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On the Superstability and Stability of the Pexiderized Exponential Equation

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Abstract. The main purpose of this paper is to establish some new results on the superstability and stability via a fixed point approach for the Pexiderized exponential equation, i.e.,

$$\|f(x+y) - g(x)h(y)\| \le \psi(x,y),$$

where f, g and h are three functions from an arbitrary commutative semigroup S to an arbitrary unitary complex Banach algebra and also $\psi : S^2 \to [0,\infty)$ is a function. Furthermore, in connection with the open problem of Th. M. Rassias and our results we generalized the theorem of Baker, Lawrence, Zorzitto and theorem of L. Székelyhidi.

Keywords: Superstability, Cauchy equation, Stability, Semigroup, Fixed point.

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1. Introduction

In 1940 S. M. Ulam [36] gave a wide ranging talk before the Mathematics club of the University of Wisconsin in which he discussed a number of

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important unsolved problems. Among those was the question concerning the stability of group homomorphisms:
Let G_1 be a group and let G_2 be a metric group with the metric d(.,.). Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $h : G_1 \to G_2$ with $d(f(x), h(x)) < \epsilon$ for all $x \in G$?
Generally, the concept of stability for a functional equation comes up when our the functional equation is replaced by an inequality which acts as a perturbation of that equation. The case of approximately additive functions was solved by D. H. Hyers [14] under the assumption that G_1 and G_2 are Banach spaces. In 1950, Hyers's Theorem was generalized by T. Aoki [2] for additive mappings and independently, in 1978, by Th. M. Rassias [31] for linear mappings considering the Cauchy difference controlled by sum of powers of norms. On the other hand, J. M. Rassias [24, 26, 27] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by P. Gavruta [13]. This stability is called Ulam-Gavruta -Rassias stability (see also [1, 8, 33, 20, 21]). In addition, J. M. Rassias considered the mixed product-sum of powers of norms as the control function. This stability is called J.M.Rassias stability (see also [15, 16, 23, 28]).
The exponential function is a powerful tool in each field of natural sciences and engineering, because many natural phenomena well known to us can be described best of all by means of it. The exponent law of exponential functions is well represented by the *exponential functional equation*

$$f(x + y) = f(x)f(y).$$

Hence, we call every solution function of the exponential functional equation an *exponential function*. A function $f : \mathbb{R} \to \mathbb{R}$ continuous at a point is an exponential function if and only if $f(x) = a^x$ for all $x \in \mathbb{R}$ or $f(x) = 0$ for all $x \in \mathbb{R}$, where $a > 0$ is a constant. (see also [6])
In 1979, another type of stability was observed by J. Baker, J. Lawrence and F. Zorzitto [6]. Indeed, they proved that if a function is approximately exponential, then it is either a true exponential function or bounded. This result was the first result concerning the superstability phenomenon of functional equations. Later, J. Baker [5] generalized this famous result as follows:
Let $(S, +)$ be an arbitrary semigroup, and let f map S into the field C of all complex numbers. Assume that f is an approximately exponential function, i.e., there exists a nonnegative number ε such that

$$\|f(x + y) - f(x)f(y)\| \leq \varepsilon \qquad for\ x, y \in S.$$

Then f is either bounded or exponential (see Baker [5], Baker, Lawrence and Zorzitto [6], and Kuczma [19]). Such a phenomenon is called the superstability of the exponential equation(see also [3, 7, 10]). In the proof of the preceding theorem, the multiplicative property of the norm was crucial. Indeed, the proof above works also for functions $f : S \to A$, where A is a normed algebra in which the norm is multiplicative, i.e., $\|xy\| = \|x\|\|y\|$ for all $x, y \in S$. Examples of such real normed algebras are the quaternions and the Cayley numbers. In the same paper Baker gives the following example to show that this result fails if the algebra does not have the multiplicative norm property. Let $\epsilon > 0$, choose $\delta > 0$ so that $|\delta - \delta^2| = \epsilon$ and let $f : C \to C \oplus C$ be defined as

$$f(\lambda) = (e^{\lambda}, \delta), \qquad \lambda \in C.$$

Then, with the nonmultiplicative norm given by $\|(\lambda, \mu)\| = \max\{|\lambda|, |\mu|\}$, we have $\|f(\lambda + \mu) - f(\lambda)f(\mu)\| = \epsilon$ for all complex λ and μ, f is unbounded, but it is not true that $f(\lambda + \mu) = f(\lambda)f(\mu)$ for all complex λ and μ. In this paper, as a consequence of our results, we establish this result positively but in a unitary complex Banach algebra. Also we present this result for the Pexiderized exponential equation.

The result of Baker, Lawrence and Zorzitto [6] was generalized by L. Székelyhidi [34] in another way and he obtained the following result.

Theorem 1.1. [34] *Let $(G, .)$ be an Abelian group with identity and let $f, m : G \to \mathbb{C}$ be functions such that there exist functions $M_1, M_2 :\to [0, \infty)$ with*

$$\|f(x.y) - f(x)m(y)\| \leq \min\{M_1(x), M_2(y)\}$$

for all $x, y \in G$. Then either f is bounded or m is an exponential and $f(x) = f(1)g(x)$ for all $x \in G$.

During the thirty-first International Symposium on Functional Equations, Th. M. Rassias [29] introduced the term *mixed stability* of the function $f : E \to \mathbb{R}$ (or $\mathbb{C}$), where E is a Banach space, with respect to two operations addition and multiplication among any two elements of the set $\{x, y, f(x), f(y)\}$. Especially, he raised an open problem concerning the behavior of solutions of the inequality

$$\|f(x.y) - f(x)f(y)\| \leq \theta(\|x\|^p + \|y\|^p).$$

In connection with this open problem, we generalized the theorem of Baker, Lawrence and Zorzitto and theorem of L. Székelyhidi; more precisely, we proved the superstability and stability of the exponential functional equation and its Pexiderized when the Cauchy difference of exponential equation is not bounded.

In the following section, first we consider the superstability and stability for the equations of the form $f(x + y) = g(x)f(y)$, in which f is a

function from a commutative semigroup to an complex Banach space and g is function from a commutative to complex field and next we consider the superstability and stability for the equations of the forms $f(x+y)=g(x)f(y)$ and $f(x+y)=g(x)h(x)$ when f,g and h are three functions from a commutative semigroup to an unitary complex Banach algebra.
For the readers convenience and explicit later use, we will recall a fundamental results in fixed point theory.

Definition 1.2. The pair (X,d) is called a generalized complete metric space if X is a nonempty set and $d : X^2 \to [0,\infty]$ satisfies the following conditions:

(1) $d(x,y) \geq 0$ and the equality holds if and only if $x=y$;
(2) $d(x,y)=d(y,x)$;
(3) $d(x,z) \leq d(x,y)+d(y,z)$;
(4) every d-Cauchy sequence in X is d-convergent.

for all $x,y \in X$.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

Theorem 1.3. [11] *Let (X,d) be a generalized complete metric space and $J : X \to X$ be strictly contractive mapping with the Lipschitz constant L. Then for each given element $x \in X$, either*

$$d(J^n(x), J^{n+1}(x)) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n(x), J^{n+1}(x)) < \infty$*, for all* $n \geq n_0$*;*
(2) *the sequence* $\{J^n(x)\}$ *converges to a fixed point* y^* *of* J*;*
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0}(x), y) < \infty\}$*;*
(4) $d(y,y^*) \leq \frac{1}{1-L} d(J(y),y)$.

2. Main result

Throughout this Section, assume that $(S,+)$ is an arbitrary commutative semigroup, E is a complex Banach space and X is a complex Banach algebra with unit 1_X, for every $A \subseteq X$ we denote the set of unitary elements in A with $U(A)$ and also $\psi : S^2 \to [0,\infty)$ is a function. In the following Theorem, we consider the superstability of the a Pexider type of exponential equation

$$f(x+y) = g(x)f(y),$$

in which f is a function from a commutative semigroup to an complex Banach space and g is function from a commutative to complex field.

Definition 2.1. Let $f : S \to \mathbb{C}$ be a function, then we define the set N_f with

$$N_f = \{a \in S \ : \ f(\alpha) \in F \backslash \{0, 1\} \ and \ |f(\alpha)| > 1\}.$$

Definition 2.2. Let $f : S \to X$ be a function, then we define the set M_f with

$$M_f = \{a \in G \ : \ f(a) \in C \backslash \{0, 1\} \times \{1_X\}\},$$

and also we introduce the function $Scf : M_f \to C$, where $f(a) = Scf(a) \times 1_X$ for all $a \in M_f$. Also we consider the set

$$\widetilde{M_f} = \{a \in M_f \ : \ |Scf(a)| > 1\}.$$

Theorem 2.3. *Suppose that $f : S \to E$, $g : S \to C$ are two functions and satisfies the inequality*

$$\|f(x + y) - g(x)f(y)\| \quad \leq \quad \psi(x, y) \tag{2.1}$$

for all $x, y \in S$. If $N_g \neq \emptyset$ and $\psi(x, y + a) \leq \psi(x, y)$ for all $x, y \in S$ and $a \in N_g$, then there is a unique function $T : S \to E$, where

$$T(x + y) = g(x)T(y),$$

$$(g(x + y) - g(x)g(y))T(z) = 0$$

and

$$\|f(y) - T(y)\| \leq \inf_{a \in N_g} \frac{\psi(a, y)}{|g(a)| - 1}$$

for all $x, y, z \in S$.

Proof. Let $a \in N_g$ be fixed and Letting $x = a$ in (2.1), we get

$$\|f(a + y) - g(a)f(y)\| \quad \leq \quad \psi(a, y) \tag{2.2}$$

for all $y \in S$. Let us consider the set $A := \{g : S \to E\}$ and introduce the generalized metric on A:

$$d(g, h) = \sup_{y \in S} \frac{\|g(y) - h(y)\|}{\psi(a, y)}.$$

It is easy to show that (A, d) is complete metric space. Now we define the function $J_a : A \to A$ with

$$J_a(h(y)) = \frac{1}{g(a)} h(y + a)$$

for all $h \in A$ and $y \in S$. So

$$\begin{aligned} d(J_a(u), J_a(h)) &= \sup_{y \in S} \frac{\|u(y+a) - h(y+a)\|}{|g(a)|\psi(a,y)} \\ &\leq \sup_{y \in S} \frac{\|u(y+a) - h(y+a)\|}{|g(a)|\psi(a,y+a)} = \frac{1}{|g(a)|} d(u,h) \end{aligned}$$

for all $u, h \in A$, that is J is a strictly contractive selfmapping of A, with the Lipschitz constant $L = \frac{1}{|g(a)|}$. From (2.2), we get

$$\|\frac{f(y+a)}{g(a)} - f(y)\| \leq \frac{\psi(a,y)}{|g(a)|}$$

for all $y \in S$, which says that $d(J(f), f) \leq L < \infty$. By Theorem (1.3), there exists a mapping $T_a : S \to E$ such that

(1) T_a is a fixed point of J, i.e.,

$$T_a(y+a) = g(a)T_a \tag{2.3}$$

for all $y \in S$. The mapping T_a is a unique fixed point of J in the set $\tilde{A} = \{h \in A \ : \ d(f,h) < \infty\}$.

(2) $d(J^n(f), T_a) \to 0$ as $n \to \infty$. This implies that

$$T_a(y) = \lim_{n \to \infty} \frac{f(y+na)}{g(a)^n}$$

for all $x \in S$.

(3) $d(f, T_a) \leq \frac{1}{1-L} d(J(f), f)$, which implies,

$$d(f, T_a) \leq \frac{1}{|g(a)| - 1}.$$

From (2.2), its easy to show that following inequality

$$\|f(y+na) - g(a)^n f(y)\| \leq \sum_{i=0}^{n-1} \psi(a, y+ia)|g(a)|^{n-1-i} \tag{2.4}$$

for all $y \in S$ and $n \in \mathbb{N}$. Now since $\psi(a, y+a) \leq \psi(a,y)$ for all $y \in S$, so

$$\psi(a, y+ma) \leq \psi(a,y)$$

for all $x \in S$ and $m \in \mathbb{N}$, thus from (2.4), we obtain

$$\|f(y+na) - g(a)^n f(y)\| \leq \psi(a,y) \frac{|g(a)|^n - 1}{|g(a)| - 1} \tag{2.5}$$

for all $y \in S$. With this inequality (2.5), we prove that $T_a = T_b$ for each $a, b \in N_g$. We have from inequality (2.5)

$$\|f(y+na) - g(a)^n f(y)\| \leq \psi(a,y)\frac{|g(a)|^n - 1}{|g(a)| - 1} \tag{2.6}$$

$$\|f(y+nb) - g(b)^n f(y)\| \leq \psi(b,y)\frac{|g(b)|^n - 1}{|g(b)| - 1} \tag{2.7}$$

for all $y \in S$. On the replacing y by $y + nb$ in (2.6) and y by $y + na$ in (2.7)

$$\|f(y+n(a+b)) - g(a)^n f(y+nb)\| \leq \psi(a,y)\frac{|g(a)|^n - 1}{|g(a)| - 1}$$

$$\|f(y+n(a+b)) - g(b)^n f(y+na)\| \leq \psi(b,y)\frac{|g(b)|^n - 1}{|g(b)| - 1}.$$

Thus,

$$\|g(a)^n f(y+nb) - g(b)^n f(y+na)\| \leq \psi(a,y)\frac{|g(a)|^n - 1}{|g(a)| - 1} + \psi(b,y)\frac{|g(b)|^n - 1}{|g(b)| - 1}$$

and dividing by $|g(a)^n g(b)^n|$

$$\|\frac{f(y+na)}{g(a)^n} - \frac{f(y+nb)}{g(b)^n}\| \leq$$

$$\frac{\psi(a,y)}{(|g(b)|-1)|g(a)|^n}(1 - \frac{1}{|g(b)|^n}) + \frac{\psi(b,y)}{(|g(a)|-1)|g(b)|^n}(1 - \frac{1}{|g(b)|^n})$$

and letting n to infinity, we obtain $T_a(y) = T_b(y)$ for all $y \in S$. Therefore, there a unique function T such that $T = T_a$ for every $a \in N_g$ and

$$\|f(y) - T(y)\| \leq \frac{\psi(a,y)}{|g(a)| - 1}$$

for all $y \in S$ and $a \in N_g$. Since $a \in N_g$ is a arbitrary element, so

$$\|f(y) - T(y)\| \leq \inf_{a \in N_g} \frac{\psi(a,y)}{|g(a)| - 1}$$

for all $y \in S$.
Let $x, y \in S$ and $a \in N_g$ be three arbitrary fixed elements, from (2.1)

$$\|f(x+y+na) - g(x)f(y+na)\| \leq \psi(x, y+na)$$

and dividing by $|g(a)|^n$,

$$\|\frac{f(x+y+na)}{g(a)^n} - g(y)\frac{f(x+na)}{g(a)^n}\| \leq \frac{\psi(x,y+na)}{|g(a)|^n} \leq \frac{\psi(x,y)}{|g(a)|^n}$$

and letting n to infinity, we get $T(x+y) = g(x)T(y)$.
Let $x, y, z \in S$ be arbitrary elements, then

$$T(x+y+z) = g(x+y)T(z)$$

and

$$T(x+y+z) = g(x)T(y+z) = g(x)g(y)T(z)$$

or

$$(g(x+y) - g(x)g(y))T(z) = 0$$

for all $x, y, z \in S$. The proof is complete.
In connection with the open problem of Th. M. Rassias and and Theorem (2.3), in the following we prove Baker's theorem [5] when the Cauchy difference of exponential equation is not bounded and also we prove a generalized version of L. Székelyhidi's theorem.

Corollary 2.4. *Let $f : S \to C$ satisfying*

$$\|f(x+y) - f(x)f(y)\| \leq \psi(x, y)$$

for all $x, y \in S$. If $\psi(x, y+a) \leq \psi(x, y)$ for all $x, y \in S$ and $a \in N_f$, then f is either bounded or is exponential.

Proof. In Theorem (2.3), if we put $E = C$ and $g(x) = f(x)$, then we will had

$$\|f(y) - T(y)\| \leq \inf_{a \in N_g} \frac{\psi(a,y)}{|f(a)|-1} \text{ and } (f(x+y) - f(x)f(y))T(z) = 0$$

for all $x, y, z \in S$. If f is a unbounded function, then $f = T$ and f is exponential.

Corollary 2.5. *Let $f, g : S \to C$, S be with identity, f be a nonzero function and*

$$\|f(x+y) - g(x)f(y)\| \leq \psi(x, y)$$

for all $x, y \in S$, where $\psi(x, y+a) \leq \psi(x, y)$ for all $x, y \in S$ and $a \in N_g$, then g is either bounded or g is exponential and $f(x) = g(x)f(0)$ for all $x \in G$.

Proof. With Theorem (2.3), if g us a unbounded function, then $f = T$, which implies $f(x) = g(x)f(0)$ for all $x \in S$ and since

$$(g(x+y) - g(x)g(y))T(z) = 0$$

for any $x, y, z \in S$ and $f(= T)$ is a nonzero function, so g is exponential.
In [5], Baker presented an example to show that

$$\|f(x+y) - f(x)f(y)\| \leq \varepsilon \qquad for\ x, y \in S$$

implies that f is either bounded or exponential fails if the algebra does not have the multiplicative norm property. Here, we establish this result positively but in a unitary complex Banach algebra.

Theorem 2.6. *Suppose that $f, g : S \to X$ satisfying*

$$\|f(x+y) - g(x)f(y)\| \quad \leq \quad \psi(x, y) \tag{2.8}$$

for all $x, y \in S$. *If* $\widetilde{M_g} \neq \emptyset$ *and* $\psi(x, y + a) \leq \psi(x, y)$ *for all* $x, y \in S$ *and* $a \in \widetilde{M_g}$, *then there exists a exactly one function* $T : S \to X$ *such that*

$$T(x + y) = g(x)f(y)$$
$$(g(x + y) - g(x)g(y))T(z) = 0$$

and satisfies

$$\|f(y) - T(y)\| \leq \inf_{a \in \widetilde{M_g}} [\frac{\psi(a, y)}{|Scg(a)| - 1}]$$

for all $x, y, z \in S$.

Proof. Let $a \in \widetilde{M_g}$ be a arbitrary fixed element, so $g(a) = Scg(a) \times 1_X$, in which $Scg : M_g \to C$ (by Definition 2.2), and from 2.8, we see that

$$\begin{aligned} \|f(a + y) - g(a)f(y)\| &= \|f(a + y) - Scg(a)(1_X f(y))\| \\ &= \|f(a + y) - Scg(a)f(y)\| \leq \psi(a, y) \end{aligned}$$

thus, $\|f(y + a) - Scg(a)f(y)\| \leq \psi(a, y)$ for all $y \in S$. Now since $N_{Scg} = \widetilde{M_g}$, so from Theorem (2.3), there is a unique function $T : S \to X$ such that

$$T(x + y) = Scg(x)T(y)$$
$$[Scg(x + y) - Scg(x)Scg(y)]T(z) = 0$$

and satisfying

$$\|f(y) - T(y)\| \leq \inf_{a \in \widetilde{M_g}} [\frac{\psi(a, y)}{|Scg(a)| - 1}]$$

for all $x, y, z \in S$. From inequality (2.8), we have

$$\|f(x + y + na) - g(x)f(y + na)\| \leq \psi(x, y + na),$$

then on the dividing by $|Scg(a)|^n$ we see that

$$\|\frac{f(x + y + na)}{Scg(a)^n} - g(x)\frac{f(y + na)}{Scg(a)^n}\| \leq \frac{\psi(x, y + na)}{|Scg(a)|^n} \leq \frac{\psi(x, y)}{|Scg(a)|^n}$$

hence, $T(x + y) = g(x)T(y)$ for all $x, y \in S$. Now let $x, y, z \in S$ be arbitrary elements, then

$$T(x + y + z) = g(x + y)T(z)$$

and

$$T(x + y + z) = g(x)T(y + z) = g(x)g(y)T(z)$$

so,

$$(g(x + y) - g(x)g(y))T(z) = 0$$

for all $x, y, z \in S$. The proof is complete.

In the following, we generalize the well-known Baker's superstability and stability result for exponential mappings with values in the field of complex numbers to the case of an arbitrary unitary complex Banach algebra.

Corollary 2.7. *Let $f : S \to X$ satisfying*

$$\|f(x+y) - f(x)f(y)\| \leq \psi(x,y)$$

for all $x, y \in S$. If $Scf(\widetilde{M_f})$ is unbounded and $\psi(x, y+a) \leq \psi(x,y)$ for all $x, y \in S$ and $a \in \widetilde{M_f}$, then f is exponential.

Proof. In Theorem (2.6), if we put $g(x) = f(x)$, then we will had $\|f(x) - T(x)\| \leq \inf_{a\in\widetilde{M_f}}[\frac{\psi(a,y)}{|Scf(a)|-1}]$ and $(f(x+y) - f(x)f(y))T(z) = 0$ for all $x, y, z \in S$. Now since Scf is unbounded, then we have $f = T$, which says that f is exponential and the proof is complete.

Corollary 2.8. *Let $f, g : S \to X$ be three functions, S be with identity and $f(0) \neq 1_X$ and also*

$$\|f(x+y) - g(x)f(y)\| \leq \psi(x,y)$$

for all $x, y \in S$, where $\psi(x, y+a) \leq \psi(x,y)$ for all $x, y \in S$ and $a \in N_g$, then Scg is either bounded or $g(x+y) = g(y)g(x)$ and $f(x) = f(0)g(x)$ for all $x \in S$.

Proof. With Theorem (2.6).
In the following Theorem, we consider the superstability of the a Pexiderized of exponential equation

$$f(x+y) = g(x)f(y),$$

in which f, g and h are three functions from a commutative semigroup to to an unitary an complex Banach algebra.

Theorem 2.9. *Let $f, g, h : S \to X$ be three functions and $g(x_0) = 1_X$ for a fixed $x_0 \in S$ and also*

$$\|f(x+y) - g(x)h(y)\| \quad \leq \quad \psi(x,y) \tag{2.9}$$

for all $x, y \in S$. If $\widetilde{M_g} \neq \emptyset$ and $\psi(x, y+a) \leq \psi(x,y)$ for all $x, y \in S$ and $a \in \widetilde{M_g}$, then there exists a exactly one function $T : S \to X$ such that

$$T(x+y) = g(x)T(y),$$
$$(g(x+y) - g(x)g(y))T(z) = 0$$

and satisfies

$$\|f(y) - T(y)\| \leq \inf_{a\in\widetilde{M_g}} \frac{\widetilde{\psi}(a,y)}{|Scg(a)| - 1},$$

$$\|h(y) - T(y)\| \leq \inf_{a \in \widetilde{M_g}} \frac{\widehat{\psi}(a,y)}{|Scg(a)| - 1}$$

for all $x, y, z \in S$*, in which* $\widetilde{\psi}(x,y) = \psi(x,y) + \|g(x)\|\psi(x_0,y)$ *and* $\widehat{\psi}(x,y) = \psi(x,y) + \psi(x_0, x+y)$ *for* $x, y \in S$*.*

Proof. Applying (2.9) we get for all $x, y \in S$

$$\begin{aligned} \|f(x+y) - g(x)f(y)\| &\leq \|f(x+y) - g(x)h(y)\| + \|g(x)f(y) - g(x)h(y)\| \\ &\leq \psi(x,y) + \|g(x)\|\psi(x_0,y) \end{aligned}$$

and

$$\begin{aligned} \|h(x+y) - g(x)h(y)\| &\leq \|h(x+y) - f(x+y)\| + \|f(x+y) - g(x)h(y)\| \\ &\leq \psi(x,y) + |g(x_0)|\psi(x_0, x+y) \end{aligned}$$

We set $\widetilde{\psi}(x,y) = \psi(x,y) + \|g(x)\|\psi(x_0,y)$ and $\widehat{\psi}(x,y) = \psi(x,y) + \psi(x_0, x+y)$ for $x, y \in S$ and these are obvious that

$$\widetilde{\psi}(x, y+1) \leq \widetilde{\psi}(x,y)$$

and

$$\widehat{\psi}(x, y+1) \leq \widehat{\psi}(x,y)$$

for $x, y \in S$. Therefore by Theorem (2.6), then there exists a exactly one function $H : S \to X$ such that

$$H(x+y) = g(x)H(y)$$

$$(g(x+y) - g(x)g(y))H(z) = 0$$

and satisfies

$$\|f(y) - H(y)\| \leq \inf_{a \in \widetilde{M_g}} \frac{\widetilde{\psi}(a,y)}{|Scg(a)| - 1}$$

for all $x, y, z \in S$, where $H(x) = \lim_{n\to\infty} \frac{f(x+na)}{Scg(a)^n}$ for all $x \in S$ and any fixed $a \in \widetilde{M_g}$. And also then there exists a exactly one function $F : S \to X$ such that

$$F(x+y) = g(x)F(y)$$

$$(g(x+y) - g(x)g(y))F(z) = 0$$

and satisfies

$$\|h(y) - F(y)\| \leq \inf_{a \in \widetilde{M_g}} \frac{\widehat{\psi}(a,y)}{|Scg(a)| - 1}$$

for all $x, y, z \in S$, where $F(x) = \lim_{n\to\infty} \frac{h(x+na)}{Scg(a)^n}$ for all $x \in S$ and any fixed $a \in \widetilde{M_g}$. Furthermore, we have

$$\|\frac{f(x+na)}{Scg(a)^n} - \frac{h(x+na)}{Scg(a)^n}\| = |Scg(a)|^{-n}\|f(x+na) - h(x+na)\| \quad (2.10)$$

$$\leq \frac{|g(x_0)|\psi(x_0, x+na)}{Scg(a)^n} \leq \frac{\psi(x_0, x)}{Scg(a)^n} \quad (2.11)$$

for all $x \in S$ and any fixed $a \in \widetilde{M_g}$. Hence, $H = F$ for all $x, y \in S$ and there exists a exactly one function $T : S \to X$ such that

$$T(x+y) = g(x)T(y),$$

$$(g(x+y) - g(x)g(y))T(z) = 0$$

and satisfies

$$\|f(y) - T(y)\| \leq \inf_{a\in\widetilde{M_g}} \frac{\widetilde{\psi}(a, y)}{|Scg(a)| - 1},$$

$$\|h(y) - T(y)\| \leq \inf_{a\in\widetilde{M_g}} \frac{\widehat{\psi}(a, y)}{|Scg(a)| - 1}$$

for all $x, y, z \in S$.
As a consequence of Theorem (2.9), we have the following results.

Corollary 2.10. *Let $f, g, h : S \to X$ be three functions, S be with identity and $g(0) = 1_X$ and also*

$$\|f(x+y) - g(x)h(y)\| \leq \psi(x, y)$$

for all $x, y \in S$. If $\psi(x, y+a) \leq \psi(x, y)$ for all $x, y \in S$ and $a \in \widetilde{M_g}$, then Scg is either bounded or $h = f$, $f(x) = f(0)g(x)$ and g is exponential.

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