leq L\beta \leq \frac{1}{2}\psi_{i2}^2,$$

and for the y-energy of pursuer x_i we get

$$w_{i2}(T) = \psi_{i2}^2 - \int_0^T p_{i2}^2(t)dt \geq \frac{1}{2}\psi_{i2}^2, \ i = 1, ..., m. \tag{3.6}$$

Using (3.5) and (3.6) we have that at the time T the total amount of x-energies of evader is still less than pursuers, and y-energy of all pursuers is positive.

In the following we are going to explain roughly how the pursuers' strategies have been constructed. For constructing the pursuers' strategies in subsection 3.1 we have to consider some factors for satisfying. First we want that position of all pursuers will be $(a, f(a))$ at some specific time, say, T. For this reason, based on Definition 2.3, Firstly we want $x_{ij}(T) = a$ or $f(a)$. So without determining any parameter ahead of time, we put $x_{ij}(T) = x_{ij}^0 + \int_0^T p_{ij}(t)dt$, Which we would know for satisfying $x_{ij}(T) = a$ or $f(a)$, we should have $p_{ij} = K(a - x_{ij}^0)$ or $p_{ij} = K(f(a) - x_{ij}^0)$ such that $KT = 1$. Second we want that x-energies of pursuer x_i becomes greater than the x-energy of each evader. This tell us with related computations that how determine β to force that the former condition happens and $w_{i2}(T)$ becomes greater than φ_{ij}^2.

Lemma 3.4. *If*

$$k_1\left(T + \frac{4L}{u_0}\right) \geq k_1(T) - \varphi_{11}^2, \tag{3.7}$$

then in the interval $\left[T, T + \frac{4L}{u_0}\right]$, *we have the completion of pursuit by pursuer* x_1.

Proof. In this proof, we use the temporary notations $x = (x_1, x_2)$ for the position $x_1 = (x_{11}, x_{12})$ and $p = (p_1, p_2)$ for the velocity $p_1 = (p_{11}, p_{12})$. Set

$$p(t) = (u_0, 0), \quad t \in \Gamma_0 = \{t > T \mid \ x_1(t) < y_1(t), \quad x_2(t) \neq z(x_1(t))\}. \tag{3.8}$$

Since

$$x(t) = x(T) + \int_T^t p(s)ds = (a + (t - T)u_0, f(a)),$$

by using (3.8) pursuer x moves along the straight line $\kappa = f(a)$ starting from the point $A = x(T) = (\bar{x_1}, \bar{x_2}) = (a, f(a))$ with the velocity $p = (u_0, 0)$ until one of the following

$$x_1(t) < y_1(t), \quad x_2(t) \neq z(x_1(t))$$

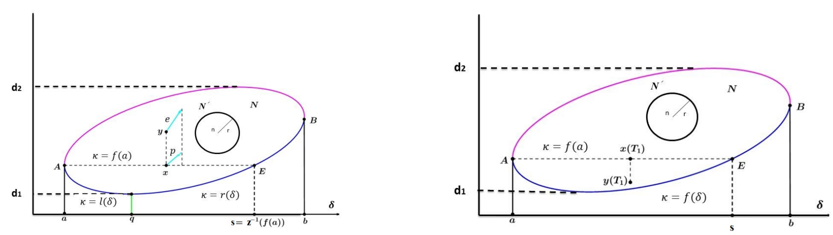
which are the conditions of the Γ_0 fails to hold. Now Consider the following two cases.

Case 1. $x_1(T_1) = y_1(T_1),\ x_1(T_1) < s$ (Figure 3) at some $T_1 \geq T$, where $s = z^{-1}(f(a))$.

Case 2. $z(x_1(T_1)) = f(a)$ (that is $z(x_1(T_1)) = x_2(T_1)$) and $x_1(T_1) \leq y_1(T_1)$ (Figure 4) at some $T_1 \geq T$.

It is easy to see that

$$T \leq T_1 \leq T + \frac{s-a}{u_0} \leq T + \frac{L}{u_0}. \tag{3.9}$$



A. $x_1(T_1) = y_1(T_1) < s$ and $y_2(T_1) > f(a)$. B. $x_1(T_1) = y_1(T_1) < s$ and $y_2(T_1) < f(a)$.

FIGURE 3. Case 1. $x_1(T_1) = y_1(T_1) < s$

We seperate case 1 as follow:

Case 1a. $x_1(T_1) = y_1(T_1),\ x_1(T_1) < s,\ y_2(T_1) < f(a)$ (Figure 3B).

Case 1b. $x_1(T_1) = y_1(T_1),\ x_1(T_1) < s,\ y_2(T_1) > f(a)$ (Figure 3A).

For the strategy of pursuer x, we set

$$p(t) = (e_1(t), -u_0), \quad t \geq T_1. \tag{3.10}$$

which proves that pursuit is completed at some time $\tau \in \left[T_1, T + \frac{2L}{u_0}\right]$. In fact, using (3.10) for all $t \geq T_1$, we have

$$x_1(t) = x_1(T_1) + \int_{T_1}^{t} p_1(s)ds = y_1(T_1) + \int_{T_1}^{t} e_1(s)ds = y_1(t), \tag{3.11}$$

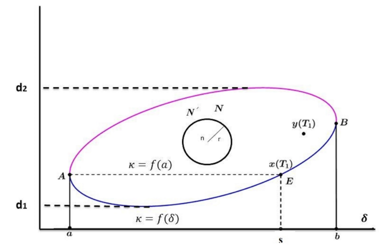


FIGURE 4. Case 2. $f(a) = z(x_1(T_1)$ and $x_1(T_1) \leq y_1(T_1)$.

It means that, for any $t \geq T_1$, the positions of pursuer x and evader y are on the same vertical line. Hence, we have the completion of pursuit if only $x_2(\tau) = y_2(\tau)$ at some $\tau \geq T_1$. To prove this,

$$x_2(t) - y_2(t) = x_2(T_1) - y_2(t) + \int_{T_1}^{t} p_2(s)ds \leq L - (t - T_1)u_0. \quad (3.12)$$

It is obvious that the right hand side of (3.12) at $t = T_1 + \frac{L}{u_0}$ is 0 , so $x_2(\tau) = y_2(\tau)$ at some $T_1 \leq \tau \leq T_1 + \frac{L}{u_0}$, which proves that $x(\tau) = y(\tau)$. It means pursuit is completed at τ. By using (3.9) for the time τ we have

$$T \leq \tau \leq T_1 + \frac{L}{u_0} \leq T + \frac{2L}{u_0}. \quad (3.13)$$

Next, we *prove admissibility of strategy (3.10).* First, we prove

$$x(t) \in int(M), \quad T_1 < t < \tau.$$

exactly we prove that $x(t)$ is an interior point of the set bounded by the lines $\kappa = f(a)$ and $\kappa = f(\delta)$, $a \leq \delta \leq s$. In fact, using (3.10)

$$x_2(t) = f(a) - (t - T_1)u_0 < f(a), \quad (3.14)$$

and so $x(t)$ is under the straight line $\kappa = f(a)$. Also, if $O \leq x_1(t) \leq s$, using (3.11) $z(x_1(t)) = z(y_1(t)) \leq y_2(t) < x_2(t)$ meaning that $x(t)$ is above the curve $\kappa = z(\delta)$, $O \leq \delta \leq s$. Similarly, $x_2(t) > l(x_1(t))$ if $a \leq x_1(t) \leq J$.

Finally, if $J < x_1(t) < O$, then $m_0 \leq y_2(t) < x_2(t) < f(a)$. Thus, $x(t) \in int(M)$, $T_1 < t < \tau$, when pursuer x applies the strategy (3.10). Hence, the boundary of the set M cannot be a barrier when pursuer applies the strategy (3.10).

We prove

$$\int_0^t p_1^2(s)ds \le \psi_{11}^2. \tag{3.15}$$

Since

$$\int_0^{T+\frac{2d}{u_0}} p_1^2(s)ds = \left(\int_0^T + \int_T^{T_1} + \int_{T_1}^{T+\frac{2L}{u_0}} \right) p_1^2(s)ds$$

and by (3.4), (3.8) and (3.9)

$$\int_0^T u_1^2(s)ds \le L\beta, \quad \int_T^{T_1} p_1^2(s)ds \le (T_1 - T)u_0^2 \le L \cdot u_0,$$

and by (3.7)

$$\int_{T_1}^{T+\frac{2L}{u_0}} e_1^2(s)ds \le \int_T^{T+\frac{4L}{u_0}} e_1^2(s)ds \le \varphi_{11}^2,$$

therefore by (3.2)

$$\int_0^{T+\frac{2L}{u_0}} p_1^2(s)ds \le L\beta + Lu_0 + \varphi_{11}^2 \le \frac{1}{2}(\psi_{11}^2 - \varphi_{11}^2) + \frac{1}{6}(\psi_{11}^2 - \varphi_{11}^2) + \varphi_{11}^2 < \psi_{11}^2.$$

Thus, the strategy (3.10) is admissible.

Next, we study both Case 1b and Case 2. Let

$$\begin{aligned} \Gamma_1 &= \{t \ge T_1 \mid \ x_2(t) \ne z(x_1(t))\}, \\ \Delta_1 &= \{t \ge T_1 \mid \ x_1(t) = y_1(t),\ x_2(t) = z(x_1(t))\}, \\ \Delta_2 &= \{t \ge T_1 \mid \ x_1(t) < y_1(t),\ x_2(t) = z(x_1(t))\}, \quad \Delta = \Delta_1 \cup \Delta_2, \end{aligned}$$

and

$$P_0(\xi) = (P_{10}(\xi), P_{20}(\xi)) = \begin{cases} \left(u_0, z'_+(\delta)u_0\right), & z'_+(\delta) \le 1 \\ \left(\frac{u_0}{z'_+(\delta)}, u_0\right), & z'_+(\delta) \ge 1 \end{cases}, \quad s \le \delta \le b. \tag{3.16}$$

For $t \ge T_1$, we set (Fig. 5)

$$p(t) = \begin{cases} (e_1(t), u_0), & t \in \Gamma_1, \\ (e_1(t), u_0), & e_1(t) < P_{10}(x_1(t)), \quad t \in \Delta_1, \\ P_0(x_1(t)), & e_1(t) \ge P_{10}(x_1(t)), \quad t \in \Delta_1, \\ P_0(x_1(t)), & t \in \Delta_2. \end{cases} \tag{3.17}$$

A. $e_1(t) < P_{10}(x_1(t)),\ t \in \Delta_1$ B. $e_1(t) \geq P_{10}(x_1(t)),\ \ t \in \Delta_1$

FIGURE 5. Structure of $p(t)$ for $t \in \Delta_1$

Property 3.5. The strategy (3.24) has the following properties:

Σ_1. Let $x_1(t_1) = y_1(t_1)$ and for $t_1 \leq t \leq t_2$

$$p(t) \;=\; \begin{cases} (e_1(t), u_0), & t \in \Gamma_1, \\ (e_1(t), u_0), & e_1(t) < P_{10}(x_1(t)), \quad t \in \Delta_1, \end{cases} \tag{3.18}$$

for some $t_1 < t_2$. Then $x_1(t) = y_1(t)$ for all $t \in [t_1, t_2]$.

Σ_2. Let $t_1 \in \Delta$. If $p(t) = P_0(x_1(t))$, $t_1 \leq t \leq t_2$, for some $t_2 > t_1$, then $x_2(t) = z(x_1(t))$, $t_1 \leq t \leq t_2$.

Σ_3. Suppose pursuer x use the strategy (3.24) on $[t_1, t_2) \subset \Delta$. If, for some $\varepsilon > 0$, $mes((t_2, t_2 + \varepsilon) \cap \Delta) = 0$, then $x_1(t_2) = y_1(t_2)$.

Σ_4. Let $x_2(\tau) = y_2(\tau)$ at some $\tau \in \Delta$. Then $x(\tau) = y(\tau)$.

Σ_5. Let $t \geq T_1$ be any time that we do not have the completion of pursuit. Then $t \in \Gamma_1 \cup \Delta$.

Σ_6. $mes(\Delta) \leq \frac{2L}{u_0}$ and $mes(\Gamma_0) \leq \frac{L}{u_0}$.

Proof. See Property 2 in [4]. □

We show the completion of pursuit. It is enough to prove that $x_2(\tau) = y_2(\tau)$ for $T_1 \leq \tau \leq T + \frac{4L}{u_0}$ to show $x(\tau) = y(\tau)$ because if $\tau \in \Gamma_1$, then $x_1(\tau) = y_1(\tau)$; and if $\tau \in \Delta$, then by property Σ_4, $x_1(\tau) = y_1(\tau)$.

If $x_2(\tau) = y_2(\tau)$ at some time τ when $x_2(\tau) \leq f(b)$, then the proof is done. Let $x_2(t) \neq y_2(t)$ when $x_2(t) \leq f(b)$. Prove that $x_2(\tau) = y_2(\tau)$ at some time τ, $x_2(\tau) > f(b)$.

In fact, by property Σ_5 any time t is in $\Gamma_1 \cup \Delta$. Assume $x_2(t) > f(b)$, $t \geq T_1$. Then $x(t)$ is not on the curve $z(\delta)$ and so $t \notin \Delta$. Therefore $t \in \Gamma_1$. In particular, $x_1(t) = y_1(t)$ and so we can use the similar proof in Case 1a for proving that the boundary of the set M can not be a barrier when pursuer applies the strategy (3.24). It is easy to see that $x_2(t) < y_2(t) \leq F(x_1(t))$. Therefore, if the pursuer position's can be any point of the curve $\kappa = F(\delta)$, $a \leq \delta \leq b$, at some time, then by that time we have $x_2(\tau) = y_2(\tau)$ for some time τ. In fact, by using Σ_6, $mes(\Delta) \leq \frac{2L}{u_0}$ and it means pursuer can move along the $\kappa = z(\delta)$, $s \leq \delta \leq b$, only a finite time not more than $\frac{2L}{u_0}$.

Using Σ_5 we have

$$x_2(t) - x_2(T_1) = \int_{T_1}^{t} p_2(s)ds = \left(\int_{\Delta\cap[T_1,t]} + \int_{\Gamma_1\cap[T_1,t]} \right) p_2(s)ds. \quad (3.19)$$

Since by (3.24), $p_2(t) \geq 0$, $t \geq T_1$, then

$$\int_{\Delta\cap[T_1,t]} p_2(s)ds \geq 0.$$

Therefore using (3.19) and the equation $\Gamma_1 \cap [T_1, t] = [T_1, t] \setminus \Delta$ getting from property Σ_5 obtain that

$$x_2(t) - x_2(T_1) \geq \int_{[T_1,t]\setminus\Delta} p_2(s)ds = u_0 \cdot mes([T_1, t] \setminus \Delta).$$

Finally, by using Σ_6 we have

$$x_2(t) - x_2(T_1) \geq u_0(mes([T, t]) - mes(\Delta)) \geq u_0 \left(t - T - \frac{2L}{u_0} \right). \quad (3.20)$$

However, $x_2(t) - x_2(T_1) \leq L$, so by (3.20) we have that at some time τ,

$$T_1 \leq \tau \leq T_1 + \frac{3L}{u_0} \leq T + \frac{4L}{u_0}, \quad (3.21)$$

we have $x_2(\tau) = y_2(\tau)$ since by the time $T + \frac{4L}{u_0}$ the position of the pursuer will be on the curve $F(\delta)$.

Therefore, we have the completion of pursuit at some time τ that satisfy the (3.13) in Case 1a and (3.21) in Cases 1b and 2. Therefore, if we have (3.7), then we have the completion of pursuit from the time T within the time $T + \frac{4L}{u_0} - T = \frac{4L}{u_0}$.

We prove that strategy of x is admissible.

Now define the following set:

$$\Delta_1' = \{t \in \Delta_1 | e_1(t) < P_{10}(x1(t))\},$$

$$\Delta_1'' = \{t \in \Delta_1 | e_1(t) \geq P_{10}(x1(t))\}$$

because the sets Γ_0, Γ_1, Δ_1', Δ_1'', and Δ_2 have mutually empty intersection,

$$\int_0^\tau p_1^2(s)ds = \left(\int_0^T + \int_{\Gamma_0} + \int_{\Gamma_1\cup\Delta_1'} + \int_{\Delta_1''\cup\Delta_2}\right) p_1^2(s)ds. \tag{3.22}$$

We have the following estimation for these integrals.

$$\int_0^T p_1^2(s)ds \leq L\beta,$$

$$\int_{\Gamma_0} p_1^2(s)ds = u_0^2\ mes(\Gamma_0) \leq Lu_0 \text{ (see property } \Sigma_6),$$

$$\int_{\Gamma_1\cup\Delta_1'} p_1^2(s)ds = \int_{\Gamma_1\cup\Delta_1'} e_1^2(s)ds = \int_T^{T+\frac{4L}{u_0}} e_1^2(s)ds \leq \varphi_{11}^2,$$

$$\int_{\Delta_1''\cup\Delta_2} p_1^2(s)ds \leq u_0^2 mes(\Delta_1''\cup\Delta_2) \leq 2Lu_0 \text{ (see property } \Sigma_6). \tag{3.23}$$

Thus, by (3.22), (3.23) and (3.2) we have

$$\int_0^\tau p_1^2(s)ds \leq L\beta + \varphi_{11}^2 + 3Lu_0 \leq \psi_{11}^2,$$

therefore, admissibility of $p(t)$ is proved for $t \geq 0$. The proof of Lemma 3.4 is done. □

Let $F(x, y, \bar{x}, T, \varphi_{11})$ be the strategy of pursuer x on $[0, T]$ defined by (3.8), (3.10) and (3.24), where $\bar{x} = (\bar{x_1}, \bar{x_2})$.

3.2. **Proof for the completion of pursuit.**

Proof. We now construct the strategies for pursuers x_1, x_2, ..., x_m as follows. If $k_1(T) > \varphi_1^2 - \varphi_{11}^2$, then we consider the strategy of pursuer x_1 as follows:

$$p_1(t) \quad = \quad \begin{cases} F(x_1, y, \bar{x}, \chi_0, k_1), & \chi_0 \leq t \leq \chi_1, \\ 0, & t > \chi_1. \end{cases} \tag{3.24}$$

where $\chi_0 = T$ and χ_1 is the first time when $k_1(\chi_1) = \varphi_1^2 - \varphi_{11}^2$. Note that $k_1(t)$, $t \geq 0$, is non increasing. By Lemma 3.4 if

$$k_1\left(\chi_0 + \frac{4L}{u_0}\right) \geq q_1(\chi_0) - \varphi_{11}^2,$$

then we have the completion of pursuit by x_1 at some $\tau_1 \in \left[\chi_0, \chi_0 + \frac{4L}{u_0}\right]$. If pursuit is not completed in this interval, then

$$k_1\left(\chi_0 + \frac{4L}{u_0}\right) < k_1(\chi_0) - \varphi_{11}^2 \leq \varphi_1^2 - \varphi_{11}^2 \quad \text{and so} \quad \chi_1 \in \left[\chi_0, \chi_0 + \frac{4L}{u_0}\right].$$

Now, set

$$p_i(t) \quad = \quad \begin{cases} 0, & t < \chi_{i-1}, \\ F(x_i, y, \bar{x}, \chi_{i-1}, k_1), & \chi_{i-1} \leq t \leq \chi_i, \\ 0, & t > \chi_i, \end{cases} \tag{3.25}$$

which is strategy of pursuer x_i, $i = 2, ..., m$, where χ_i is the first time (not stated clearly) when $k_1(\chi_i) = k_1(\chi_{i-1}) - \varphi_{i,1}^2$, $i = 2, ..., m$. If

$$k_1\left(\chi_{i-1} + \frac{4L}{u_0}\right) \geq k_1(\chi_{i-1}) - \varphi_{i,1}^2,$$

then by Lemma 3.4 we have the completion of pursuit by pursuer x_i at some $\tau_i \in \left[\chi_{i-1}, \chi_{i-1} + \frac{4L}{u_0}\right]$. If pursuit is not completed in $\left[\chi_{i-1}, \chi_{i-1} + \frac{4L}{u_0}\right]$, then we must have

$$k_1\left(\chi_{i-1} + \frac{4L}{u_0}\right) < k_1(\chi_{i-1}) - \varphi_{i,1}^2, \quad i = 1, 2, ..., m. \tag{3.26}$$

Since the right hand side of this inequality at $i = m$ is

$$k_1(\chi_{m-1}) - \varphi_{m,1}^2 = k_1(\chi_{m-2}) - \varphi_{m-1,1}^2 - \varphi_{m,1}^2 = \varphi_1^2 - \varphi_{1,1}^2 - \cdots - \varphi_{m,1}^2 = 0.$$

Therefore, for $i = m$, (3.26) is equivalent to

$$k_1\left(\chi_{m-1} + \frac{4L}{u_0}\right) < 0, \text{ or the same} \quad \int_0^{\chi_{m-1} + \frac{4L}{u_0}} e_1^2(s)ds > \varphi_1^2,$$

on the other hand we have the admissibility of the control of evader $e(t)$ and the above inequalities contradict this admissibility. So the inequality (3.26) fails to hold at some $i = r$ and then by Lemma 3.4 pursuit is completed by the pursuer x_r in $\left[\chi_{r-1}, \chi_{r-1} + \frac{4L}{u_0}\right]$.

Let now $k_1(T) \leq \varphi_1^2 - \varphi_{11}^2$ and let the number k is chosen to satisfy the inequality

$$\varphi_1^2 - \varphi_{11}^2 - \cdots - \varphi_{k,1}^2 \leq k_1(T) \leq \varphi_1^2 - \varphi_{11}^2 - \cdots - \varphi_{k-1,1}^2. \tag{3.27}$$

Then for the strategies of pursuers we set $p_i(t) = 0$, $t \geq T$, $i = 1, ..., (k-1)$. For $i = k$, the strategy of pursuer x_k is defined as follows

$$p_k(t) \quad = \quad \begin{cases} F(x_k, y, \bar{x}, \chi_0, k_1), & T \leq t \leq \chi_k, \\ 0, & t > \chi_k, \end{cases} \tag{3.28}$$

where χ_k is the first time when $k_1(\chi_k) = \varphi_1^2 - \varphi_{11}^2 - \cdots - \varphi_{k,1}^2$.

Now consider the strategies of pursuers x_i, $k < i \leq m$ by formula (3.25). The rest of the proof that pursuit can be completed is the same as above. The proof of the theorem is done. □

4. Conclusion

For the completion of the differential game of m pursuers and one evader with the constraints we have got a sufficient condition. Pursuit differential game has been investigated for any nonconvex set M. We exactly gave the strategies of the pursuers and proved that the completion of pursuit can be done from any initial state in M.

The following is the explanation that pursuers used. The pursuers by using (3.3) want to reach to the point $\bar{x} = (a, f(a))$, so they use just their initial states and $\bar{x}$ when $0 \leq t \leq T$. For getting whether $x_{i1}(t) = y_1(t)$, the pursuer x_i uses $x_i(t)$ and $y(t)$ when $T \leq t < T_1$. In the case $t \geq T_1$, the pursuer x_i uses $x_i(t)$, $y(t)$, $k_1(t)$ and $e(t)$.

we have displayed the equation $k_1(\chi_i) = k_1(\chi_{i-1}) - \varphi_{i,1}^2$. For determining χ_i, there is no usage of χ_{i-1}. Indeed, to determine χ_i, $i = 1, 2, ..., m$, we use the following

$$k_1(\chi_i) = k_1(\chi_{i-1}) - \varphi_{i,1}^2 = \varphi_1^2 - \varphi_{1,1}^2 - \cdots - \varphi_{i,1}^2.$$

We gave the strategies with the detailed costruction in the proof of Theorem 3.1. That being said, we used the numbers ψ_{i2}, $i = 1, ..., m$, which are positive in the proof of Theorem 3.1.

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