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Geodesic vectors of invariant (α, β)-metrics on nilpotent Lie groups of five dimensional

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Abstract. In this paper, we consider invariant (α, β)- metrics which are induced by invariant Riemannian metrics and invariant vector fields on homogeneous spaces. We first study geodesic vectors and investigates the set of all homogeneous geodesics of (α, β)-metrics. Then we study the geometry of simply connected two-step nilpotent Lie groups of dimension five equipped with a left invariant (α, β)- metrics and we examine Lie algebras with 1-dimensional center, 2-dimensional center and 3-dimensional center.

Keywords: (α, β)-metrics, geodesic vector, two-step nilpotent Lie group, invariant metric.

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1. Introduction

Finsler geometry is just the Riemannian geometry without the quadratic restriction. Finsler generalized Riemann's theory in his doctoral thesis, but his name was established in differential geometry by Cartan [2].

In 1972, Matsumoto had introduced the concept of (α, β)-metric in Finsler geometry [10]. A Finsler metric of the form

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$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-manifold M and $\beta = b_i(x)y^i$ is a 1-form on M, is called an (α, β)-metric. (α, β)-metrics are the generalizations of the Randers metric, introduced by G. Randers [13]. There are various applications of (α, β)-metrics in information geometry, physics and biology.

The concept of geodesics is a very important subjects in geometry. Indeed, geodesics are the generalization of a straight line in an Euclidean space. Geodesic can be viewed as a curve that minimizes the distance between two points on the manifold. A geodesic in a homogeneous Finsler space $(G/H, F)$ is called homogeneous geodesic if it is an orbit of a one-parameter subgroup of G.

In [7], Latifi has extended the concept of homogeneous geodesics in homogeneous Finsler spaces and he has given a criterion for the characterization of geodesic vectors. In [8], Latifi and Razavi study homogeneous geodesics in a three-dimensional connected Lie group with a left invariant Randers metric.

A connected Riemannian manifold (M, g) is said to be homogeneous if a connected group of isometries G acts transitively on it. Then M can be viewed as a coset space G/H with a G -invariant metrics, where H is the isotropy subgroup at a fixed point o of M. A geodesic $\gamma(t)$ through the origin $o = eH$ is called a homogeneous geodesic if it is an orbit of a one-parameter subgroup of G. Indeed,

$$\gamma(t) = \exp(tZ)(o),$$

where Z is a non-zero vector in the Lie algebra $\mathfrak{g}$ of G.

Homogeneous geodesic in a Lie group were studied by V. V. Kajzer in [5] where he proved that a Lie group G with a left-invariant metric has at least one homogeneous geodesic through the identity.

A connected Riemannian manifold which admits a transitive nilpotent Lie group G of isometries is called a nilmanifold [3]. E. Wilson showed that for a given homogeneous nilmanifold M, there exists a unique nilpotent Lie subgroup N of $I(M)$ acting simply transitively on M, and N is normal in $I(M)$ [14]. J. Lauret classified, up to isometry, all homogeneous nilmanifolds of dimension 3 and 4 (not necessarily two-step nilpotent) and computed the corresponding isometry groups. He also studied, as example, the structure of specific five-dimensional two-step nilmanifolds with two-dimensional center [9].

The Lie algebra $\mathfrak{g}$ is called two-step nilpotent Lie algebra if $[x, [y, z]] = 0$ for any $x, y, z \in \mathfrak{g}$. A Lie group G is said to be two-step nilpotent if its Lie algebra $\mathfrak{g}$ is two-step nilpotent. Two-step nilpotent Lie groups endowed with a left-invariant metric, often called two-step homogeneous nilmanifolds are studied in the last years [11, 12].

In this paper, we study the existence of invariant vector fields on homogeneous Finsler spaces with (α, β)- metrics. Also, we study the geometry of simply connected two-step nilpotent Lie groups of dimension five endowed with left invariant (α, β)- metrics. We consider homogeneous geodesics in an invariant (α, β)- metrics on simply connected two-step nilpotent Lie groups of dimensional five.

2. Preliminaries

In this section, we recall briefly some known facts about Finsler spaces. For details, see [1].

Definition 2.1. Let M be a n- dimensional C^∞ manifold and $TM = \cup_{x \in M} T_x M$ be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_x M$ and $\lambda > 0$.
(3) The $n \times n$ Hessian matrix

$$[g_{ij}] = \frac{1}{2}\Big[\frac{\partial^2 F^2}{\partial y^i \partial y^j}\Big]$$

is positive definite at every point $(x, y) \in TM^0$.

The following bilinear symmetric form $g_y : T_x M \times T_x M \longrightarrow R$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0}.$$

We recall that, by the homogeneity of F we have

$$g_y(u, v) = g_{ij}(x, y) u^i v^j, \quad F = \sqrt{g_{ij}(x, y) y^i y^j}.$$

Definition 2.2. Let $\alpha = \sqrt{\widetilde{a}_{ij}\,(x)\, y^i y^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta\,(x, y) = b_i(x) y^i$ be a 1-form on an n- dimensional manifold M. Suppose

$$b := \|\beta(x)\|_\alpha := \sqrt{\widetilde{a}^{ij}(x) b_i(x) b_j(x)},$$

and let the function F is defined as follows

$$F := \alpha\varphi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\varphi = \varphi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\varphi(s) - s\varphi'(s) + \left(b^2 - s^2\right)\varphi''(s) > 0, \qquad |s| \leq b < b_0.$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form 2.1 is called an (α, β)- metric.

The Riemannian metric $\widetilde{a}$ induces an inner product on any cotangent space T_x^*M such that

$$\langle dx^i(x), dx^j(x)\rangle = \widetilde{a}^{ij}(x).$$

The induced inner product on T_x^*M induced a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\widetilde{X}$ on M such that

$$\widetilde{a}\left(y, \widetilde{X}(x)\right) = \beta(x, y).$$

Also, we have

$$\|\beta(x)\|_\alpha = \|\widetilde{X}(x)\|_\alpha.$$

Therefore we can write (α, β)- metrics as follows:

$$F(x, y) = \alpha(x, y)\,\varphi\left(\frac{\widetilde{a}(\widetilde{X}(x), y)}{\alpha(x, y)}\right), \tag{2.2}$$

where for any $x \in M$,

$$\sqrt{\widetilde{a}\left(\widetilde{X}(x), \widetilde{X}(x)\right)} = \|\widetilde{X}(x)\|_\alpha < b_0.$$

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM^0 \to M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the Chern connection whose coefficients are denoted by Γ^i_{jk} [1]. This connection is almost g-compatible and has no torsion. Since, in general, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must define a fixed reference vector.

Let $\sigma(t)$ be a smooth regular curve in M, with velocity field T. Let $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along σ. The expression

$$\Big[\frac{dW^i}{dt} + W^jT^k(\Gamma^i_{jk})_{(\sigma,T)}\Big]\frac{\partial}{\partial x^i}|_{\sigma(t)},$$

would have defined the covariant derivative $D_T W$ with reference vector T. A curve $\sigma(t)$, with velocity $T = \dot{\sigma}(t)$ is a Finslerian geodesic if

$$D_T\Big[\frac{T}{F(T)}\Big] = 0, \quad \text{with reference vector } T,$$

such that the constant speed geodesics are precisely the solution of

$$D_T T = 0, \quad \text{with reference vector } T.$$

Since $T = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}$, this differential equations that describe constant speed geodesics are

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})_{(\sigma,T)} = 0.$$

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra and G is the simply connected Lie group with Lie algebra $\mathfrak{g}$. A Finsler metric $F : TG \longrightarrow [0, \infty)$ will be called left-invariant if

$$F\big((L_a)_{*e}X\big) = F(X), \quad \forall a \in G, \quad \forall X \in \mathfrak{g},$$

where L_a is the left translation and e is the unit element of the Lie group.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We may identify the tangent bundle TG with $G \times \mathfrak{g}$ by means of the diffeomorphism that sends (g, X) to $(L_g)_* X \in T_g G$.

Definition 2.4. A Finsler function $F : TG \longrightarrow R_+$ will be called G-invariant if F is constant on all G-orbits in $TG = G \times \mathfrak{g}$. Indeed,

$$F(g, X) = F(e, X), \ \forall g \in G \text{ and } X \in \mathfrak{g}.$$

The G-invariant Finsler functions on TG may be identified with the Minkowski norms on $\mathfrak{g}$. If $F : TG \longrightarrow R_+$ is a G-invariant Finsler function, then we may define $\tilde{F} : \mathfrak{g} \longrightarrow R_+$ by

$$\tilde{F}(X) = F(e, X),$$

where e denotes the identity in G. Conversely, if we are given a Minkowski norm $\tilde{F} : \mathfrak{g} \longrightarrow R_+$, then $\tilde{F}$ arises from a G-invariant Finsler function $F : TG \longrightarrow R_+$ given by

$$F(g, X) = \tilde{F}(X), \text{ for all } (g, X) \in G \times \mathfrak{g}.$$

3. Homogeneous Geodesics

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of the Lie groups G and H respectively. Then the direct sum decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ such that $Ad(h)(\mathfrak{m}) \subset \mathfrak{m}$, $\forall h \in H$, is called a reductive decomposition of $\mathfrak{g}$.

Definition 3.1. A Finsler space (M, F) is called a homogeneous Finsler space if the group of isometries of (M, F), $I(M, F)$, acts transitively on M.

We recall that, any homogeneous Finsler manifold $M = G/H$ is a reductive homogeneous space.

Definition 3.2. Let $(G/H, F)$ be a homogeneous Finsler space and e be the identity of G. A non-zero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve $exp(tX).eH$ is a geodesic of $(G/H, F)$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{a}$ be a left invariant Riemannian metric on G. In [6], the author proved that a vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}(Y, [Y, Z]) = 0, \quad \forall Z \in \mathfrak{g}. \tag{3.1}$$

In [7], the second author proved the following result that gives a criterion for a non-zero vector to be a geodesic vector in a homogeneous Finsler space.

Lemma 3.3. *A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{Y_\mathfrak{m}} = (Y_\mathfrak{m}, [Y, Z]_\mathfrak{m}) = 0, \quad \forall Z \in \mathfrak{g}. \tag{3.2}$$

Now we have the following results for geodesic vector of (α, β)-metrics:

Theorem 3.4. *Let G/H be a homogeneous (α, β)-metric F defined by the $F = \alpha\phi(s)$, $s = \beta/\alpha$ induced by an invariant Riemannian metric $\tilde{a}$ and an invariant vector field $\tilde{X}$ such that $\tilde{X}(H) = X$. Then, a non-zero vector $y \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\tilde{a}(Ay_\mathfrak{m} + BX, [y, z]_\mathfrak{m}) = 0, \quad \forall z \in \mathfrak{g}, \tag{3.3}$$

where

$$A = \phi^2(q) - \phi(q)\phi'(q)q, \quad B = \phi'(q)F(y) \quad \text{and} \quad q = \frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}.$$

Proof. We know that,

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}F^2(y + su + tv)|_{s=t=0}. \tag{3.4}$$

Let

$$F(x, y) = \alpha(x, y)\phi\Big(\frac{\tilde{a}(X(x), y)}{\alpha(x, y)}\Big).$$

By using the formula 3.4 and after some calculations, we get

$$g_y(u,v) = \tilde{a}(u,v)\phi^2(q) + \tilde{a}(y,u)\phi(q)\phi'(q)\Big(\frac{\tilde{a}(X,v)}{\sqrt{\tilde{a}(y,y)}} - \frac{\tilde{a}(X,y)\tilde{a}(y,v)}{(\tilde{a}(y,y))^{\frac{3}{2}}}\Big)$$
$$+ \Big((\phi'(q))^2 + \phi(q)\phi''(q)\Big)\Big(\frac{\tilde{a}(X,v)}{\sqrt{\tilde{a}(y,y)}} - \frac{\tilde{a}(X,y)\tilde{a}(y,v)}{(\tilde{a}(y,y))^{\frac{3}{2}}}\Big)$$
$$\times \Big(\tilde{a}(X,u)\sqrt{\tilde{a}(y,y)} - \frac{\tilde{a}(y,u)\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}\Big)$$
$$+ \frac{\phi(q)\phi'(q)}{\sqrt{\tilde{a}(y,y)}}\Big(\tilde{a}(X,u)\tilde{a}(y,v) - \tilde{a}(u,v)\tilde{a}(X,y)\Big),$$

where $q = \frac{\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}}$. So for all $z \in \mathfrak{g}$ we have:

$$\begin{aligned} g_{y_\mathfrak{m}}(y_\mathfrak{m}, [y,z]_\mathfrak{m}) &= \tilde{a}(y_\mathfrak{m}, [y,z]_\mathfrak{m})\Big(\phi^2(q) - \phi(q)\phi'(q)q\Big) \\ &\quad + \tilde{a}(X, [y,z]_\mathfrak{m})\Big(\phi'(q)F(y)\Big) \\ &= \tilde{a}(Ay_\mathfrak{m} + BX, [y,z]_\mathfrak{m}), \end{aligned} \tag{3.5}$$

where

$$A = \phi^2(q) - \phi(q)\phi'(q)q, \quad B = \phi'(q)F(y).$$

Then by lemma 3.3, $g_{y_\mathfrak{m}}(y_\mathfrak{m}, [y,z]_\mathfrak{m}) = 0$ if and only if

$$\tilde{a}(Ay_\mathfrak{m} + BX, [y,z]_\mathfrak{m}) = 0, \quad \forall z \in \mathfrak{g}.$$

□

Corollary 3.5. *Let G/H be a homogeneous (α,β)-metric F with $F = \alpha\phi(s)$, $s = \beta/\alpha$ defined by an invariant Riemannian metric $\tilde{a}$ and an invariant vector field $\tilde{X}$ such that $\tilde{X}(H) = X$. Suppose that $y \in \mathfrak{g} - \{0\}$ and $\tilde{a}(X, [y,z]_\mathfrak{m}) = 0$. Then y is a geodesic vector of (M,F) if and only if it is a geodesic vector of $(M,\tilde{a})$.*

Proof. From equation 3.5, we have

$$g_{y_\mathfrak{m}}(y_\mathfrak{m}, [y,z]_\mathfrak{m}) = \tilde{a}(y_\mathfrak{m}, [y,z]_\mathfrak{m})\Big(\phi^2(q) - \phi(q)\phi'(q)q\Big).$$

By the definition of (α,β)-metrics $\phi^2(q) - \phi(q)\phi'(q)q$ is positive. Then

$$g_{y_\mathfrak{m}}(y_\mathfrak{m}, [y,z]_\mathfrak{m}) = 0,$$

if and only if $\tilde{a}(y_\mathfrak{m}, [y,z]_\mathfrak{m}) = 0$. □

4. simply connected two-step nilpotent Lie groups of dimension five

In this section we study simply connected two-step nilpotent Lie groups of dimension five equipped with left-invariant (α, β)- metrics and has 1-dimensional, 2-dimensional and 3-dimensional center.

4.1. **Lie algebras with 1-dimensional center.** Let $\mathfrak{g}$ denotes a 5-dimensional 2-step nilpotent Lie algebra with 1- dimensional center $\mathfrak{k}$ and let G be the corresponding simply connected Lie group. We assume that $\mathfrak{g}$ is equipped with an inner product $\langle , \rangle$. Let e_5 be a unit vector in $\mathfrak{k}$ and let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. In [4], S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda e_5, \quad [e_3, e_4] = \mu e_5, \tag{4.1}$$

where $\lambda \geqslant \mu > 0$. Also the other commutators are zero.

For Example, $O(2) \times SO(2)$ be a Lie group for $\lambda \neq \mu$ and $U(2) \times \mathbb{Z}_2$ be a Lie group for $\lambda = \mu$ [12].

Let F be a left invariant (α, β)-metric on simply connected two-step nilpotent Lie group G defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$. We want to describe all geodesic vectors of (G, F).

By using Theorem 3.4, a vector $y = \sum_{i=1}^{5} y_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}(A \sum_{i=1}^{5} y_i e_i + B \sum_{i=1}^{5} x_i e_i, [\sum_{i=1}^{5} y_i e_i, e_j]) = 0, \tag{4.2}$$

where

$$A = \phi^2(q) - \phi(q)\phi'(q)q, \quad B = \phi'(q)F(y) \;\text{ and }\; q = \frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}},$$

for each $j = 1, 2, 3, 4, 5$.

So we get:

$$\lambda y_1(Ay_5 + Bx_5) = 0,$$

$$\lambda y_2(Ay_5 + Bx_5) = 0,$$

$$\mu y_3(Ay_5 + Bx_5) = 0,$$

$$\mu y_4(Ay_5 + Bx_5) = 0. \tag{4.3}$$

Corollary 4.1. *Let F be the (α, β)-metric induced by the Riemannina metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{5} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then geodesic vectors depending only on $\tilde{a}(X, e_5)$.*

Corollary 4.2. *Let F be the (α, β)-metric induced by the invariant Riemannian metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then a vector $y \in \mathfrak{g}$ is a geodesic vector if and only if $y \in span\{e_1, e_2, e_3, e_4\}$ or $y = \beta e_5$ for $\beta \neq 0$.*

Corollary 4.3. *Let (M, F) be the (α, β)-metric induced by an invariant Riemannina metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with one dimensional center. Then X is a geodesic vector of $(M, \tilde{a})$ if and only if X is a geodesic vector of (M, F).*

Theorem 4.4. *Let (M, F) be the (α, β)-metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on simply connected two-step nilpotent Lie groups of dimension five with one dimensional center. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. From 4.1, $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$. Let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of $(M, \tilde{a})$. By using equation 3.1 we have:

$$\tilde{a}(y, [y, e_i]) = 0,$$

for each $i = 1, 2, 3, 4, 5$. Then by using equation 4.2, y is a geodesic vector of (M, F).

Conversely, let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of (M, F). Since $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$ by using 4.2 we have:

$$\tilde{a}(y, [y, e_i]) = 0, \quad i = 1, 2, 3, 4, 5.$$

□

4.2. **Lie algebras with 2-dimensional center.** In [4] S. Homolya and O. Kowalski, showed that there exist an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda e_4, \quad [e_1, e_3] = \mu e_5, \tag{4.4}$$

where $\{e_4, e_5\}$ is a basis for the center of $\mathfrak{g}$, the other commutators are zero and $\lambda \geq \mu > 0$.

For Example, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ be a Lie group for $\lambda \neq \mu$ and $O(2) \times \mathbb{Z}_2$ be a Lie group for $\lambda = \mu$ [12].

Let F be a left invariant (α, β)- metric on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$.

By using Theorem 3.4, a vector $y = \sum_{i=1}^{5} y_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}(A \sum_{i=1}^{5} y_i e_i + B \sum_{i=1}^{5} x_i e_i, [\sum_{i=1}^{5} y_i e_i, e_j]) = 0, \tag{4.5}$$

where

$$A = \phi^2(q) - \phi(q)\phi'(q)q, \quad B = \phi'(q)F(y) \text{ and } q = \frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}},$$

for each $j = 1, 2, 3, 4, 5$.

So we get:

$$\lambda y_2(Ay_4 + Bx_4) + \mu y_3(Ay_5 + Bx_5) = 0,$$

$$\lambda y_1(Ay_4 + Bx_4) = 0,$$

$$\mu y_1(Ay_5 + Bx_5) = 0. \tag{4.6}$$

Corollary 4.5. *Let F be the (α, β)- metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center. Then geodesic vectors dependig only on $\tilde{a}(X, e_4), \tilde{a}(X, e_5), \lambda$ and μ.*

Corollary 4.6. *Let (M, F) be the (α, β)- metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with two dimensional center. Then X is a geodesic vector of $(M, \tilde{a})$ if and only if X is a geodesic vector of (M, F).*

Theorem 4.7. *Let (M, F) be the (α, β)- metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{3} x_i e_i$ on simply connected two-step nilpotent Lie groups of dimension five with two dimensional center. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(N, \tilde{a})$.*

Proof. From 4.4, $\tilde{a}(X,[y,e_i]) = 0$ for each $i = 1,2,3,4,5$. Let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of $(M,\tilde{a})$. By using equation 3.1 we have:

$$\tilde{a}(y,[y,e_i]) = 0,$$

for each $i = 1,2,3,4,5$. Then by using equation 4.5, y is a geodesic vector of (M,F).
Conversely, let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of (M,F). Since $\tilde{a}(X,[y,e_i]) = 0$ for each $i = 1,2,3,4,5$ by using 4.5 we have:

$$\tilde{a}(y,[y,e_i]) = 0, \quad i = 1,2,3,4,5.$$

□

4.3. **Lie algebras with 3-dimensional center.** In this section, we study simply connected two-step nilpotent Lie group of dimension five with 3-dimensional center equipped with left-invariant (α,β)- metric. In [4], S. Homolya and O. Kowalski showed that there exist an orthonormal basis $\{e_1,e_2,e_3,e_4,e_5\}$ of $\mathfrak{g}$ such that:

$$[e_1,e_2] = \lambda e_3, \tag{4.7}$$

where $\{e_3,e_4,e_5\}$ is a basis for the center of $\mathfrak{g}$, the other commutators are zero and $\lambda > 0$.

For Example, $H_3 \times \mathbb{R}^2$ or $O(2) \times O(2)$ be a Lie group with the metric Heisenberg Lie algebra $\mathfrak{h}_3(\lambda) \oplus \mathbb{R}^2$ [12].

Let F be a left invariant (α,β)- metric on simply connected two-step nilpotent Lie groups of dimension five with three dimensional center defined by the Riemannian metric $\tilde{a}$ and the vector field $X = \sum_{i=1}^{5} x_i e_i$. We want to describe all geodesic vectors of (M,F).

By using Theorem 3.4, a vector $y = \sum_{i=1}^{5} y_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if

$$\tilde{a}(A\sum_{i=1}^{5} y_i e_i + B\sum_{i=1}^{5} x_i e_i, [\sum_{i=1}^{5} y_i e_i, e_j]) = 0, \tag{4.8}$$

where

$$A = \phi^2(q) - \phi(q)\phi'(q)q, \quad B = \phi'(q)F(y) \;\text{ and }\; q = \frac{\tilde{a}(X,y)}{\sqrt{\tilde{a}(y,y)}},$$

for each $j = 1,2,3,4,5$.

So we get:

$$\lambda y_1(Ay_3 + Bx_3) = 0,$$

$$\lambda y_2(Ay_3 + Bx_3) = 0. \tag{4.9}$$

Then, we conclude the following results:

Corollary 4.8. *Let (M, F) be the (α, β)- metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = \sum_{i=1}^{5} x_i e_i$ on simply connected two-step nilpotent Lie groups of dimension five with three dimensional center. Then geodesic vectors dependig only on $\tilde{a}(X, e_3)$.*

Corollary 4.9. *Let (M, F) be the (α, β)- metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field X on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then X is a geodesic vector of $(M, \tilde{a})$ if and only if X is a geodesic vector of (M, F).*

Theorem 4.10. *Let (M, F) be the (α, β)- metric defined by an invariant metric $\tilde{a}$ and an left invariant vector field $X = x_1e_1 + x_2e_2 + x_4e_4 + x_5e_5$ on simply connected two-step nilpotent Lie group of dimension five with three dimensional center. Then $y \in \mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_3, e_4, e_5\}$ or $y \in Span\{e_1, e_2, e_4, e_5\}$.*

Theorem 4.11. *Let (M, F) be the (α, β)- metric induced by the Riemannian metric $\tilde{a}$ and the left invariant vector field $X = x_1e_1 + x_2e_2 + x_4e_4 + x_5e_5$ on simply connected two-step nilpotent Lie groups of dimension five with three dimensional center. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(N, \tilde{a})$.*

Proof. From 4.7, $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$. Let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of $(M, \tilde{a})$. By using equation 3.1 we have:

$$\tilde{a}(y, [y, e_i]) = 0,$$

for each $i = 1, 2, 3, 4, 5$. Then by using equation 4.8, y is a geodesic vector of (M, F).

Conversely, let $y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of (M, F). Since $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4, 5$ by using 4.8 we have:

$$\tilde{a}(y, [y, e_i]) = 0, \quad i = 1, 2, 3, 4, 5.$$

□

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