
New Ostrowski type inequalities for differentiable harmonically quasi-convex functions via fractional integral

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ABSTRACT. In this paper, we prove a new integral identity, and then we establish some new Ostrowski's inequalities for fonctions whose first derivatives are harmonically quasi-convex via Riemann-Liouville fractional integrals. **Keywords:** integral inequality, Ostrowski inequality, harmonically quasi-convex functions.

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1. INTRODUCTION

Ostrowski [19] proved the following integral inequality

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Theorem 1.1. Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \rightarrow \mathbb{R}$, be a differentiable mapping in the interior I° of I , and $a, b \in I^\circ$ with $a < b$. If $|f'| \leq M$ for all $x \in [a, b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq (b-a) \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] M.$$

The above inequalities has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature, one can mention [1–6, 9–18, 20], and references therein.

İşcan [7], gave the following Ostrowski's inequalities for harmonically quasi-convex functions

Theorem 1.2. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$ and $f \in L([a, b])$. If $|f'|^q$ ($q \geq 1$) is harmonically quasi-convex on $[a, b]$, then for all $x \in [a, b]$, we have

$$\begin{aligned} & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \\ & \leq \frac{ab}{b-a} \left\{ (x-a)^2 (C_1(a, x, \theta, \rho) \sup \{|f'(x)|^q, |f'(a)|^q\})^{\frac{1}{q}} \right. \\ & \quad \left. + (b-x)^2 (C_2(b, x, \theta, \rho) \sup \{|f'(x)|^q, |f'(b)|^q\})^{\frac{1}{q}} \right\}, \end{aligned}$$

where

$$\begin{cases} C_1(a, x, \theta, \rho) = \frac{B(\rho+1, 1)}{x^{2\theta}} \times {}_2F_1(2\theta, \rho+1; \rho+2; 1-\frac{a}{x}) \\ C_2(b, x, \theta, \rho) = \frac{B(1, \rho+1)}{x^{2\theta}} \times {}_2F_1(2\theta, 1; \rho+2; 1-\frac{x}{b}) \end{cases}$$

where $B(., .)$ and ${}_2F_1(., .; .; .)$ are Beta and hypergeometric functions.

Theorem 1.3. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° , $a, b \in I$ with $a < b$ and $f \in L([a, b])$. If $|f'|^q$ ($q \geq 1$) is harmonically quasi-convex on $[a, b]$, then for all $x \in [a, b]$, we have

$$\begin{aligned} & \left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(t)}{t^2} dt \right| \\ & \leq \frac{ab}{b-a} \left\{ C_3(a, x) (x-a)^2 (\sup \{|f'(x)|^q, |f'(a)|^q\})^{\frac{1}{q}} \right. \\ & \quad \left. + C_3(b, x) (b-x)^2 (\sup \{|f'(x)|^q, |f'(b)|^q\})^{\frac{1}{q}} \right\}, \end{aligned}$$

where

$$C_3(i, x) = \frac{1}{x-i} \left(\frac{1}{i} - \frac{\ln x - \ln i}{x-i} \right).$$

Motivated by the above results, in this paper, we prove a new integral identity, and then we establish some new Ostrowski's inequalities

for functions whose first derivatives are harmonically quasi-convex via Riemann-Liouville fractional integrals.

2. PRELIMINARIES

In this sections we recall some definitions and lemma

Definition 2.1. [22] A function $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \sup\{f(x), f(y)\}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 2.2. [8] Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $I_{a+}^\alpha f$ and $I_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

$$I_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$, is the Gamma function and

$$I_{a+}^0 f(x) = I_{b-}^0 f(x) = f(x).$$

We also recall that the hypergeometric function is defined as follows

$${}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt,$$

where $c > b > 0$, $|z| < 1$, and B is the Euler beta function given by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

Lemma 2.3. [21] For any $0 \leq a < b$ in \mathbb{R} and $0 < p \leq 1$, we have

$$|a^p - b^p| \leq (b-a)^p.$$

3. MAIN RESULTS

Lemma 3.1. Let $f : [a, b] \subset (0, +\infty) \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L([a, b])$, then the following equality holds

$$\frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a}\right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h) \left(\frac{1}{b}\right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h) \left(\frac{1}{a}\right) \right) - f(x) \quad (3.1)$$

$$= \frac{ab(b-a)}{2} \left(\int_0^1 \frac{k}{((1-t)a+tb)^2} f' \left(\frac{ab}{(1-t)a+tb} \right) dt + \int_0^1 \frac{(t^\delta - (1-t)^\delta)}{((1-t)a+tb)^2} f' \left(\frac{ab}{(1-t)a+tb} \right) dt \right)$$

where $h(x) = \frac{1}{x}$ and

$$k = \begin{cases} 1 & \text{if } t \in \left[0, \frac{a(b-x)}{x(b-a)} \right] \\ -1 & \text{if } t \in \left[\frac{a(b-x)}{x(b-a)}, 1 \right], \end{cases} \quad (3.2)$$

for all $x \in [a, b]$.

Proof. Let

$$I_1 = \int_0^1 \frac{k}{((1-t)a+tb)^2} f' \left(\frac{ab}{(1-t)a+tb} \right) dt,$$

and

$$I_2 = \int_0^1 \frac{(t^\delta - (1-t)^\delta)}{((1-t)a+tb)^2} f' \left(\frac{ab}{(1-t)a+tb} \right) dt.$$

Clearly,

$$\begin{aligned} I_1 &= \int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{((1-t)a+tb)^2} f' \left(\frac{ab}{(1-t)a+tb} \right) dt - \int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{((1-t)a+tb)^2} f' \left(\frac{ab}{(1-t)a+tb} \right) dt \\ &= \frac{1}{ab(a-b)} f \left(\frac{ab}{(1-t)a+tb} \right) \Big|_{t=0}^{t=\frac{a(b-x)}{x(b-a)}} - \frac{1}{ab(a-b)} f \left(\frac{ab}{(1-t)a+tb} \right) \Big|_{t=\frac{a(b-x)}{x(b-a)}}^{t=1} \\ &= \frac{1}{ab(b-a)} (f(b) + f(a)) - \frac{2}{ab(b-a)} f(x). \end{aligned} \quad (3.3)$$

By integration by parts, I_2 gives

$$\begin{aligned} I_2 &= \frac{(t^\delta - (1-t)^\delta)}{ab(a-b)} f \left(\frac{ab}{(1-t)a+tb} \right) \Big|_{t=0}^{t=1} \\ &\quad - \frac{\delta}{ab(a-b)} \int_0^1 (t^{\delta-1} + (1-t)^{\delta-1}) f \left(\frac{ab}{(1-t)a+tb} \right) dt \\ &= -\frac{1}{ab(b-a)} (f(a) + f(b)) \\ &\quad - \frac{\delta}{ab(a-b)} \left(\int_0^1 t^{\delta-1} f \left(\frac{ab}{(1-t)a+tb} \right) dt + \int_0^1 (1-t)^{\delta-1} f \left(\frac{ab}{(1-t)a+tb} \right) dt \right). \end{aligned} \quad (3.4)$$

Making the change of variable $\frac{(1-t)a+tb}{ab} = u$, (3.4) becomes

$$\begin{aligned}
I_2 &= -\frac{1}{ab(b-a)} (f(a) + f(b)) - \frac{\Gamma(\delta+1)}{ab(a-b)} \left(\frac{ab}{b-a}\right)^\delta \\
&\quad \times \left(\frac{1}{\Gamma(\delta)} \int_{\frac{1}{b}}^{\frac{1}{a}} \left(u - \frac{1}{b}\right)^{\delta-1} f\left(\frac{1}{u}\right) du + \frac{1}{\Gamma(\delta)} \int_{\frac{1}{a}}^{\frac{1}{b}} \left(\frac{1}{a} - u\right)^{\delta-1} f\left(\frac{1}{u}\right) du \right) \\
&= -\frac{1}{ab(b-a)} (f(a) + f(b)) \\
&\quad + \frac{\Gamma(\delta+1)}{ab(b-a)} \left(\frac{ab}{b-a}\right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h)\left(\frac{1}{b}\right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h)\left(\frac{1}{a}\right) \right) \quad (3.5)
\end{aligned}$$

Summing (3.3) and (3.5), and then multiplying the resulting equality by $\frac{ab(b-a)}{2}$, we get the desired result. \square

Theorem 3.2. *Let $f : [a, b] \subset (0, +\infty) \rightarrow \mathbb{R}$ be differentiable mapping on (a, b) with $a < b$, such that $f \in L([a, b])$. If $|f'|$ is harmonically quasi-convex, then for all $x \in [a, b]$ the following fractional inequality holds*

$$\begin{aligned}
&\left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a}\right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h)\left(\frac{1}{b}\right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h)\left(\frac{1}{a}\right) \right) - f(x) \right| \\
&\leq \frac{ab(b-a)}{2} \left(\frac{b-x}{ab(b-a)} \max\{|f'(x)|, |f'(b)|\} + \frac{x-a}{ab(b-a)} \max\{|f'(a)|, |f'(x)|\} \right. \\
&\quad \left. + \frac{1}{2a^2(\delta+1)} \max\left\{ \left| f'\left(\frac{2ab}{a+b}\right) \right|, |f'(b)| \right\} {}_2F_1\left(2, 1; \delta+2; \frac{1}{2}\left(1 - \frac{b}{a}\right)\right) \right. \\
&\quad \left. + \frac{1}{2b^2(\delta+1)} \max\left\{ \left| f'\left(\frac{2ab}{a+b}\right) \right|, |f'(a)| \right\} {}_2F_1\left(2, 1; \delta+2; \frac{1}{2}\left(1 - \frac{a}{b}\right)\right) \right),
\end{aligned}$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the hypergeometric function.

Proof. Using Lemma 3.1 and properties of modulus, and then by Applying Lemma 2.3 and the harmonic quasi-convexity of $|f'|$ to the resulting inequality, we get

$$\begin{aligned}
&\left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a}\right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h)\left(\frac{1}{b}\right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h)\left(\frac{1}{a}\right) \right) - f(x) \right| \\
&\leq \frac{ab(b-a)}{2} \left(\int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{((1-t)a+tb)^2} \left| f'\left(\frac{ab}{(1-t)a+tb}\right) \right| dt \right. \\
&\quad \left. + \int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{((1-t)a+tb)^2} \left| f'\left(\frac{ab}{(1-t)a+tb}\right) \right| dt \right)
\end{aligned}$$

$$\begin{aligned}
& + \int_0^{\frac{1}{2}} \frac{((1-t)^\delta - t^\delta)}{((1-t)a+tb)^2} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right| dt + \int_{\frac{1}{2}}^1 \frac{(t^\delta - (1-t)^\delta)}{((1-t)a+tb)^2} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right| dt \\
\leq & \frac{ab(b-a)}{2} \left(\max \{ |f'(x)|, |f'(b)| \} \int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{((1-t)a+tb)^2} dt \right. \\
& + \max \{ |f'(a)|, |f'(x)| \} \int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{((1-t)a+tb)^2} dt \\
& + \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|, |f'(b)| \right\} \int_0^{\frac{1}{2}} \frac{(1-2t)^\delta}{((1-t)a+tb)^2} dt \\
& \left. + \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|, |f'(a)| \right\} \int_{\frac{1}{2}}^1 \frac{(2t-1)^\delta}{((1-t)a+tb)^2} dt \right) \\
= & \frac{ab(b-a)}{2} \left(\frac{1}{b-a} \max \{ |f'(x)|, |f'(b)| \} \int_0^{\frac{a(b-x)}{x(b-a)}} \frac{b-a}{(a+t(b-a))^2} dt \right. \\
& + \frac{1}{b-a} \max \{ |f'(a)|, |f'(x)| \} \int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{b-a}{(a+t(b-a))^2} dt \\
& + \frac{1}{2a^2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|, |f'(b)| \right\} \int_0^{\frac{1}{2}} \frac{(1-t)^\delta}{(1-\frac{1}{2}t(1-\frac{b}{a}))^2} dt \\
& \left. + \frac{1}{2b^2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|, |f'(a)| \right\} \int_{\frac{1}{2}}^1 \frac{(1-t)^\delta}{(1-\frac{1}{2}t(1-\frac{a}{b}))^2} dt \right) \\
= & \frac{ab(b-a)}{2} \left(\frac{b-x}{ab(b-a)} \max \{ |f'(x)|, |f'(b)| \} + \frac{x-a}{ab(b-a)} \max \{ |f'(a)|, |f'(x)| \} \right. \\
& + \frac{1}{2a^2(\delta+1)} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|, |f'(b)| \right\} {}_2F_1 \left(2, 1; \delta+2; \frac{1}{2} \left(1 - \frac{b}{a} \right) \right) \\
& \left. + \frac{1}{2b^2(\delta+1)} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|, |f'(a)| \right\} {}_2F_1 \left(2, 1; \delta+2; \frac{1}{2} \left(1 - \frac{a}{b} \right) \right) \right).
\end{aligned}$$

The proof is completed. \square

Theorem 3.3. Let $f : [a, b] \subset (0, +\infty) \rightarrow \mathbb{R}$ be differentiable mapping on (a, b) with $a < b$, such that $f \in L([a, b])$. If $|f'|^q$ is harmonically quasi-convex where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following fractional inequality

$$\left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a} \right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h) \left(\frac{1}{b} \right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h) \left(\frac{1}{a} \right) \right) - f(x) \right|$$

$$\begin{aligned}
&\leq \frac{ab(b-a)}{2} \left(\left(\frac{b^{2p-1}-x^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}} \right)^{\frac{1}{p}} \left(\frac{a(b-x)}{x(b-a)} \max \{|f'(x)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\frac{x^{2p-1}-a^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}} \right)^{\frac{1}{p}} \left(\frac{b(x-a)}{x(b-a)} \max \{|f'(x)|^q, |f'(a)|^q\} \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{{}_2F_1(2p, 1; \delta p + 2; \frac{1}{2}(1-\frac{b}{a}))}{2a^2(\delta p + 1)} \right)^{\frac{1}{p}} \left(\frac{1}{2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \\
&\quad \left. + \left(\frac{{}_2F_1(2p, 1; \delta p + 2; \frac{1}{2}(1-\frac{a}{b}))}{2b^2(\delta p + 1)} \right)^{\frac{1}{p}} \left(\frac{1}{2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} \right)
\end{aligned}$$

holds for all $x \in [a, b]$.

Proof. From Lemma 3.1, properties of modulus, and Hölder inequality, we have

$$\begin{aligned}
&\left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a} \right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta - (f \circ h) \left(\frac{1}{b} \right) + I_{\left(\frac{1}{b}\right)^+}^\delta + (f \circ h) \left(\frac{1}{a} \right) \right) - f(x) \right| \\
&\leq \frac{ab(b-a)}{2} \left(\left(\int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{((1-t)a+tb)^{2p}} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{a(b-x)}{x(b-a)}} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{((1-t)a+tb)^{2p}} dt \right)^{\frac{1}{p}} \left(\int_{\frac{a(b-x)}{x(b-a)}}^1 \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^{\frac{1}{2}} \frac{((1-t)^\delta - t^\delta)^p}{((1-t)a+tb)^{2p}} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 \frac{(t^\delta - (1-t)^\delta)^p}{((1-t)a+tb)^{2p}} dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \right). \quad (3.6)
\end{aligned}$$

Now, using Lemma 2.3, and the harmonic quasi-convexity of $|f'|^q$, we obtain

$$\left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a} \right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta - (f \circ h) \left(\frac{1}{b} \right) + I_{\left(\frac{1}{b}\right)^+}^\delta + (f \circ h) \left(\frac{1}{a} \right) \right) - f(x) \right|$$

$$\begin{aligned}
 &\leq \frac{ab(b-a)}{2} \left(\left(\int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{(a+t(b-a))^{2p}} dt \right)^{\frac{1}{p}} \left(\frac{a(b-x)}{x(b-a)} \max \{|f'(x)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}} \right. \\
 &\quad + \left(\int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{(a+t(b-a))^{2p}} dt \right)^{\frac{1}{p}} \left(\frac{b(x-a)}{x(b-a)} \max \{|f'(x)|^q, |f'(a)|^q\} \right)^{\frac{1}{q}} \\
 &\quad + \left(\frac{1}{2a^2} \int_0^1 \frac{(1-t)^{\delta p}}{\left(1-\frac{1}{2}t\left(1-\frac{b}{a}\right)\right)^{2p}} dt \right)^{\frac{1}{p}} \left(\frac{1}{2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \\
 &\quad + \left. \left(\frac{1}{2b^2} \int_0^1 \frac{(1-t)^{\delta p}}{\left(1-\frac{1}{2}t\left(1-\frac{a}{b}\right)\right)^{2p}} dt \right)^{\frac{1}{p}} \left(\frac{1}{2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} \right) \\
 &= \frac{ab(b-a)}{2} \left(\left(\frac{b^{2p-1}-x^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}} \right)^{\frac{1}{p}} \left(\frac{a(b-x)}{x(b-a)} \max \{|f'(x)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}} \right. \\
 &\quad + \left(\frac{x^{2p-1}-a^{2p-1}}{(b-a)(2p-1)a^{2p-1}b^{2p-1}} \right)^{\frac{1}{p}} \left(\frac{b(x-a)}{x(b-a)} \max \{|f'(x)|^q, |f'(a)|^q\} \right)^{\frac{1}{q}} \\
 &\quad + \left(\frac{{}_2F_1\left(2p, 1; \delta p + 2; \frac{1}{2}\left(1-\frac{b}{a}\right)\right)}{2a^2(\delta p + 1)} \right)^{\frac{1}{p}} \left(\frac{1}{2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \\
 &\quad + \left. \left(\frac{{}_2F_1\left(2p, 1; \delta p + 2; \frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2b^2(\delta p + 1)} \right)^{\frac{1}{p}} \left(\frac{1}{2} \max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} \right),
 \end{aligned}$$

which is the desired result. □

Theorem 3.4. *Let $f : [a, b] \subset (0, +\infty) \rightarrow \mathbb{R}$ be differentiable mapping on (a, b) with $a < b$, such that $f \in L([a, b])$. If $|f'|^q$, $q \geq 1$ is harmonically quasi-convex, then for all $x \in [a, b]$ the following fractional inequality holds*

$$\begin{aligned}
 &\left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a} \right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h) \left(\frac{1}{b} \right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h) \left(\frac{1}{a} \right) \right) - f(x) \right| \\
 &\leq \frac{ab(b-a)}{2} \left(\frac{b-x}{(b-a)ab} (\max \{|f'(x)|^q, |f'(b)|^q\})^{\frac{1}{q}} \right. \\
 &\quad + \frac{x-a}{(b-a)ab} (\max \{|f'(x)|^q, |f'(b)|^q\})^{\frac{1}{q}} \\
 &\quad + \left. \frac{{}_2F_1\left(2, 1; \delta + 2; \frac{1}{2}\left(1-\frac{b}{a}\right)\right)}{2a^2(\delta + 1)} (\max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(b)|^q \right\})^{\frac{1}{q}} \right)
 \end{aligned}$$

$$+ \frac{{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2b^2(\delta+1)} \left(\max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}},$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the hypergeometric function.

Proof. From Lemma 3.1, properties of modulus, and power mean inequality, we have

$$\begin{aligned} & \left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a} \right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h) \left(\frac{1}{b} \right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h) \left(\frac{1}{a} \right) \right) - f(x) \right| \\ & \leq \frac{ab(b-a)}{2} \left(\left(\int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{((1-t)a+tb)^2} dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{a(b-x)}{x(b-a)}} \frac{1}{((1-t)a+tb)^2} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{((1-t)a+tb)^2} dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{a(b-x)}{x(b-a)}}^1 \frac{1}{((1-t)a+tb)^2} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^{\frac{1}{2}} \frac{(1-t)^\delta - t^\delta}{((1-t)a+tb)^2} dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \frac{(1-t)^\delta - t^\delta}{((1-t)a+tb)^2} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 \frac{t^\delta - (1-t)^\delta}{((1-t)a+tb)^2} dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 \frac{t^\delta - (1-t)^\delta}{((1-t)a+tb)^2} \left| f' \left(\frac{ab}{(1-t)a+tb} \right) \right|^q dt \right)^{\frac{1}{q}} \right). \end{aligned} \tag{3.7}$$

Now, using Lemma 2.3, harmonic quasi-convexity of $|f'|^q$, we deduces

$$\begin{aligned} & \left| \frac{\Gamma(\delta+1)}{2} \left(\frac{ab}{b-a} \right)^\delta \left(I_{\left(\frac{1}{a}\right)^-}^\delta (f \circ h) \left(\frac{1}{b} \right) + I_{\left(\frac{1}{b}\right)^+}^\delta (f \circ h) \left(\frac{1}{a} \right) \right) - f(x) \right| \\ & \leq \frac{ab(b-a)}{2} \left(\frac{b-x}{(b-a)ab} (\max \{ |f'(x)|^q, |f'(b)|^q \})^{\frac{1}{q}} \right. \\ & \quad + \frac{x-a}{(b-a)ab} (\max \{ |f'(x)|^q, |f'(b)|^q \})^{\frac{1}{q}} \\ & \quad + \frac{{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)}{2a^2(\delta+1)} \left(\max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \\ & \quad \left. + \frac{{}_2F_1\left(2,1;\delta+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)}{2b^2(\delta+1)} \left(\max \left\{ \left| f' \left(\frac{2ab}{a+b} \right) \right|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} \right), \end{aligned}$$

which is the desired result. \square

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