Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://10.22080/CJMS.2022.20492.1566

Caspian J Math Sci. **12**(1)(2023), 68-80 (Research Article)

Approximation of common solutions of the infinite family of equilibrium problems in Banach Spaces

Rasoul Jahed [1], Hamid Vaezi [2] and Hossein Piri [3]
[1] Department of Mathematics, Germi Branch, Islamic Azad University, Germi, Iran.
[2] Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, University of Tabriz, Tabriz, Iran.
[3] Department of Mathematics, University of Bonab, Bonab, Iran.

Abstract. In this paper, we study convergence analysis of the sequence generated by extragradient method for an infinite family of the equilibrium problems in Banach spaces. We first prove weak convergence of the generated sequence to a common solution of the infinite family of the equilibrium problems. Then we use Halpern type regularization method in order to prove strong convergence of the generated sequence to a common equilibrium point.

Keywords: Equilibrium problem, Extragradient method, Halpern regularization, Pseudo-monotone bifunction.

2010 Mathematics subject classification: 74G10, 47H05.

[1]rjahed@iaugermi.ac.ir
[2]Corresponding author: hvaezi@tabrizu.ac.ir
[3]h.piri@bonabu.ac.ir
Received: 12 December 2020
Revised: 09 September 2022
Accepted: 10 September 2022
How to Cite: Vaezi, Hamid; Jahed, Rasoul; Piri, Hossein. Approximation of common solutions of the infinite family of equilibrium problems in Banach Spaces, Casp.J. Math. Sci.,**12**(1)(2023), 68-80.

1. Preliminaries

Let E be a real Banach space with norm $\|.\|$ and E^* denote the dual of E. We denote the value of $v \in E^*$ at $x \in E$ by $\langle x, v\rangle$. When $\{x^n\}$ is a sequence in E, we denote strong convergence of $\{x^n\}$ to $x \in E$ by $x^n \to x$ and weak convergence by $x^n \rightharpoonup x$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{v \in E^* : \langle x, v\rangle = \|x\|^2 = \|v\|^2\}$$

for $x \in E$. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| < 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists for all $x, y \in U = \{z \in E : \|z\| = 1\}$. It is also said to be uniformly smooth if the limit (1.1) is attained uniformly for $x, y \in U$.

Let E be a smooth Banach space. We use the following function studied in Alber [1], Kamimura and Takahashi [11] and Reich [18]:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{1.2}$$

for all $x, y \in E$. It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. Notice that the duality mapping is the identity operator in Hilbert spaces. Therefore, if E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$.

Proposition 1.1. [11] *Let E be a uniformly convex and smooth Banach space and let $\{x^n\}$ and $\{y^n\}$ be two sequences of E. If $\lim_{n\to\infty} \phi(x^n, y^n) = 0$ and either $\{x^n\}$ or $\{y^n\}$ is bounded, then $\lim_{n\to\infty} \|x^n - y^n\| = 0$.*

Proposition 1.2. [11] *Let E be a reflexive, strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and $x \in E$. Then there exists a unique element $\bar{x} \in C$ such that*

$$\phi(\bar{x}, x) = \inf\{\phi(z, x) : z \in C\}.$$

Regarding Proposition 1.2, we denote the unique element $\bar{x} \in C$ by $P_C(x)$, where the mapping P_C is called the generalized projection from E onto C. It is obvious that in Hilbert spaces, P_C is coincident with the metric projection from E onto C. We also need the following proposition to prove strong convergence in Section 3.

Proposition 1.3. [11] *Let E be a smooth Banach space, C be a convex subset of E, $x \in E$ and $\bar{x} \in C$. Then*

$$\phi(\bar{x}, x) = \inf\{\phi(z, x) : z \in C\}$$

if and only if

$$\langle z - \bar{x}, Jx - J\bar{x}\rangle \leq 0, \quad \forall z \in C.$$

Throughout this paper we assume that E is a real Banach space which is uniformly convex and uniformly smooth unless otherwise specified. Let $f : E \times E \to \mathbb{R}$. f is called a bifunction. Let $K \subset E$ be nonempty, closed and convex. An equilibrium problem for f and K as briefly $EP(f; K)$ consists of finding $x^* \in K$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K. \tag{1.3}$$

x^* is called a solution of the problem or an equilibrium point. We denote the set of all equilibrium points for (1.3) by $S(f; K)$. Equilibrium problems extend and unify many problems in optimization, variational inequalities, fixed point theory, complementarity problems, Nash equilibria and many other problems in nonlinear analysis.

The following conditions on bifunctions may be used throughout the paper, therefore we exhibit them as:
A_1: f is pseudo-monotone, i.e. whenever $f(x, y) \geq 0$ with $x, y \in E$ it holds that $f(y, x) \leq 0$;
A_2: f is ϕ-Lipschitz-type continuous, i.e. there exist two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\phi(y, x) - c_2\phi(z, y), \quad \forall x, y, z \in E;$$

A_3: $f(\cdot, y)$ is upper semicontinuous for all $y \in E$;
A_4: $f(x, \cdot) : K \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in E$.
Equilibrium problems for monotone and pseudo-monotone bifunctions studied extensively in Hilbert, Banach as well as in topological vector spaces by many authors (see [2, 3, 4, 7, 8, 9, 12, 14, 15]). Recently the extragradient method for equilibrium problems in Hadamard spaces has been studied in [12]. Also in [6], Hieu studied the solutions of equilibrium problems and proved strong convergence of the sequence generated by extragradient method to a solution of the problem in Hilbert spaces.

The strong convergence of the sequence generated by the hybrid proximal point method to a common fixed point of a family of quasi ϕ-nonexpansive mappings have studid in [10] by Jahed, Vaezi and Piri.

In this paper, we will deal with the extragradient method for equilibrium problems in Banach spaces. The results of this paper improve the results in [6] in three senses:

a) We will deal with a rather general class of Banach spaces, while [6] only considers Hilbert spaces.

b) Convergence analysis of the method in [6] requires weak continuity of $f(\cdot,\cdot)$, which seldom holds in infinite dimensional spaces, but our continuity assumptions (lower semicontinuity of $f(x,\cdot)$ and weak upper semicontinuity of $f(\cdot,y)$ for all $x,y\in E$) are much less demanding, and covers the important concave-convex case.

c) We will deal with an infinite family of pseudo-monotone bifunctions, while [6] only considers a finite family of bifunctions.

This paper is organized as follows. In Section 2, we prove weak convergence of the sequence generated by extragradient method to a common equilibrium point of an infinite family of pseudomonotone equilibrium problems. In Section 3, we study strong convergence of a Halpern type regularization of the extragradient method to a common solution of the equilibrium problems for an infinite family of bifunctions in Banach spaces.

2. Weak convergence by extragradient method

In this section, we use the extragradient method for equilibrium problems in Banach spaces. This method first introduced by Korpelevich in [17]. After him, the extragradient method were studied extensively for approximating solutions of variational inequalities and equilibrium problems by many authors (see for example [6], [8] and references therein). Now we study convergence analysis of the sequence generated by the extragradient method to a common solution of an infinite family of equilibrium problems. We first introduce the algorithm, then we prove that the generated sequence converges weakly to a common solution of the problem. We suppose that the sequence $\{f_n\}$ of bifunctions satisfy A_1, A_2, A_3, A_4 and $\bigcap_k S(f_k;K)\neq\varnothing$.

In order to prove optimality of weak cluster points of the sequence generated by Algorithm 2.1, we need the following assumption, which has been introduced in [12]. We recall it in the following

$$\begin{cases} \textit{For each arbitrary sequence } \{z^k\} \textit{ and each subsequence } \{z^{k_n}\} \textit{ of } \{z^k\}, \\ \textit{if } z^{k_n}\rightharpoonup z \text{ and } \limsup f_{k_n}(z^{k_n},y)\geq 0,\ \forall y\in K, \text{ then } z\in\bigcap_k S(f_k;K). \end{cases} \tag{2.1}$$

When $f_n\equiv f$, it is easy to see that if $f(.,y)$ is weak upper semicontinuous for all $y\in K$ then f satisfies the condition (2.1). But, the converse is not hold in general (see [13]). It is enough to take $E=l^2$, $K=\{\xi=(\xi_1,\xi_2,...)\in l^2:\xi_i\geq 0,\forall i=1,2,...\}$ and $f(x,y)=(y_1-x_1)\sum_{i=1}^{\infty}(x_i)^2$. Then, Remark 2.1 in [13] shows that the condition (2.1) is a suitable condition which is weaker than the weakly upper semicontinuity of bifunctions $\{f_n\}$ respect to the first arguments.

Algorithm 2.1.

Initialize: Take $n = 0$, $0 < \alpha \le \lambda_k \le \beta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$ for all k and $x^0 \in E$.
Step 1: Solve the minimization problem and let y^n be its solution, i.e.

$$y^n \in \mathrm{Argmin}_{y\in K}\{f_n(x^n, y) + \frac{1}{2\lambda_n}\|y\|^2 - \frac{1}{\lambda_n}\langle y, Jx^n\rangle\}. \tag{2.2}$$

Step 2: Solve the minimization problem and let x^{n+1} be its solution, i.e.

$$x^{n+1} \in \mathrm{Argmin}_{y\in K}\{f_n(y^n, y) + \frac{1}{2\lambda_n}\|y\|^2 - \frac{1}{\lambda_n}\langle y, Jx^n\rangle\}. \tag{2.3}$$

Step 3: Take $n := n + 1$ and go back Step 1.

In order to prove the weak convergence of the sequences generated by Algorithm 2.1, we need the following lemmas.

Lemma 2.2. *The sequences $\{x^n\}$ and $\{y^n\}$ generated by Algorithm 2.1 are well defined.*

Proof. We define $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ as

$$\varphi(y) = \begin{cases} f_n(x^n, y) + \frac{1}{2\lambda_n}\|y\|^2 - \frac{1}{\lambda_n}\langle y, Jx^n\rangle & y \in K \\ +\infty & y \notin K. \end{cases} \tag{2.4}$$

It is clear that φ is proper, convex and lower semicontinuous. Therefore by Rockafellar's theorem [19, 20], the subdifferential of φ is maximal monotone operator and hence onto, by virtue of Minty's theorem. Thus, $\partial\varphi$ has some zero, which is a minimizer of φ. Hence y^n exists in (2.2). Now, the result is trivial. □

Lemma 2.3. *Assume that $\{x^n\}$ and $\{y^n\}$ generated by Algorithm 2.1 and $x^* \in \bigcap_k S(f_k; K)$, then*

$$\phi(x^*, x^{n+1}) \le \phi(x^*, x^n) - (1 - 2c_1\lambda_n)\phi(y^n, x^n) - (1 - 2c_2\lambda_n)\phi(x^{n+1}, y^n). \tag{2.5}$$

Proof. Since x^{n+1} solve the minimization problem in (2.3), we have:

$$x^{n+1} \in \mathrm{Argmin}_{y\in K}\{f_n(y^n, y) + \frac{1}{2\lambda_n}\|y\|^2 - \frac{1}{\lambda_n}\langle y, Jx^n\rangle\}.$$

Therefore,

$$0 \in \partial\{f_n(y^n, \cdot) + \frac{1}{2\lambda_n}\|\cdot\|^2 - \frac{1}{\lambda_n}\langle \cdot, Jx^n\rangle\}(x^{n+1}) + N_K(x^{n+1}),$$

where $N_K(x^{n+1})$ is the normal cone for K at $x^{n+1} \in K$, i.e.

$$N_K(x^{n+1}) = \{v \in E^* : \langle y - x^{n+1}, v\rangle \le 0, \forall y \in K\}.$$

Thus, there exist $w^n \in \partial f_n(y^n, \cdot)(x^{n+1})$ and $\bar{w} \in N_K(x^{n+1})$ such that

$$0 = w^n + \frac{1}{\lambda_n} Jx^{n+1} - \frac{1}{\lambda_n} Jx^n + \bar{w}.$$

Therefore, we have $\langle y - x^{n+1}, -w^n - \frac{1}{\lambda_n} Jx^{n+1} + \frac{1}{\lambda_n} Jx^n\rangle \le 0$. Hence, $\frac{1}{\lambda_n}\langle y - x^{n+1}, Jx^n - Jx^{n+1}\rangle \le \langle y - x^{n+1}, w^n\rangle \le f_n(y^n, y) - f_n(y^n, x^{n+1})$. Therefore,

$$f_n(y^n, x^{n+1}) - f_n(y^n, y) \le \frac{1}{\lambda_n}\langle y - x^{n+1}, Jx^{n+1} - Jx^n\rangle. \tag{2.6}$$

Similar to this argument, since y^n solve the minimization problem in (2.2), we have:

$$f_n(x^n, y^n) - f_n(x^n, y) \le \frac{1}{\lambda_n}\langle y - y^n, Jy^n - Jx^n\rangle. \tag{2.7}$$

Now, take $x^* \in \bigcap_k S(f_k; K)$. Note that, since $f_n(x^*, y^n) \ge 0$, pseudo-monotonicity of f_n implies that $f_n(y^n, x^*) \le 0$. Then set $y = x^*$ in (2.6) and $y = x^{n+1}$ in (2.7), we obtain respectively

$$f_n(y^n, x^{n+1}) \le \frac{1}{\lambda_n}\langle x^* - x^{n+1}, Jx^{n+1} - Jx^n\rangle \tag{2.8}$$

and

$$\frac{1}{\lambda_n}\langle y^n - x^{n+1}, Jy^n - Jx^n\rangle \le f_n(x^n, x^{n+1}) - f_n(x^n, y^n). \tag{2.9}$$

On the other hand, since f_n is ϕ-Lipschitz-type continuous, we have:

$$-c_1\phi(y^n, x^n) - c_2\phi(x^{n+1}, y^n) + f_n(x^n, x^{n+1}) - f_n(x^n, y^n) \le f_n(y^n, x^{n+1}). \tag{2.10}$$

Note that by (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned} -c_1\lambda_n\phi(y^n, x^n) - c_2\lambda_n\phi(x^{n+1}, y^n) + \langle y^n - x^{n+1}, Jy^n - Jx^n\rangle \le \\ \langle x^* - x^{n+1}, Jx^{n+1} - Jx^n\rangle. \end{aligned} \tag{2.11}$$

From (2.11) and the definition of ϕ, it is easy to see

$$\phi(x^*, x^{n+1}) \le \phi(x^*, x^n) - (1 - 2c_1\lambda_n)\phi(y^n, x^n) - (1 - 2c_2\lambda_n)\phi(x^{n+1}, y^n).$$

□

In order to prove uniqueness of the weak limit point in the following theorem, we need the following condition on Banach space E:
If $\{y^n\}$ and $\{z^n\}$ are sequences in K that converge weakly to y and z, respectively and $y \ne z$, then

$$\liminf_{n\to\infty} |\langle y - z, Jy^n - Jz^n\rangle| > 0. \tag{2.12}$$

For example, it is known that ℓ_p spaces for $1 < p < \infty$ satisfies in the above condition.

Theorem 2.4. *Assume that $\{f_n\}$ satisfy A_1, A_2, A_3 and A_4. In addition the solution set $\bigcap_k S(f_k; K)$ is nonempty. Then all weak cluster points of the sequence $\{x^n\}$ generated by Algorithm 2.1 belong to $\bigcap_k S(f_k; K)$. In addition, if E satisfies (2.12), then the sequence $\{x^n\}$ converges weakly to a point of $\bigcap_k S(f_k; K)$.*

Proof. Let $x^* \in \bigcap_k S(f_k; K)$. From Lemma 2.3, we conclude that

$$\phi(x^*, x^{n+1}) \leq \phi(x^*, x^n). \tag{2.13}$$

Therefore, $\lim_{n\to\infty} \phi(x^*, x^n)$ exists and $\{x^n\}$ is bounded. Also, by Lemma 2.3, we have:

$$\lim_{n\to\infty} \phi(y^n, x^n) = \lim_{n\to\infty} \phi(x^{n+1}, y^n) = 0. \tag{2.14}$$

Now, Proposition 1.1 implies that

$$\lim_{n\to\infty} \|y^n - x^n\| = \lim_{n\to\infty} \|x^{n+1} - y^n\| =_{n\to\infty} \lim \|x^{n+1} - x^n\| = 0. \tag{2.15}$$

In the sequel, by (2.8), (2.9) and (2.10),

$$-c_1\lambda_n\phi(y^n, x^n) - c_2\lambda_n\phi(x^{n+1}, y^n) + \langle y^n - x^{n+1}, Jy^n - Jx^n\rangle \leq f_n(y^n, x^{n+1}) \leq$$

$$\langle x^* - x^{n+1}, Jx^{n+1} - Jx^n\rangle \leq \frac{1}{2}(\phi(x^*, x^n) - \phi(x^*, x^{n+1}) - \phi(x^{n+1}, x^n)). \tag{2.16}$$

Taking limit from (2.16) and using (2.14) and (2.15), we have

$$\lim_{n\to\infty} f_n(y^n, x^{n+1}) = 0. \tag{2.17}$$

Note that by (2.6),

$$\frac{-1}{\lambda_n}\|y - x^{n+1}\|\|Jx^{n+1} - Jx^n\| \leq f_n(y^n, y) - f_n(y^n, x^{n+1}). \tag{2.18}$$

Uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, from (2.15), we get

$$\lim_{n\to\infty} \|Jx^{n+1} - Jx^n\| = 0. \tag{2.19}$$

Now, since $\{x^n\}$ is bounded, by taking liminf from (2.18) and using (2.17), we have

$$\liminf_{n\to\infty} f_n(y^n, y) \geq 0, \qquad \forall y \in K. \tag{2.20}$$

Also, since $\{y^n\}$ is bounded, there exists subsequences $\{x^{n_i}\}$ of $\{x^n\}$, $\{y^{n_i}\}$ of $\{y^n\}$ and $p \in E$ such that $x^{n_i} \rightharpoonup p$. Note that $y^{n_i} \rightharpoonup p$ by (2.15). Now, (2.1) and (2.20) imply that $p \in \bigcap_k S(f_k; K)$.
In the sequel, we prove the uniqueness of the weak cluster point of $\{x^n\}$ by condition (2.12). Let q be an other weak cluster point of $\{x^n\}$. Then

there exists subsequence $\{x^{n_j}\}$ such that $x^{n_j} \rightharpoonup q$. We have already proved that q is an element of $\bigcap_k S(f_k;K)$, also $\lim_{n\to\infty} \phi(p,x^n)$ and $\lim_{n\to\infty} \phi(q,x^n)$ exist by Lemma 2.3. Note that

$$\begin{aligned} 2\langle p-q, Jx^{n_i} - Jx^{n_j}\rangle &= 2\langle p, Jx^{n_i}\rangle - 2\langle q, Jx^{n_i}\rangle - 2\langle p, Jx^{n_j}\rangle + 2\langle q, Jx^{n_j}\rangle \\ &= -\phi(p,x^{n_i}) + \phi(q,x^{n_i}) + \phi(p,x^{n_j}) - \phi(q,x^{n_j}). \end{aligned}$$

Taking limit when $i \to +\infty$ and then when $j \to +\infty$, we obtain $p = q$, i.e. $\{x^n\}$ weakly converges to a point of $\bigcap_k S(f_k;K)$. □

3. Halpern type of extragradient method

In this section, we perform a modification on Algorithm 2.1, which ensures strong convergence of the generated sequence to a common solution of the infinite family of equilibrium problems. In Hilbert spaces, this procedure, called Halpern's regularization (see [5, 13, 15, 22]). In order to find a common equilibrium point of bifunctions $\{f_n\}$, we propose the Algorithm 3.1 and analyze the convergence of the iteration sequences. In the sequel, we assume that the bifunctions $\{f_n\}$ satisfy A_1, A_2, A_3, A_4, (2.1) and $\bigcap_k S(f_k;K) \neq \emptyset$.

Algorithm 3.1.

Initialize: Take $u, x^0 \in E$, $n := 0$, $\{\alpha_i\} \subset (0,1)$ such that $\lim_{i\to\infty} \alpha_i = 0$ and $\sum_{i=0}^{\infty} \alpha_i = +\infty$, $0 < \alpha \le \lambda_k \le \beta < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, for all k.
Step 1: Solve the minimization problem and let y^n be its solution, i.e.

$$y^n \in \text{Argmin}_{y\in K}\{f_n(x^n,y) + \frac{1}{2\lambda_n}\|y\|^2 - \frac{1}{\lambda_n}\langle y, Jx^n\rangle\}. \tag{3.1}$$

Step 2: Solve the minimization problem and let z^n be its solution, i.e.

$$z^n \in \text{Argmin}_{y\in K}\{f_n(y^n,y) + \frac{1}{2\lambda_n}\|y\|^2 - \frac{1}{\lambda_n}\langle y, Jx^n\rangle\}. \tag{3.2}$$

Step 3: Determine the next approximation x^{n+1} as

$$x^{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz^n). \tag{3.3}$$

Step 4: Take $n := n+1$ and go back Step 1.

In order to prove the strong convergence of the sequences generated by Algorithm 3.1, we need the following lemmas.

Lemma 3.2. *The sequences $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$ generated by Algorithm 3.1 are well defined.*

Proof. See Lemma 2.2. □

Lemma 3.3. *Assume that* $\{x^n\}$, $\{y^n\}$ *and* $\{z^n\}$ *are generated by Algorithm 3.1 and* $x^* \in \bigcap_k S(f_k; K)$. *Then*

$$\phi(x^*, z^n) \leq \phi(x^*, x^n) - (1-2c_1\lambda_n)\phi(y^n, x^n) - (1-2c_2\lambda_n)\phi(z^n, y^n). \quad (3.4)$$

Proof. It is enough to replace x^{n+1} by z^n in the proof of Lemma 2.3. □

Lemma 3.4. [21] *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers,* $\{\alpha_n\}$ *be a sequence of real numbers in* $(0, 1)$ *with* $\sum_{n=1}^{\infty} \alpha_n = +\infty$ *and* $\{t_n\}$ *be a sequence of real numbers. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n, \qquad \forall n \in \mathbb{N}.$$

If $\limsup_{k\to\infty} t_{n_k} \leq 0$ *for every subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *satisfying*

$$\liminf_{k\to\infty}(s_{n_k+1} - s_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} s_n = 0$.

Let E be a strictly convex, smooth and reflexive Banach space and J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one and surjective and it is the duality mapping from E^* into E.

We make use of the following mapping V studied in Alber [1]

$$V(x, v) = \|x\|^2 - 2\langle x, v\rangle + \|v\|^2. \quad (3.5)$$

for all $x \in E$ and $v \in E^*$. In other words, $V(x, v) = \phi(x, J^{-1}v)$ for all $x \in E$ and $v \in E^*$.

Lemma 3.5. [16] *Let* E *be a strictly convex, smooth and reflexive Banach space and let* V *be as in* (3.5). *Then*

$$V(x, v) \leq V(x, v + w) - 2\langle J^{-1}(v) - x, w\rangle, \quad (3.6)$$

for all $x \in E$ *and* $v, w \in E^*$.

Theorem 3.6. *Assume that* $\{f_n\}$ *satisfy* A_1, A_2, A_3, A_4 *and* (2.1). *In addition the solution set* $\bigcap_k S(f_k; K)$ *is nonempty. Then the sequence* $\{x^n\}$ *generated by Algorithm 3.1 converges strongly to* $P_{\bigcap_k S(f_k;K)}u$.

Proof. Let $x^* = P_{\bigcap_k S(f_k;K)}u$. By Lemma 3.3,

$$\phi(x^*, z^n) \leq \phi(x^*, x^n). \quad (3.7)$$

By (3.7) and (3.3) we have:

$$\begin{aligned}\phi(x^*, x^{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz^n)) \\ &= V(x^*, \alpha_n Ju + (1 - \alpha_n)Jz^n) \leq \alpha_n V(x^*, Ju) + (1 - \alpha_n)V(x^*, Jz^n) \\ &\leq \alpha_n \phi(x^*, u) + (1 - \alpha_n)\phi(x^*, z^n) \leq \alpha_n \phi(x^*, u) + (1 - \alpha_n)\phi(x^*, x^n) \\ &\leq \max\{\phi(x^*, u), \phi(x^*, x^n)\} \leq ... \leq \max\{\phi(x^*, u), \phi(x^*, x^0)\},\end{aligned}$$

which follows $\{x^n\}$ is bounded. Thus, by (3.7), $\{z^n\}$ is also bounded. On the other hand, by Lemma 3.5, we have

$$\begin{aligned}\phi(x^*, x^{n+1}) &= V(x^*, \alpha_n Ju + (1-\alpha_n)Jz^n)\\ &\le V(x^*, \alpha_n Ju + (1-\alpha_n)Jz^n - \alpha_n(Ju - Jx^*))\\ &- 2\langle J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz^n) - x^*, -\alpha_n(Ju - Jx^*)\rangle\\ &= V(x^*, (1-\alpha_n)Jz^n + \alpha_n Jx^*) + 2\langle x^{n+1} - x^*, \alpha_n(Ju - Jx^*)\rangle\\ &\le (1-\alpha_n)V(x^*, Jz^n) + \alpha_n V(x^*, Jx^*) + 2\alpha_n\langle x^{n+1} - x^*, Ju - Jx^*\rangle\\ &= (1-\alpha_n)\phi(x^*, z^n) + 2\alpha_n\langle x^{n+1} - x^*, Ju - Jx^*\rangle\\ &\le (1-\alpha_n)\phi(x^*, x^n) + 2\alpha_n\langle x^{n+1} - x^*, Ju - Jx^*\rangle.\end{aligned}$$

We are going to prove that $\phi(x^*, x^n) \to 0$. By Lemma 3.4, it suffices to show that $\limsup_{k\to\infty}\langle x^{n_k+1} - x^*, Ju - Jx^*\rangle \le 0$ for every subsequence $\{\phi(x^*, x^{n_k})\}$ of $\{\phi(x^*, x^n)\}$ satisfying $\liminf_{k\to\infty}(\phi(x^*, x^{n_k+1}) - \phi(x^*, x^{n_k})) \ge 0$. Suppose that $\{\phi(x^*, x^{n_k})\}$ is a subsequence of $\{\phi(x^*, x^n)\}$ such that $\liminf_{k\to\infty}(\phi(x^*, x^{n_k+1}) - \phi(x^* - x^{n_k})) \ge 0$. Then

$$\begin{aligned}0 &\le \liminf_{k\to\infty}(\phi(x^*, x^{n_k+1}) - \phi(x^*, x^{n_k}))\\ &= \liminf_{k\to\infty}(V(x^*, \alpha_{n_k}Ju + (1-\alpha_{n_k})Jz^{n_k}) - \phi(x^*, x^{n_k}))\\ &\le \liminf_{k\to\infty}(\alpha_{n_k}V(x^*, Ju) + (1-\alpha_{n_k})V(x^*, Jz^{n_k}) - \phi(x^*, x^{n_k}))\\ &= \liminf_{k\to\infty}(\alpha_{n_k}\phi(x^*, u) + (1-\alpha_{n_k})\phi(x^*, z^{n_k}) - \phi(x^*, x^{n_k}))\\ &= \liminf_{k\to\infty}(\alpha_{n_k}(\phi(x^*, u) - \phi(x^*, z^{n_k})) + \phi(x^*, z^{n_k}) - \phi(x^*, x^{n_k}))\\ &\le \limsup_{k\to\infty}\alpha_{n_k}(\phi(x^*, u) - \phi(x^*, z^{n_k})) + \liminf_{k\to\infty}(\phi(x^*, z^{n_k}) - \phi(x^*, x^{n_k}))\\ &= \liminf_{k\to\infty}(\phi(x^*, z^{n_k}) - \phi(x^*, x^{n_k}))\\ &\le \limsup_{k\to\infty}(\phi(x^*, z^{n_k}) - \phi(x^*, x^{n_k})) \le 0.\end{aligned}$$

So,

$$\lim_{k\to\infty}(\phi(x^*, z^{n_k}) - \phi(x^*, x^{n_k})) = 0. \tag{3.8}$$

Hence, by Lemma 3.3, we conclude that

$$\lim_{k\to\infty}\phi(x^{n_k}, y^{n_k}) = \lim_{k\to\infty}\phi(y^{n_k}, z^{n_k}) = 0. \tag{3.9}$$

In the sequel, by Proposition 1.1, we obtain

$$\lim_{k\to\infty}\|x^{n_k} - y^{n_k}\| = \lim_{k\to\infty}\|y^{n_k} - z^{n_k}\| = \lim_{k\to\infty}\|x^{n_k} - z^{n_k}\| = 0. \tag{3.10}$$

Uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, from (3.10), we get

$$\lim_{k\to\infty} \|Jx^{n_k} - Jz^{n_k}\| = 0. \tag{3.11}$$

Note that since y^n and z^n respectively solve the minimization problems in (3.1) and (3.2), similar to (2.6) and (2.7), again we can obtain the following relations:

$$f_n(y^n, z^n) - f_n(y^n, y) \leq \frac{1}{\lambda_n}\langle y - z^n, Jz^n - Jx^n\rangle \tag{3.12}$$

and

$$f_n(x^n, y^n) - f_n(x^n, y) \leq \frac{1}{\lambda_n}\langle y - y^n, Jy^n - Jx^n\rangle. \tag{3.13}$$

Replacing y by x^* in (3.12), since $f_n(y^n, x^*) \leq 0$, we have

$$f_n(y^n, z^n) \leq \frac{1}{\lambda_n}\langle x^* - z^n, Jz^n - Jx^n\rangle. \tag{3.14}$$

Also, replacing y by z^n in (3.13), we get

$$\frac{1}{\lambda_n}\langle y^n - z^n, Jy^n - Jx^n\rangle \leq f_n(x^n, z^n) - f_n(x^n, y^n). \tag{3.15}$$

On the other hand, since f is ϕ-Lipschitz-type continuous, we have:

$$-c_1\phi(y^n, x^n) - c_2\phi(z^n, y^n) + f_n(x^n, z^n) - f_n(x^n, y^n) \leq f_n(y^n, z^n). \tag{3.16}$$

Hence, by (3.14), (3.15) and (3.16), we obtain

$$-c_1\lambda_n\phi(y^n, x^n) - c_2\lambda_n\phi(z^n, y^n) + \langle y^n - z^n, Jy^n - Jx^n\rangle \leq f_n(y^n, z^n) \leq$$

$$\langle x^* - z^n, Jz^n - Jx^n\rangle = \frac{1}{2}(\phi(x^*, x^n) - \phi(x^*, z^n) - \phi(z^n, x^n)). \tag{3.17}$$

Replacing n by n_k in (3.17) and using (3.8), (3.9) and (3.10), we have

$$\lim_{k\to\infty} f_{n_k}(y^{n_k}, z^{n_k}) = 0. \tag{3.18}$$

From (3.12), we get

$$\frac{-1}{\lambda_n}\|y - z^n\|\|Jz^n - Jx^n\| \leq f_n(y^n, y) - f_n(y^n, z^n). \tag{3.19}$$

Replacing n by n_k in (3.19) and taking liminf, by (3.11) and (3.18), we have

$$\liminf_{k\to\infty} f_{n_k}(y^{n_k}, y) \geq 0, \qquad \forall y \in K. \tag{3.20}$$

On the other hand, there exists a subsequence $\{x^{n_{k_t}}\}$ of $\{x^{n_k}\}$ and $p \in K$ such that $x^{n_{k_t}} \rightharpoonup p$ and

$$\limsup_{k\to\infty}\langle x^{n_k} - x^*, Ju - Jx^*\rangle = \lim_{t\to\infty}\langle x^{n_{k_t}} - x^*, Ju - Jx^*\rangle = \langle p - x^*, Ju - Jx^*\rangle. \tag{3.21}$$

Since $y^{n_{k_t}} \rightharpoonup p$ by (3.10), hence, (2.1) and (3.20) imply that $p \in \bigcap_k S(f_k; K)$. Now, since $\bigcap_k S(f_k; K)$ is closed and convex, $x^{n_{k_t}} \rightharpoonup p$ and $x^* = P_{\bigcap_k S(f_k;K)}u$. Therefore by Proposition 1.3, we have $\langle p - x^*, Ju - Jx^* \rangle \leq 0$. Hence,

$$\limsup_{k\to\infty} \langle x^{n_k} - x^*, Ju - Jx^* \rangle = \langle p - x^*, Ju - Jx^* \rangle \leq 0. \tag{3.22}$$

Note that

$$\begin{aligned}\phi(z^{n_k}, x^{n_k+1}) &= V(z^{n_k}, \alpha_{n_k} Ju + (1-\alpha_{n_k}) Jz^{n_k}) \\ &\leq \alpha_{n_k} V(z^{n_k}, Ju) + (1-\alpha_{n_k}) V(z^{n_k}, Jz^{n_k}) = \alpha_{n_k}\phi(z^{n_k}, u).\end{aligned}$$

Taking the limit, we get

$$\lim_{k\to\infty} \phi(z^{n_k}, x^{n_k+1}) = 0.$$

So, Proposition 1.1 implies that

$$\lim_{k\to\infty} \|z^{n_k} - x^{n_k+1}\| = 0.$$

Hence, $\limsup_{k\to\infty}\langle x^{n_k+1} - x^*, Ju - Jx^* \rangle \leq 0$ by (3.22). Now, by Lemma 3.4, $\phi(x^*, x^n) \to 0$ and Proposition 1.1 implies that $x^n \to x^* = P_{\bigcap_k S(f_k;K)}u$.

□

References

[1] Y. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications. Theory and applications of nonlinear operators of accretive and monotone type, *Lecture Notes in Pure and Appl. Math., 178, Dekker, New York, (1996), 15-50.*

[2] M. Bianchi and S. Schaible, Generalized monotone bifunctions and equilibrium problems, *J. Optim. Theory Appl.* **90** *(1996), 31-43.*

[3] O. Chadli, Z. Chbani and H. Riahi, Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities, *J. Optim. Theory Appl.* **105** *(2000), 299-323.*

[4] P. L. Combetes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.* **6** *(2005), 117–136.*

[5] B. Halpern, Fixed point of nonexpanding maps, *Bull. Am. Math. Soc.* **73** *(1967), 957-961.*

[6] D. V. Hieu, Common solutions to pseudomonotone equilibrium problems, *Bull. Iranian Math. Soc.* **42** *(2016), 1207-1219.*

[7] A. N. Iusem, G. Kassay, W. Sosa, On certain conditions for the existence of solutions of equilibrium problems, *Math. Program., Ser. B* **116** *(2009), 259-273.*

[8] A.N. Iusem, V. Mohebbi, Extragradient methods for nonsmooth equilibrium problems in Banach spaces, *Optimization (2018) doi: 10.1080/02331934.2018.1462808.*

[9] A. N. Iusem, W. Sosa, On the proximal point method for equilibrium problems in Hilbert spaces, *Optimization,* **59** *(2010), 1259-1274.*

[10] R. Jahed, H. Vaezi and H. Piri, Strong convergence of the iterations of quasi ϕ-nonexpansive mappings and its applications in Banach space, *Sahand commun Math. Anal.* **17** *(3) (2020), 71-80.*

[11] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.* **13** *(3) (2002), 938-945.*

[12] H. Khatibzadeh, V. Mohebbi, Approximating solutions of equilibrium problems in Hadamard spaces, *Miskolc Math. Notes (to be published).*

[13] H. Khatibzadeh, V. Mohebbi, Proximal point algorithm for infinite pseudomonotone bifunctions, *Optimization* **65** *(2016), 1629-1639.*

[14] H. Khatibzadeh, V. Mohebbi, M. H. Alizadeh, On the cyclic pseudomonotonicity and the proximal point algorithm, *Numer. Algebra Control Optim.* **8** *(2018) 441–449.*

[15] H. Khatibzadeh, V. Mohebbi, S. Ranjbar, Convergence analysis of the proximal point algorithm for pseudo-monotone equilibrium problems, *Optim. Methods Soft.* **30** *(2015), 1146–1163.*

[16] F. Kohsaka, W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, *Abstr. Appl. Anal. (2004), 239-249.*

[17] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, *Ekonomika Matematcheskie Metody* **12** *(1976) 747–756.*

[18] S. Reich, A weak convergence theorem for the alternating method with Bregman distances, Theory and applications of nonlinear operators of accretive and monotone type, *Lecture Notes in Pure and Appl. Math., 178, Dekker, New York, (1996), 313-318.*

[19] R. T. Rockafellar, Characterization of the subdifferentials of convex functions, *Pacific J. Math.* **17** *(1966), 497–510.*

[20] R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* **33** *(1970), 209–216.*

[21] S. Saejung, P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, *Nonlinear Anal.* **75** *(2012) 742-750.*

[22] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.,* **66** *(2002), 240-256.*