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B-Focal Curves Of Biharmonic B-General Helices In Heisenberg Group

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Abstract. In this paper, we study $B-$focal curves of biharmonic B-general helices according to Bishop frame in the Heisenberg group Heis^3. Finally, we characterize the $B-$focal curves of biharmonic B-general helices in terms of Bishop frame in the Heisenberg group Heis^3.

Keywords: Biharmonic curve, Bishop frame, Heisenberg group, Parallel transport, Helix.

2000 Mathematics subject classification: 31B30, 58E20.

1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

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The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \operatorname{tr} R\left(\mathcal{T}(\phi), d\phi\right) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps, [7,8].

In this paper, we study $\mathcal{B}-$focal curves of biharmonic $\mathfrak{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. Finally, we characterize the $\mathcal{B}-$focal curves of biharmonic $\mathfrak{B}$-general helices in terms of Bishop frame in the Heisenberg group Heis^3.

2. The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \frac{1}{2}\overline{x}y + \frac{1}{2}x\overline{y})$$

(2.1)

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products [11]

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

We obtain

$$\begin{aligned}
\nabla_{\mathbf{e}_1}\mathbf{e}_1 &= \nabla_{\mathbf{e}_2}\mathbf{e}_2 = \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0, \\
\nabla_{\mathbf{e}_1}\mathbf{e}_2 &= -\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, \\
\nabla_{\mathbf{e}_1}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, \\
\nabla_{\mathbf{e}_2}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1.
\end{aligned}$$

3. Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In The Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, 3.1 \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1,\ g(\mathbf{N}, \mathbf{N}) = 1,\ g(\mathbf{B}, \mathbf{B}) = 1,\ 3.2 \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as [1]

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, 3.3 \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1,\ g(\mathbf{M}_1, \mathbf{M}_1) = 1,\ g(\mathbf{M}_2, \mathbf{M}_2) = 1,\ 3.4 \\ g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{T}, \mathbf{M}_2) = g(\mathbf{M}_1, \mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures. where $\theta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\theta(s), 3.5 \\ k_2 &= \kappa(s)\sin\theta(s). \end{aligned} \tag{3.5}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, 3.6 \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.6}$$

To separate a general helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathfrak{B}$-general helix.

Theorem 3.1. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic* $\mathfrak{B}$-*general helix. Then the parametric equation of $\gamma_{\mathfrak{B}}$ are*

$$
\begin{aligned}
x_{\mathfrak{B}}(s) &= \frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2,\\
y_{\mathfrak{B}}(s) &= -\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3, 3.7(3.7)\\
z_{\mathfrak{B}}(s) &= (\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\\
&\quad -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4,
\end{aligned}
$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 are constants of integration.

4. $\mathcal{B}-$Focal Curve of Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In The Heisenberg Group Heis^3

Denoting the focal curve by $\mathfrak{focal}^{B}_{\gamma_{\mathfrak{B}}}$ of $\gamma_{\mathfrak{B}}$, we can write

$$\mathfrak{focal}^{B}_{\gamma_{\mathfrak{B}}}(s) = (\gamma + \mathfrak{f}_1$$

BM$_1 + \mathfrak{f}_2^{\mathcal{B}}\mathbf{M}_2)(s)$, (4.1)where the coefficients $\mathfrak{f}_1^{\mathcal{B}}$, $\mathfrak{f}_2^{\mathcal{B}}$ are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively.

To separate a focal curve according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the focal curve defined above as $\mathcal{B}$-focal curve.

Theorem 4.1. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic* $\mathfrak{B}$-*general helix with non-zero natural curvatures. Then, the position* *vector of* $\mathfrak{focal}^B_{\gamma_{\mathfrak{B}}}$ *is*

$$\begin{aligned}
\mathfrak{focal}^B_{\gamma_{\mathfrak{B}}}(s) = {} & [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
& +\mathfrak{p}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
& +\frac{1-\mathfrak{p}k_1}{k_2}\cos\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2]\mathbf{e}_1 \\
& +[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
& -\mathfrak{p}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
& +\frac{1-\mathfrak{p}k_1}{k_2}\cos\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]\mathbf{e}_2 \\
& +[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
& [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
& +(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\
& -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
& -\frac{1-\mathfrak{p}k_1}{k_2}\sin\theta+\zeta_4]\mathbf{e}_3,
\end{aligned}$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 *are constants of integration.*

Proof. Assume that γ is a unit speed biharmonic curve and $\mathfrak{focal}^B_{\gamma_{\mathfrak{B}}}$ its $\mathcal{B}-$focal curve on Heis^3.

By differentiating of the formula (4.1), we get

$$\mathfrak{F}^{\mathcal{B}}_{\gamma}(s)' = (1-\mathfrak{f}^{\mathcal{B}}_1k_1-\mathfrak{f}^{\mathcal{B}}_2k_2)\mathbf{T}+\left(\mathfrak{f}^{\mathcal{B}}_1\right)'\mathbf{M}_1+\left(\mathfrak{f}^{\mathcal{B}}_2\right)'\mathbf{M}_2.$$

Using above equation, the first 2 components vanish, we get

$$\begin{aligned}
\mathfrak{f}^{\mathcal{B}}_1k_1+\mathfrak{f}^{\mathcal{B}}_2k_2 &= 1, \\
\left(\mathfrak{f}^{\mathcal{B}}_1\right)' &= 0.
\end{aligned}$$

Considering second equation above system, we chose

$$\mathfrak{f}_1^{\mathcal{B}} = \mathfrak{p} = \text{constant} \neq 0.$$

Then

$$\mathfrak{f}_2^{\mathcal{B}} = \frac{1 - \mathfrak{p}k_1}{k_2}.$$

On the other hand, we have

$$\begin{aligned}\mathbf{T} &= \sin\theta\cos[(\frac{k_1^2 + k_2^2}{\sin^2\theta} - \cos\theta)^{\frac{1}{2}}s + \zeta_0]\mathbf{e}_1 \\ &+ \sin\theta\sin[(\frac{k_1^2 + k_2^2}{\sin^2\theta} - \cos\theta)^{\frac{1}{2}}s + \zeta_0]\mathbf{e}_2 + \cos\theta\mathbf{e}_3.\end{aligned}$$

Thus, it is seen that

$$\mathfrak{focal}_\gamma^B(s) = (\gamma + \mathfrak{p}\mathbf{M}_1 + \frac{1 - \mathfrak{p}k_1}{k_2}\mathbf{M}_2)(s),$$

By means of obtained equations, we express (4.2). This completes the proof.

In the light of Theorem 4.1, we express:

Theorem 4.2. *Let* $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ *be a unit speed biharmonic* $\mathfrak{B}$*-general helix and* $\mathfrak{focal}_{\gamma_{\mathfrak{B}}}^B$ *its* $\mathcal{B}-$*focal curve on* $Heis^3$*. Then, the* *parametric equations of* $\mathfrak{focal}^B_{\gamma_{\mathfrak{B}}}$ *are given by*

$$
\begin{aligned}
x_{\mathfrak{focal}^B_\gamma}(s) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+\mathfrak{p}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+\frac{1-\mathfrak{p}k_1}{k_2}\cos\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2], \\
z_{\mathfrak{focal}^B_\gamma}(s) &= [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&-\mathfrak{p}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+\frac{1-\mathfrak{p}k_1}{k_2}\cos\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3], \\
z_{\mathfrak{focal}^B_\gamma}(s) &= [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+\mathfrak{p}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+\frac{1-\mathfrak{p}k_1}{k_2}\cos\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&-\mathfrak{p}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&+\frac{1-\mathfrak{p}k_1}{k_2}\cos\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
&+[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
&[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
&+(\cos\theta)\,s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\
&-\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]-\frac{1-\mathfrak{p}k_1}{k_2}\sin\theta+\zeta_4],
\end{aligned}
$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 *are constants of integration.*

Proof. Substituting (2.1) into (4.2), we obtain above system. This completes the proof.

If we use Mathematica both $\gamma_{\mathfrak{B}}$ and its focal curve, we have

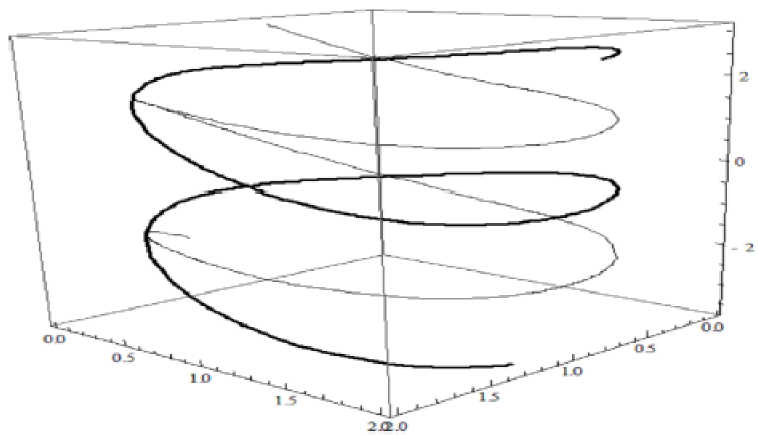

FIGURE 1.

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