Caspian Journal of Mathematical Sciences(CJMS)
University of Mazandaran, Iran
http://www.cjms.umz.ac.ir
ISSN: 1735-0611

CJMS. **1**(1)(2012), 13-22

EXISTENCE OF POSITIVE SOLUTIONS FOR FOURTH-ORDER BOUNDARY VALUE PROBLEMS WITH THREE- POINT BOUNDARY CONDITIONS

N. NYAMORADI [1,*]

ABSTRACT. In this work, by employing the Krasnosel'skii fixed point theorem, we study the existence of positive solutions of a three-point boundary value problem for the following fourth-order differential equation

$$\begin{cases} u^{(4)}(t) - f(t, u(t), u''(t)) = 0 \qquad 0 \le t \le 1, \\ u(0) = u(1) = 0, \qquad \alpha u''(0) - \beta u'''(0) = 0, \qquad u''(1) - \alpha u''(\eta) = 0, \end{cases}$$

where $\beta > 0, 0 < \eta < 1, 0 < \alpha\eta < 1, (1 - \alpha\eta) + \beta(1 - \alpha) > 0.$

Keywords: Positive solution, Fourth-order boundary value problem, Three-point boundary conditions, Fixed point theorem.

1. INTRODUCTION

In this paper, we will study the existence of positive solutions of a three-point boundary value problem for the following fourth-order differential equation

$$\begin{cases} u^{(4)}(t) - f(t, u(t), u''(t)) = 0 \qquad 0 \le t \le 1, \\ u(0) = u(1) = 0, \qquad \alpha u''(0) - \beta u'''(0) = 0, \qquad u''(1) - \alpha u''(\eta) = 0, \end{cases} \tag{1.1}$$

where $\beta > 0, 0 < \eta < 1, 0 < \alpha\eta < 1, \Delta := (1 - \alpha\eta) + \beta(1 - \alpha) > 0.$

Recently, motivated by the wide application of the BVPs in physical and applied mathematics, the study of multi-point boundary value problems has received increasing interest. Many approaches, such as the Leray– Schauder continuation theorem, nonlinear alternatives of Leray–Schauder, fixed-point theorems, and coincidence degree theory, are used to acquire the existence and multiplicity results, see [1, 2, 3, 4]. The study of multi-point boundary value problems of linear second-order ordinary differential equations was initiated by Il'in and Moiseev [5]. Then Gupta [6] studied three-point boundary value problems of nonlinear ordinary differential equations. The first work about positive solutions of multi-point boundary value problems is due to

Received: 26 Nov 2011.
Revised: 13 Dec 2011
Accepted: 18 Dec 2011
* *Corresponding author: nyamoradi@razi.ac.ir.*

Ma [7]. In [7], under the assumption that the nonlinearity of f is either super-linear or sub-linear, the existence of at least one positive solution was showed by applying the Guo-Krasnoselskii's fixed point theorem. See Liu [8], Li [9], Yao [10], Wei and Pang [11], Zhonga et al. [12] for more information.

Here, by a positive solution u^* of BVP (1) we mean a solution u^* of BVP (1) which satisfies $u^* > 0$, $0 < t < 1$. We give the following assumptions:
(H1) $\beta > 0, 0 < \eta < 1, 0 < \alpha < \frac{1+\beta}{\eta+\beta}(\leq \frac{1}{\eta})$ and $\Delta := (1-\alpha\eta) + \beta(1-\alpha) > 0$.
(H2) $f \in C([0,1] \times [0,\infty) \times (-\infty, 0], [0.\infty))$ and

$$0 < \int_0^1 (s+\beta)(1-s)ds < +\infty.$$

Inspired and motivated by the works mentioned above, in this work we will consider the existence of positive solutions to BVP (1). we shall first give a new form of the solution, and then determine the properties of the Green's function for associated linear boundary value problems; finally, by employing the Guo-Krasnosel'skii fixed point theorem, some sufficient conditions guaranteeing the existence of a positive solution. The rest of the article is organized as follows: in Section 2, we present some preliminaries and the Guo-Krasnosel'skii fixed point theorem that will be used in Section 3. The main results and proofs will be given in Section 3. Finally, in Section 4, we shall give two examples to illustrate our main results.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proofs of the main results.

Definition 2.1. Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:
(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Let $E = \{u \in C^2[0,1]; u(0) = u(1) = 0\}$. Then we have the following lemma:

Lemma 2.2. *For $u \in E$, $\|u\|_\infty \leq \|u'\|_\infty \leq \|u''\|_\infty$, where $\|u\|_\infty = \sup_{t\in[0,1]} |u(t)|$.*

thus, E is a Banach space when it is endowed with the norm $\|u\| = \|u''\|_\infty$.

Lemma 2.3. *If $Y(t) \in C([0,1])$, then the following boundary value problem*

$$\begin{cases} y''(t) + Y(t) = 0 & 0 \leq t \leq 1, \\ y(0) - \beta y'(0) = 0, & y(1) - \alpha y(\eta) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$y(t) = \int_0^1 G(t,s)Y(s)ds, \tag{2.2}$$

where

$$G(t,s)=\begin{cases}\frac{1}{\Delta}(s+\beta)((1-t)+\alpha(t-\eta)), & s\le \min\{t,\eta\};\\ \frac{1}{\Delta}((s+\beta)(1-t)+\alpha(t-s)(\eta+\beta)), & 0<\eta\le s\le t\le 1;\\ \frac{1}{\Delta}(t+\beta)((1-s)+\alpha(s-\eta)), & 0\le t\le s\le \eta\le 1;\\ \frac{1}{\Delta}(t+\beta)(1-s), & \max\{t,\eta\}\le s.\end{cases} \tag{2.3}$$

Proof. In fact, if $u(t)$ is a solution of the BVP (2.2), then we may assume that

$$y(t)=-\int_0^t (t-s)Y(s)ds+At+B.$$

By the boundary conditions (2.1), we get

$$\begin{aligned} A &= \frac{1}{\Delta}\Big\{\int_0^1 (1-s)Y(s)ds-\alpha\int_0^\eta(\eta-s)Y(s)ds\Big\},\\ B &= \frac{\beta}{\Delta}\Big\{\int_0^1 (1-s)Y(s)ds-\alpha\int_0^\eta(\eta-s)Y(s)ds\Big\},\end{aligned}$$

Therefore, BVP (2.1) has a unique solution

$$\begin{aligned} y(t)=-\int_0^t (t-s)Y(s)ds &+ \frac{t}{\Delta}\Big\{\int_0^1 (1-s)Y(s)ds-\alpha\int_0^\eta(\eta-s)Y(s)ds\Big\}\\ &+ \frac{\beta}{\Delta}\Big\{\int_0^1 (1-s)Y(s)ds-\alpha\int_0^\eta(\eta-s)Y(s)ds\Big\}.\end{aligned}$$

The proof is complete.

Lemma 2.4. *If $u\in C[0,1]$, then the following boundary value problem*

$$\begin{cases} u''(t)=-y(t) & 0\le t\le 1,\\ u(0)=u(1)=0,\end{cases} \tag{2.4}$$

has a unique solution

$$u(t)=\int_0^1 H(t,s)y(s)ds, \tag{2.5}$$

where

$$H(t,s)=\begin{cases} s(1-t), & s\le t\\ t(1-s), & t\le s\end{cases}.$$

Proof. In fact, if $u(t)$ is a solution of the BVP (2.4), then we may suppose that

$$u(t)=-\int_0^t (t-s)y(s)+At+B.$$

By the boundary conditions (2.4), we get $B=0$ and

$$A=\int_0^1 (1-s)y(s)ds.$$

Therefore, BVP (2.4) has a unique solution

$$u(t)= - \int_0^t (t-s)y(s)ds+t\int_0^1 (1-s)y(s)ds=\int_0^1 H(t,s)y(s)ds.$$

The proof is complete.

Remark 1. By Lemma 2.3 and Lemma 2.4, the BVP (1.1) has a unique solution

$$u(t) = \int_0^1 H(t,s)\int_0^1 G(s,r)f(r,u(r),u''(r))drds, \tag{2.6}$$

where

$$y(t) := -u''(t) = \int_0^1 G(t,s)f(s,u(s),u''(s))ds. \tag{2.7}$$

We need some properties of the function H and G in order to discuss the existence of positive solutions.

Lemma 2.5. *(See [13]) $H(t,s) \geq 0$ and $G(t,s) \geq 0$, for all $(t,s) \in [0,1]\times[0,1]$.*

Lemma 2.6. *(See [13]). For all $t,s \in [0,1]\times[0,1]$, we have*

$$k_1(t)G(s,s) \leq G(t,s) \leq k_2(s+\beta)(1-s),$$

where

$$k_1(t) = \min\big\{1, \alpha(1-\eta), t, 1-t\big\}, \qquad k_2 = \frac{\max\big\{1+\alpha, \frac{\alpha(1-\eta)}{1-\alpha\eta}\big\}}{(1-\alpha\eta)+\beta(1-\alpha)}.$$

Obviously $k_1(t) \geq 0$ is a nonnegative function and k_2 is positive constant.

Lemma 2.7. *The unique solution $u(t)$ of the BVP (1.1) is nonnegative and satisfies $\min_{t\in[\eta,1]}(-u''(t)) \geq \lambda\|u\|$, where $\lambda = \min\big\{\eta, \alpha\eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}\big\}$*

Proof. It is obvious that $u(t)$ is nonnegative and $y(t) = -u''(t)$ is concave on $[0,1]$ since, by Lemma 2.3, $y''(t) = -f(t,u(t),u''(t)) \leq 0$. By (2.5), Lemma 2.3 and Lemma 2.5, we easily know that $y(0) \geq 0$. Let $\|y\|_\infty = y(t^*), t^* \in [0,1]$. If $0 \leq \alpha \leq \eta < 1$, then $\min_{\eta\leq t\leq 1} y(t) = y(1)$.
(1) For $0 \leq t^* \leq \eta < 1$, by concavity of y, we easily know

$$\|y\|_\infty = y(t^*) \leq y(1) + (y(\eta)-y(1))\frac{t^*-1}{\eta-1} \leq \frac{1-\alpha\eta}{\alpha(1-\eta)}y(1).$$

(2) For $\eta \leq t^* < 1$, we have

$$\|y\|_\infty = y(t^*) \leq y(0) + (y(\eta)-y(0))\frac{t^*}{\eta} \leq \frac{1}{\alpha\eta}y(1).$$

If $1 \leq \alpha < \frac{\alpha+\beta}{\alpha\eta+\beta} \leq \frac{1}{\eta}$, then $\min_{\eta\leq t\leq 1} y(t) = y(\eta)$. Thus, we have

$$\|y\|_\infty = y(t^*) \leq y(\eta)\frac{t^*}{\eta} < \frac{y(\eta)}{\eta}.$$

From the above discussion, we get

$$\min_{\eta\leq t\leq 1} y(t) \geq \min\big\{\eta, \alpha\eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}\big\}\|y\|_\infty = \lambda\|y\|_\infty.$$

Thus, by Lemma 2.2, we have

$$\min_{\eta\leq t\leq 1}(-u''(t)) \geq \lambda\|-u''\|_\infty = \lambda\|u\|.$$

Then, we achieve the desired result.
Denote

$$P = \{u \in E; u(t) \geq 0, \min(-u''(t))_{t\in[\eta,1]} \geq \lambda\|u\|\}.$$

It is obvious that P is cone.
Define the operator T by

$$Tu(t) = \int_0^1 H(t,s)\int_0^1 G(s,r)f(r,u(r),u''(r))drds. \tag{2.8}$$

By Remark 1, BVP (1.1) has a positive solution $u = u(t)$ if and only if u is a fixed point of T.

Lemma 2.8. *The operator defined in (2.8) is completely continuous and satisfies $T(P) \subseteq P$.*

Proof. The operator defined in (2.8) by an application of the Arzela- Ascoli theorem, is completely continuous and by Lemma 6, we know that $T(P) \subseteq P$.
Our approach is based on the following Guo-Krasnosel'skii fixed point theorem of cone expansion-compression type ([14] and [15]).

Theorem 2.9. *([14]) Let E be a Banach space and $P \subseteq E$ a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : P\bigcap(\overline{\Omega}_2\backslash\Omega_1) \to P$ be a completely continuous operator. In addition suppose either*
(A) $\|Tu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2$ or
(B) $\|Tu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2$
holds. Then T has a fixed point in $P\bigcap(\overline{\Omega}_2\backslash\Omega_1)$.

3. MAIN RESULTS

In this section, we discuss the existence of a positive solution of BVP (1.1). For convenience we set

$$\begin{aligned}
\max f_0 &= \lim_{-v\to o^+}\max_{t\in[0,1]}\sup_{u\in[0,+\infty]}\frac{f(t,u,v)}{-v},\\
\min f_0 &= \lim_{-v\to o^+}\min_{t\in[0,1]}\inf_{u\in[0,+\infty]}\frac{f(t,u,v)}{-v},\\
\max f_\infty &= \lim_{-v\to +\infty}\max_{t\in[0,1]}\sup_{u\in[0,+\infty]}\frac{f(t,u,v)}{-v},\\
\min f_\infty &= \lim_{-v\to +\infty}\min_{t\in[0,1]}\inf_{u\in[0,+\infty]}\frac{f(t,u,v)}{-v}.
\end{aligned}$$

Theorem 3.1. *Suppose that $f \in C([0,1]\times[0,\infty)\times(-\infty,0],[0.\infty))$ and f is superlinear, i.e., $\max f_0 = 0$ and $\min f_\infty = +\infty$; then the BVP (1.1) has at least one positive solution.*

Proof. Since $\max f_0 = 0$, then for any ϵ satisfying $k_2\epsilon \int_0^1 (s+\beta)(1-s)ds \le 1$, there exists $R_1 > 0$ such that

$$f(t,u,v) \le \epsilon(-v), \quad \text{for } t \in [0,1], \quad u \in [0,+\infty), \quad 0 \le -v \le R_1. \tag{3.1}$$

Set $\Omega_1 = \{u \in P : \|u\| < R_1\}$. Then, for any $u \in P \cap \partial\Omega_1$, from Lemma 2.6, Lemma 2.8 and using (3.1) we have

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 G(t,s) f(s,u(s),u''(s))ds \\ &\le k_2 \int_0^1 (s+\beta)(1-s) f(s,u(s),u''(s))ds \\ &\le k_2\epsilon \int_0^1 (s+\beta)(1-s)(-u''(s))ds \\ &\le \|u''\|_\infty = \|u\|, \end{aligned}$$

which implies that

$$\|Tu\| \le \|u\| \qquad \text{for } u \in P \cap \partial\Omega_1. \tag{3.2}$$

On the other hand, since $\min f_\infty = +\infty$, then for any ϵ satisfying $\epsilon k_1(\frac{1}{2}(\eta+1))\lambda \int_0^1 G(s,s)ds \ge 1$, there exists $R_2 > R_1 > 0$ such that

$$f(t,u,v) \ge \epsilon(-v), \quad \text{for } t \in [0,1], \quad u \in [0,+\infty), \quad -v \ge R_2. \tag{3.3}$$

Set $\Omega_2 = \{u \in P : \|u\| < R_2\}$. For any $u \in P \cap \partial\Omega_2$, by Lemma 2.7 one has $\min_{t\in[\tau,1]} u(t) \ge \lambda\|u\|$. Thus, from (2.8) and (3.3) we can conclude that

$$\begin{aligned} -(Tu)''(\frac{1}{2}(\eta+1)) &= \int_0^1 G(\frac{1}{2}(\eta+1),s) f(s,u(s),u''(s))ds \\ &\ge k_1(\frac{1}{2}(\eta+1)) \int_0^1 G(s,s) f(s,u(s),u''(s))ds \\ &\ge \epsilon k_1(\frac{1}{2}(\eta+1)) \int_0^1 G(s,s)(-u''(s))ds \\ &\ge \epsilon k_1(\frac{1}{2}(\eta+1))\lambda\|u\| \int_0^1 G(s,s)ds \\ &\ge \|u\|, \end{aligned}$$

and thus

$$\|Tu\| \ge \|u\| \qquad \text{for } u \in P \cap \partial\Omega_2. \tag{3.4}$$

Therefore, by (3.2), (3.4) and the first part of Theorem 1 we know that the operator T has a fixed point in $P \bigcap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.2. *Suppose that* $f \in C([0,1]\times[0,\infty)\times(-\infty,0],[0.\infty))$ *and* f *is sublinear, i.e.,* $\min f_0 = +\infty$ *and* $\max f_\infty = 0$*; then the BVP (1.1) has at least one positive solution.*

Proof. Since $\min f_0 = +\infty$, then for any ϵ satisfying $\epsilon k_1(\frac{1}{2}(\eta+1))\lambda \int_0^1 G(s,s)ds \geq 1$, there exists $R_1 > 0$ such that

$$f(t,u,v) \geq \epsilon(-v), \quad \text{for } t \in [0,1], \quad u \in [0,+\infty), \quad 0 \leq -v \leq R_1. \tag{3.5}$$

Set $\Omega_1 = \{u \in P : \|u\| < R_1\}$. For any $u \in P \cap \partial\Omega_1$, by Lemma 2.7 one has $\min_{t\in[\tau,1]} u(t) \geq \lambda\|u\|$. Thus, from (2.9) and (3.5) we can conclude that

$$\begin{aligned}
-(Tu)''(\frac{1}{2}(\eta+1)) &= \int_0^1 G(\frac{1}{2}(\eta+1),s)f(s,u(s),u''(s))ds \\
&\geq k_1(\frac{1}{2}(\eta+1))\int_0^1 G(s,s)f(s,u(s),u''(s))ds \\
&\geq \epsilon k_1(\frac{1}{2}(\eta+1))\int_0^1 G(s,s)(-u''(s))ds \\
&\geq \epsilon k_1(\frac{1}{2}(\eta+1))\lambda\|u\|\int_0^1 G(s,s)ds \\
&\geq \|u\|,
\end{aligned}$$

and thus

$$\|Tu\| \geq \|u\| \qquad \text{for } u \in P \cap \partial\Omega_1. \tag{3.6}$$

Next, since $\max f_\infty = 0$, we consider two cases:
Case(i): Suppose that $f(t,u,v)$ is bounded, i.e., there exist a positive constant M such that $f(t,u,v) \leq M$. Take $\max\{k_2 M \int_0^1 (s+\beta)(1-s)ds, R_1\} \leq R_2$. For $u \in P$ with $\|u\| = R_2$,

$$\begin{aligned}
-(Tu)''(t) &= \int_0^1 G(t,s)f(s,u(s),u''(s))ds \\
&\leq k_2\int_0^1 (s+\beta)(1-s)f(s,u(s),u''(s))ds \\
&\leq k_2 M\int_0^1 (s+\beta)(1-s)ds \\
&\leq R_2 = \|u\|,
\end{aligned}$$

Case(ii): Suppose that $f(t,u,v)$ is unbounded. Since $\max f_\infty = 0$, then for any ϵ satisfying $k_2\epsilon\int_0^1 (s+\beta)(1-s)ds \leq 1$, there exist $R_0 > R_1$ such that

$$f(t,u,v) \leq \epsilon(-v), \quad \text{for } t \in [0,1], \quad u \in [0,+\infty), \quad -v \geq R_0. \tag{3.7}$$

Then we define a function $f^*(r) : [0,\infty) \to [0,\infty)$ by

$$f^*(r) = \max\{f(t,u,v) : t \in [0,1], 0 \leq u \leq r, 0 \leq -v \leq r\}.$$

It is easy that $f^*(r)$ is non-decreasing and $\lim_{r\to+\infty} \frac{f^*(r)}{r} = 0$. There exists R_0 such that

$$f^*(r) \leq \epsilon r, \quad \text{for} \quad r \in [R_0,+\infty). \tag{3.8}$$

Taking $R_2 > R_0$, from (3.7) and (3.8)

$$f(t,u,v) \leq f^*(R_2) \leq \epsilon R_2, \quad \text{for } t \in [0,1], \quad 0 \leq u \leq R_2, \quad 0 \leq -v \leq R_2. \tag{3.9}$$

On the other hand, for $u \in P$ with $\|u\| = R_2$, from Lemma 2.2 we know that

$$\|u\|_\infty \leq R_2. \tag{3.10}$$

From Lemma 2.6 and by using (3.9) and (3.10), for $u \in P$ with $\|u\| = R_2$

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 G(t,s) f(s, u(s), u''(s)) ds \\ &\leq k_2 \int_0^1 (s+\beta)(1-s) f(s, u(s), u''(s)) ds \\ &\leq k_2 \epsilon R_2 \int_0^1 (s+\beta)(1-s) ds \\ &\leq R_2 = \|u\|, \end{aligned}$$

Therefore, in either case, we set $\Omega_2 = \{u \in P : \|u\| < R_2\}$ such that

$$\|Tu\| \leq \|u\| \qquad \text{for } u \in P \cap \partial\Omega_2. \tag{3.11}$$

Therefore, by (3.6), (3.11) and the second part of Theorem 2.9 we know that the operator T has a fixed point in $P \bigcap (\overline{\Omega}_2 \backslash \Omega_1)$.

4. APPLICATION

Example 4.1. Consider the following boundary value problem system:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)) \qquad 0 \leq t \leq 1, \\ u(0) = u(1) = 0, \qquad u''(0) - u'''(0) = 0, \qquad u''(1) - \frac{1}{2}u''(\frac{1}{2}) = 0, \end{cases}, \tag{4.1}$$

where $f(t, u(t), u''(t)) = \frac{1}{\sqrt{1+u}} - (u''^{-3} + \sin \pi t$. Clearly,

$$0 < \int_0^1 (s + \frac{1}{2})(1-s) ds < +\infty, \quad \min f_0 = +\infty, \quad \max f_\infty = 0.$$

By Theorem 3.2, system (4.1) has at least one positive solution.

Example 4.2. Consider the following boundary value problem system:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)) \qquad 0 \leq t \leq 1, \\ u(0) = u(1) = 0, \qquad u''(0) - u'''(0) = 0, \qquad u''(1) - \frac{1}{2}u''(\frac{1}{2}) = 0, \end{cases} \tag{4.2}$$

where, $f(t, u(t), u''(t)) = \frac{(-u''^3}{\sqrt{1+u}} - (u''^5 + (-u''^3 \sin \pi t$. Clearly,

$$0 < \int_0^1 (s + \frac{1}{2})(1-s) ds < +\infty, \quad \max f_0 = 0, \quad \min f_\infty = +\infty.$$

Then by Theorem 3.1, system (4.2) has at least one positive solution.

ACKNOWLEDGMENTS

The author is very grateful to the referee for his/her valuable suggestions and comments.

REFERENCES

[1] R. P. Agarwal, D. O'Regan, P. J. Y. Wong, Positive Solutions of Differential, Difference, and Integral Equations, Kluwer Academic Publishers, Boston, 1999.

[2] R. I. Avery, A generalization of the Leggett–Williams fixed point theorem, Math. Sci. Res. Hotline 2 (1998), 9-14.

[3] R. I. Avery, A. C. Peterson, Three positive fixed points of nonlinear operators on ordered Banach spaces, Comput. Math. Appl. 42 (2001), 313-322.

[4] I. Rachxunkova, Upper and lower solutions and topological degree, J. Math. Anal. Appl. 234 (1999), 311-327.

[5] V. A. Il'in, E. I. Moiseev, Non-local boundary value problem of the first kind for a Sturm–Liouville operator in its differential and finite difference aspects, Differential Equations 23 (1987), 803-810.

[6] C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, J. Math. Anal. Appl. 168 (1992), 540-551.

[7] Y. R. Ma, Multiplicity of positive solutions for second-order three-point boundary value problems, Comput. Math. Appl. 40 (2000), 193-204.

[8] B. Liu, Positive solutions of fourth-order two point boundary value problems, Appl. Math. Comput. 148 (2004), 407-420.

[9] Y. Li, Positive solutions of fourth-order boundary value problems with two parameters, J. Math. Anal. Appl. 281 (2003), 477-484.

[10] Q. Yao, Positive solutions for eigenvalue problems of fourth-Order elastic beam equations, Appl. Math. Lett. 17 (2004), 237-243.

[11] Z. Wei, C. Pang, Positive solutions and multiplicity of fourth-order m-point boundary value problems with two parameters, Nonlinear. Anal. 67 (2007), 1586-1598.

[12] Y. Zhonga, S. Chena, C. Wangb, Existence results for a fourth-order ordinary differential equation with a four-point boundary condition, Appl. Math. Lett. 21 (2008), 465-470.

[13] Y. Sun, L. Liu, J. Zhang, R. P. Agarwal Positive solutions of singular three-point boundary value problems for second-order differential equations, J. Copmut. Appl. Math. 230 (2009), 738-750.

[14] M. A. Krasnosel'skii, Positive solutions of operator equations, Noordhoff, Groningen, Netherlands, 1964.

[15] D. Guo, V. Lakshmikantham, Nonlinear problem in Abstract Cones, Academic Press, New York, 1988.

[1]*Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran.*