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Mountain pass solution for a $p(t)$-biharmonic Kirchhoff-type equation

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Abstract. In this paper we deal with the existence of weak solution for a $p(t)$-Kirchhoff-type problem of the following form

$$\begin{cases} -\left(\alpha-\beta\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt\right)\Delta(|\Delta\vartheta|^{p(t)-2}\Delta\vartheta)= & \\ \lambda|\vartheta|^{p(t)-2}\vartheta+g(t,\vartheta) & \text{in } \Gamma, \\ \vartheta=\Delta\vartheta=0 & \text{on } \partial\Gamma. \end{cases}$$

Using the Mountain Pass Theoem, we establish conditions ensuring the existence result.

Keywords: $p(t)$-biharmonic equation, Kirchhoff-type problems, Mountain Pass Theorem.

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1. Introduction

In this paper we study the following problem

$$\begin{cases} -\left(\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta|^{p(t)}\, dt\right) \Delta(|\Delta\vartheta|^{p(t)-2}\Delta\vartheta) = \\ \lambda|\vartheta|^{p(t)-2}u + g(t,\vartheta) & \text{in } \Gamma, \\ \vartheta = \Delta\vartheta = 0 & \text{on } \partial\Gamma. \end{cases} \tag{1.1}$$

where $\Gamma \subset \mathbb{R}^N$, $N \geq 2$ is a bounded smooth domain with smooth boundary $\partial\Gamma$, $p(t) \in C(\overline{\Gamma})$, $\alpha, \beta > 0$ are constants, g is a continuous function, λ is a real parameter. We impose these conditions on the nonlinearity $g(t,s) \in C(\overline{\Gamma}, \mathbb{R})$:

($\mathbf{g_1}$) The the Carathéodory function $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfies the subcritical growth condition, i.e. there exists a constant $c_1 \geq 0$ so that

$$|g(t,s)| \leq c_1(1 + |s|^{q(t)-1}),$$

for all $(t,s) \in \Gamma \times \mathbb{R}$ where $q(t) \in C_+(\overline{\Gamma})$ and $q(t) < p_k^*(t)$.

($\mathbf{g_2}$) $g(x,s) = o(|s|^{p(t)-2}s)$ as $s \to 0$ uniformly with respect to $t \in \Gamma$.

($\mathbf{g_3}$) There exist $M > 0$ and $\theta \in \left(p^+, \dfrac{2(p^-)^2}{p^+}\right)$ so that $0 < \theta G(t,s) \leq sg(t,s)$, for all $|s| \geq M$ and $t \in \Gamma$ where $G(t,s) = \int_0^s g(t,\tau)\, d\tau$.

Nonlocal $p(t)$-biharmonic elliptic problems are an interesting area of nonlinear analysis, connecting many different mathematical fields such as partial differential equations (PDEs), functional analysis and the calculus of variations. By utilizing both nonlocal operators and space for variables in the exponent of equations, these problems are extensions of classical biharmonic equation. The applications of nonlocal $p(t)$-biharmonic operators are vast and impactful, addressing complex problems across multiple disciplines. Numerous papers have been published, focusing on various aspects such as existence and multiplicity of solutions, qualitative properties, and applications of these problems in different contexts, (see [1, 2, 3, 8, 9]).

We concentrate on a new Kirchhoff problem related to the $p(t)$-biharmonic operator, that is, the form with a nonlocal coefficient $(\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta|^{p(t)}\, dt)$. Its background is derived from nagative Young's modulus, when the atoms are separated into two pieces instead of being compressed, leading to a negative strain.
As we know, the eigenvalues of $p(t)$-biharmonic problem with Navier-boundary conditions

$$\begin{cases} \Delta(|\Delta\vartheta|^{p(t)-2}\Delta\vartheta) = \lambda|\vartheta|^{p(t)-2}\vartheta & \text{in } \Gamma, \\ \vartheta = \Delta\vartheta = 0 & \text{on } \partial\Gamma. \end{cases}$$

were studied in [3], and the first eigenvalue is determined by the following Rayleigh quotient

$$\lambda_1 = \inf_{X\setminus\{0\}} \frac{\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt}{\int_\Gamma \frac{1}{p(t)}|\vartheta|^{p(t)}\,dt} \tag{1.2}$$

where $X = W^{2,p(t)}(\Gamma)\bigcap W_0^{1,p(t)}(\Gamma)$. Moreover, under some special conditions, λ_1 is positive.

The authors in [11] for the first time, studied this form of the Kirchhoff-type problem

$$\begin{cases} -\left(a - b\int_\Gamma |\nabla u|^2\,dx\right)\Delta u = \lambda|u|^{p-2}u & \text{in } \Gamma, \\ u = 0 & \text{on } \partial\Gamma, \end{cases}$$

with $2 < p < 2^* := (2N)/(N-2)$, and they obtained the existence of solutions by using the mountain pass theorem. Furthermore, some interesting results have been obtained for this kind of Kirchhoff-type problem. We refer the readers to [1, 10, 13] and the references therein.

Now, we state our main result:

Theorem 1.1. *Assume that the function $q \in C(\overline{\Gamma})$ satisfies*

$$1 < p^- < p(t) < p^+ < 2p^- < q^- < q(t) < p_k^*(t) := \frac{Np(t)}{N-kp(t)} \tag{1.3}$$

and $2p^- < \theta$.

Then considering conditions $(\mathbf{g_1})$-$(\mathbf{g_3})$, for all $\lambda \in \mathbb{R}$, problem (1.1) admits a nontrivial weak solution.

2. Notations and preliminaries

Let Γ be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Gamma}) = \{p(t);\ p(x) \in C(\overline{\Gamma}),\ p(t) > 1,\ \forall t \in \overline{\Gamma}\},$$

$$p^+ = \max\{p(t);\ t \in \overline{\Gamma}\}, \quad p^- = \min\{p(t);\ t \in \overline{\Gamma}\};$$

$$L^{p(t)}(\Gamma) = \{\vartheta : \Gamma \to \mathbb{R} \text{ measurable and } \int_\Gamma |\vartheta(t)|^{p(t)}\,dt < \infty\},$$

with the norm $|\vartheta|_{L^{p(t)}(\Gamma)} = |\vartheta|_{p(t)} = \inf\left\{\mu > 0;\ \int_\Gamma \left|\frac{\vartheta(t)}{\mu}\right|^{p(t)} dx \le 1\right\}$.

Proposition 2.1 (See [6])**.** *The space $(L^{p(t)}(\Gamma), |\cdot|_{p(t)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(t)}(\Gamma)$ where $q(t)$ is the conjugate function of $p(t)$, i.e., $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$, for all $t \in \Gamma$. For $\vartheta \in L^{p(t)}(\Gamma)$ and $\upsilon \in L^{q(t)}(\Gamma)$, we have $\left|\int_\Gamma \vartheta\upsilon\,dt\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|\vartheta|_{p(t)}|\upsilon|_{q(t)} \le 2|\vartheta|_{p(t)}|\upsilon|_{q(t)}$.*

The Sobolev space with variable exponent $W^{k,p(t)}(\Gamma)$ is defined as follows: $W^{k,p(t)}(\Gamma) = \{\vartheta \in L^{p(t)}(\Gamma) : \ D^\alpha \vartheta \in L^{p(t)}(\Gamma), \ |\alpha| \le k\}$, where $D^\alpha \vartheta = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial t_2^{\alpha_2} \ldots \partial t_N^{\alpha_N}} \vartheta$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(t)}(\Gamma)$ equipped with the norm $\|\vartheta\|_{k,p(t)} = \sum_{|\alpha| \le k} |D^\alpha \vartheta|_{p(t)}$, also becomes a separable and reflexive Banach space. For more details, we refer the reader to [5, 6].

Proposition 2.2 (See [6]). *For $p, r \in C_+(\overline{\Gamma})$ such that $r(t) \le p_k^*(t)$ for all $t \in \overline{\Gamma}$, there is a continuous embedding $W^{k,p(t)}(\Gamma) \hookrightarrow L^{r(t)}(\Gamma)$. If we replace $\le$ with $<$, the embedding is compact.*

We denote by $W_0^{k,p(t)}(\Gamma)$ the closure of $C_0^\infty(\Gamma)$ in $W^{k,p(t)}(\Gamma)$. Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space $X = W^{2,p(t)}(\Gamma) \cap W_0^{1,p(t)}(\Gamma)$ equipped with the norm $\|\vartheta\| = \inf \left\{ \mu > 0 : \ \int_\Gamma \left| \frac{\Delta \vartheta(t)}{\mu} \right|^{p(t)} \ dx \le 1 \right\}$.

Remark 2.3. According to [12], the norm $\| \cdot \|_{2,p(t)}$ is equivalent to the norm $|\Delta \cdot|_{p(t)}$ in the space X. Consequently, the norms $\| \cdot \|_{2,p(t)}$, $\| \cdot \|$ and $|\Delta \cdot|_{p(t)}$ are equivalent.

We consider the functional $\rho(\vartheta) = \int_\Gamma |\Delta \vartheta|^{p(t)} \, dt$ and give the following fundamental proposition.

Proposition 2.4 (See [4]). *For $\vartheta \in X$ and $\vartheta_n \subset X$, we have*

(**1**) $\|\vartheta\| < 1$ *(respectively* $= 1; > 1$) $\Longleftrightarrow \rho(\vartheta) < 1$ *(respectively* $= 1; > 1$*);*
(**2**) *if* $\|\vartheta\| > 1$*, then* $\|\vartheta\|^{p^-} \le \rho(\vartheta) \le \|\vartheta\|^{p^+}$*;*
(**3**) *if* $\|\vartheta\| < 1$*, then* $\|\vartheta\|^{p^+} \le \rho(\vartheta) \le \|\vartheta\|^{p^-}$*;*
(**4**) $\|\vartheta_n\| \to 0$ *(respectively* $\to \infty$) $\Longleftrightarrow \rho(\vartheta_n) \to 0$*(respectively* $\to \infty$*).*

Let us define the functional

$$\mathcal{K}(\vartheta) = \int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)} \, dx.$$

It is well known that $\mathcal{K}$ is well defined, even and C^1 in X. Moreover, the operator $L = \mathcal{K}' : X \to X^*$ defined as

$$\langle L(\vartheta), \upsilon \rangle = \int_\Gamma |\Delta \vartheta|^{p(t)-2} \Delta \vartheta \Delta \upsilon \, dt$$

for all $\vartheta, \upsilon \in X$ satisfies the following assertions.

Proposition 2.5 (See El Amrouss et al. [4]). *The derivative operator L has the following properties:*

(**1**) *L is continuous, bounded and strictly monotone;*
(**2**) *L is a mapping of (S_+)-type, namely: $\vartheta_n \rightharpoonup \vartheta$ and $\limsup_{n\to+\infty} L(\vartheta_n)(\vartheta_n - \vartheta) \leq 0$, implies $\vartheta_n \to \vartheta$;*
(**3**) *L is a homeomorphism.*

3. Proof of the main result

Definition 3.1. We say that $\vartheta \in X$ is a weak solution of problem (1.1), if

$$(\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta|^{p(t)}\, dt) \int_\Gamma |\Delta\vartheta|^{p(t)-2} \Delta\vartheta \Delta\varphi\, dt - \lambda \int_\Gamma |\vartheta|^{p(t)-2} \vartheta\varphi\, dt = \int_\Gamma g(t,\vartheta)\varphi\, dt,$$

for any $\varphi \in X$.

The problem (1.1) has a variational form with the energy functional $J : X \to \mathbb{R}$, defined as follows:

$$J(\vartheta) = \alpha \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta|^{p(t)}\, dt - \frac{\beta}{2} \left(\int_\Gamma \frac{1}{p(t)} |\Delta\vartheta|^{p(t)}\, dt \right)^2 - \lambda \int_\Gamma \frac{1}{p(t)} |\vartheta|^{p(t)}\, dt - \int_\Gamma G(t,\vartheta)\, dt, \tag{3.1}$$

for all $\vartheta \in X$. Moreover, the functional J is well defind and of class C^1 in X. Furthermore, we have

$$\langle J'(\vartheta), \varphi \rangle = (\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta|^{p(t)}\, dt) \int_\Gamma |\Delta\vartheta|^{p(t)-2} \Delta\vartheta\Delta\varphi\, dt - \lambda \int_\Gamma |\vartheta|^{p(t)-2}\vartheta\varphi\, dt - \int_\Gamma g(t,\vartheta)\varphi\, dt, \tag{3.2}$$

for every $\varphi \in X$. Hence, we can observe that the critical points of J are weak solutions of problem (1.1).

3.1. Compactness condition.

Definition 3.2. Let $(X, \|\cdot\|)$ be a Banach space and $J \in C^1(X)$. We say that J satisfies the Palais-Smale condition at level c ($(PS)_c$ in short), if any sequence $\{u_n\} \subset X$ satisfying

$$J(\vartheta_n) \to c \quad \text{and} \quad J'(\vartheta_n) \to 0 \quad \text{in} \quad X^* \quad \text{as} \quad n \to \infty, \tag{3.3}$$

has a convergent subsequence.

Lemma 3.3. *Assume that $(\mathbf{g_1})$- $(\mathbf{g_3})$ hold. Then the functional J satisfies the $(PS)_c$ condition, where $c < \dfrac{\alpha^2}{2\beta}$.*

Proof. We proceed in two steps.

Step1. We prove that $\{\vartheta_n\}$ is bounded in X. Let $\{\vartheta_n\} \subset X$ be a $(PS)_c$ sequence such that $c < \dfrac{\alpha^2}{2\beta}$.

• For $\lambda > 0$. Arguing by contradiction, we assume that, passing eventually to a subsequence, still denote by $\{\vartheta_n\}$, we have $\|\vartheta_n\| \to +\infty$ as $n \to +\infty$. Using (3.3) and $(\mathbf{g_3})$, for n large enough, we can write

$$\begin{aligned}
C + \|\vartheta_n\| &\geq \theta J(\vartheta_n) - \langle J'(\vartheta_n), \vartheta_n\rangle \\
&\geq \theta\Big(\alpha \int_\Gamma \frac{1}{p(t)}|\Delta\vartheta_n|^{p(t)}\, dt - \frac{\beta}{2}\Big(\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta_n|^{p(t)}\, dt\Big)^2 \\
&- \lambda \int_\Gamma \frac{1}{p(t)}|\vartheta_n|^{p(t)}\, dt - \int_\Gamma G(t,\vartheta_n)\, dt\Big) \\
&- \Big(\Big[\alpha - \beta\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta_n|^{p(t)}\, dx\Big]\int_\Gamma |\Delta\vartheta_n|^{p(t)}\, dt - \lambda\int_\Gamma |\vartheta_n|^{p(t)}\, dt \\
&- \int_\Gamma g(t,\vartheta_n)\vartheta_n\, dt\Big) \\
&\geq \alpha(\frac{\theta}{p^+} - 1)\int_\Gamma |\Delta\vartheta_n|^{p(t)}\, dt + \\
&\beta(\frac{-\theta}{2(p^-)^2} + \frac{1}{p^+})\Big(\int_\Gamma |\Delta\vartheta_n|^{p(t)}\, dt\Big)^2 - \lambda(\frac{\theta}{p^-} - 1)\int_\Gamma |\vartheta_n|^{p(t)}\, dt - C|\Gamma|,
\end{aligned}$$

where $|\Gamma| = \int_\Gamma dt$. Therefore, we deduce that

$$\begin{aligned}
C + \|\vartheta_n\| + \lambda(\frac{\theta}{p^-} - 1)\|\vartheta_n\|^{p^+} \quad &\geq \alpha(\tfrac{\theta}{p^+} - 1)\|\vartheta_n\|^{p^-} + \beta(\tfrac{-\theta}{2(p^-)^2} + \tfrac{1}{p^+})\|\vartheta_n\|^{2p^-} \\
&-C|\Gamma|.
\end{aligned}$$

Dividing the above inequality by $\|\vartheta_n\|^{p^+}$, taking into account (1.3) holds and passing to the limit as $n \to +\infty$, we obtain a contradiction. It follows that $\{\vartheta_n\}$ is bounded in X.

• For $\lambda \leq 0$. From (3.3) and $(\mathbf{g_3})$, for n large enough, we have

$$C + \|\vartheta_n\| \geq \alpha(\frac{\theta}{p^+} - 1)\|\vartheta_n\|^{p^-} + \beta(\frac{-\theta}{2(p^-)^2} + \frac{1}{p^+})\|\vartheta_n\|^{2p^-} - C|\Gamma|.$$

It follows from (1.3) that $\{\vartheta_n\}$ is bounded in X.

Step2. Now, we will prove that $\{\vartheta_n\}$ has a convergent subsequence in X. Up to a subsequence, for some $\vartheta \in X$ we have

$$\begin{cases} \vartheta_n \rightharpoonup \vartheta, \text{ in } X; \\ \vartheta_n \to \vartheta, \text{ in } L^{p(t)}(\Gamma); \\ \vartheta_n \to \vartheta, \text{ in } L^{q(t)}(\Gamma); \\ \vartheta_n(t) \to \vartheta(t), \text{ a.e. in } \Gamma. \end{cases}$$

By Hölder inequality and Proposition 2.2, we obtain

$$\begin{aligned} \Big| \int_\Gamma |\vartheta_n|^{p(t)-2}\vartheta_n(\vartheta_n - \vartheta)\, dt \Big| &\le \int_\Gamma |\vartheta_n|^{p(t)-1}|\vartheta_n - \vartheta|\, dt \\ &\le \| |\vartheta_n|^{p(t)-1}\|_{\frac{p(t)}{p(t)-1}} \|\vartheta_n - \vartheta\|_{p(t)} \\ &\to 0, \quad \text{as} \quad n \to +\infty, \end{aligned}$$

and then,

$$\lim_{n\to+\infty} \int_\Gamma |\vartheta_n|^{p(t)-2}\vartheta_n(\vartheta_n - \vartheta)\, dt = 0. \tag{3.4}$$

Now, let $\epsilon > 0$ be small enough. By assumptions ($\mathbf{g_1}$) and ($\mathbf{g_2}$), we have

$$|g(t, \vartheta_n)| \le \epsilon|\vartheta_n|^{p(t)-1} + c(\epsilon)|\vartheta_n|^{q(t)-1}. \tag{3.5}$$

Using (3.5), Hölder inequality and Proposition 2.2, we deduce that

$$\begin{aligned} \Big| \int_\Gamma g(t, \vartheta_n)(\vartheta_n - \vartheta)\, dt \Big| &\le \int_\Gamma \epsilon|\vartheta_n|^{p(t)-1}|\vartheta_n - u| \\ &\quad + c(\epsilon)|\vartheta_n|^{q(t)-1}|\vartheta_n - \vartheta|\, dt \\ &\le \epsilon\| |\vartheta_n|^{p(t)-1}\|_{\frac{p(t)}{p(t)-1}} \|\vartheta_n - \vartheta\|_{p(t)} \\ &\quad + c(\epsilon)\| |\vartheta_n|^{q(t)-1}\|_{\frac{q(t)}{q(t)-1}} \|\vartheta_n - \vartheta\|_{q(t)} \\ &\to 0, \quad \text{as} \quad n \to +\infty, \end{aligned}$$

and then,

$$\lim_{n\to+\infty} \int_\Gamma g(t, \vartheta_n)(\vartheta_n - \vartheta)\, dt = 0. \tag{3.6}$$

From (3.3), we conclude that

$$\langle J'(\vartheta_n), \vartheta_n - \vartheta \rangle \to 0.$$

Therefore

$$\begin{aligned}\langle J'(\vartheta_n), \vartheta_n - \vartheta\rangle = &\Big(\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta_n|^{p(t)}\, dt\Big) \\ &\times \int_\Gamma |\Delta\vartheta_n|^{p(t)-2}\Delta\vartheta_n(\Delta\vartheta_n - \Delta\vartheta)\, dt \\ &- \lambda \int_\Gamma |\vartheta_n|^{p(t)-2}\vartheta_n(\vartheta_n - \vartheta)\, dt - \int_\Omega g(t,\vartheta_n)(\vartheta_n - \vartheta)\, dt \\ \to 0.\end{aligned}$$

So, considering (3.4) and (3.6), we obtain

$$\Big(\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta_n|^{p(t)}\, dt\Big) \int_\Gamma |\Delta\vartheta_n|^{p(t)-2}\Delta\vartheta_n(\Delta\vartheta_n - \Delta\vartheta)\, dt \to 0. \quad (3.7)$$

Similar to the proof of Lemma 3.1 in [7], we can deduce that the sequence

$$\Big\{\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta_n|^{p(t)}\, dx\Big\} \quad \text{is bounded}$$

and we have

$$\alpha - \beta \int_\Gamma \frac{1}{p(t)} |\Delta\vartheta_n|^{p(t)}\, dx \nrightarrow 0, \quad \text{as} \quad n \to +\infty.$$

This fact combined with (3.7) implies that

$$\int_\Gamma |\Delta\vartheta_n|^{p(t)-2}\Delta\vartheta_n(\Delta\vartheta_n - \Delta\vartheta)\, dt \to 0.$$

Since L is of (S_+) by Proposition 2.5, we obtain $\vartheta_n \to \vartheta$ in X. The proof is complete. □

3.2. **Proof of Theorem 1.1.**

Lemma 3.4. *Assume that g satisfies $(\mathbf{g_1})$- $(\mathbf{g_3})$. Then J satisfies the Mountain Pass geometry, that is,*

(i) *there exists $\rho, \delta > 0$ such that $J(\vartheta) \geq \delta > 0$, for any $\vartheta \in X$ with $\|\vartheta\| = \rho$.*
(ii) *there exists $e \in X$ with $\|e\| > \rho$ such that $J\,(e) < 0$.*

Proof. First we prove the statement **(i)**.
• Assume $\lambda \leq 0$. Using $(\mathbf{g_1})$ and $(\mathbf{g_3})$, we can write

$$|G(t,\vartheta)| \leq \frac{\epsilon}{p(t)} |\vartheta|^{p(t)} + \frac{c(\epsilon)}{q(t)}. \quad (3.8)$$

Let $\epsilon = \frac{1}{2}\alpha\,\lambda_1$, $\rho \in (0,1)$ and $u \in X$ be such that $\|\vartheta\| = \rho$. By Propositions 2.2 and 2.4, we have

$$\begin{aligned}
J(\vartheta) &= \alpha\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt - \frac{\beta}{2}\left(\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt\right)^2 \\
&\quad - \lambda\int_\Gamma \frac{1}{p(t)}|\vartheta|^{p(t)}\,dt - \int_\Gamma G(t,\vartheta)\,dt \\
&\geq \alpha\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt - \frac{\beta}{2}\left(\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt\right)^2 - \epsilon\int_\Gamma \frac{|\vartheta|^{p(t)}}{p(t)}\,dt \\
&\quad - c(\epsilon)\int_\Gamma \frac{|\vartheta|^{q(t)}}{q(t)}\,dt \\
&\geq (\alpha - \frac{\epsilon}{\lambda_1})\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt - \frac{\beta}{2}\left(\int_\Gamma \frac{1}{p(t)}|\Delta\vartheta|^{p(t)}\,dt\right)^2 \\
&\quad - \frac{Cc(\epsilon)}{q^-}\int_\Gamma |\Delta\vartheta|^{q(t)}\,dt \\
&\geq \frac{1}{p^+}(\alpha - \frac{\epsilon}{\lambda_1})\|\vartheta\|^{p^+} - \frac{\beta}{2(p^-)^2}\|\vartheta\|^{2p^-} - \frac{Cc(\epsilon)}{q^-}\|\vartheta\|^{q^-} \\
&\geq \left(\frac{\alpha}{2p^+} - \frac{\beta}{2(p^-)^2}\|\vartheta\|^{2p^- - p^+} - \frac{Cc(\epsilon)}{q^-}\|\vartheta\|^{q^- - p^+}\right)\|\vartheta\|^{p^+}.
\end{aligned}$$

Considering (1.3), we can choose $\rho > 0$ and then there exists $\delta > 0$ such that $J(\vartheta) \geq \delta > 0$ for every $\vartheta \in X$ with $\|\vartheta\| = \rho$.

- Assume $\lambda > 0$. Let $\epsilon > 0$ be small enough such that $\frac{1}{2p^+}(\alpha - \frac{\lambda}{\lambda_1}) = \frac{\epsilon}{\lambda_1 p^-}$. Consider $\rho \in (0,1)$ and $\vartheta \in X$ such that $\|\vartheta\| = \rho$. By Propositions 2.2 and 2.4, we deduce that

$$\begin{aligned} J(\vartheta) &= \alpha \int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dt - \frac{\beta}{2} \left(\int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dt \right)^2 \\ &\quad - \lambda \int_\Gamma \frac{1}{p(t)} |\vartheta|^{p(t)}\, dt - \int_\Gamma G(t,\vartheta)\, dt \\ &\geq \alpha \int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dt - \frac{\beta}{2} \left(\int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dt \right)^2 \\ &\quad - \frac{\lambda}{\lambda_1} \left(\int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dx \right) - \epsilon \int_\Gamma \frac{|u\vartheta|^{p(t)}}{p(t)}\, dt - c(\epsilon) \int_\Gamma \frac{|\vartheta|^{q(t)}}{q(t)}\, dt \\ &\geq (\alpha - \frac{\lambda}{\lambda_1}) \int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dt - \frac{\beta}{2} \left(\int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dt \right)^2 \\ &\quad - \frac{\epsilon}{\lambda_1} \int_\Gamma \frac{1}{p(t)} |\Delta \vartheta|^{p(t)}\, dx - \frac{Cc(\epsilon)}{q^-} \int_\Gamma |\Delta \vartheta|^{q(t)}\, dt \\ &\geq \left(\frac{1}{p^+} (\alpha - \frac{\lambda}{\lambda_1}) - \frac{\epsilon}{\lambda_1 p^-} \right) \|\vartheta\|^{p^+} - \frac{\beta}{2(p^-)^2} \|\vartheta\|^{2p^-} - \frac{Cc(\epsilon)}{q^-} \|\vartheta\|^{q^-} \\ &\geq \left(\frac{1}{2p^+} (\alpha - \frac{\lambda}{\lambda_1}) - \frac{\beta}{2(p^-)^2} \|\vartheta\|^{2p^- - p^+} - \frac{Cc(\epsilon)}{q^-} \|\vartheta\|^{q^- - p^+} \right) \|\vartheta\|^{p^+}. \end{aligned}$$

Considering (1.3), there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, there exists $\delta > 0$ such that for any $\vartheta \in X$ with $\|\vartheta\| = \rho$ we have $J(\vartheta) \geq \delta > 0$.

Now, we prove the statement **(ii)**.

By $(\mathbf{g_3})$, we know that for all $M > 0$, there exists $C_M > 0$ so that

$$G(x, \vartheta) \geq M|\vartheta|^\theta - C_M, \quad \text{for all} \quad (x, \vartheta) \in \Gamma \times \mathbb{R}. \tag{3.9}$$

Let $\varphi \in C_0^\infty(\Gamma)$, $\varphi > 0$ and $\eta > 1$. Using (3.9), we obtain

$$
\begin{aligned}
J(\eta\varphi) = \alpha \int_\Gamma \frac{1}{p(t)} |\eta\Delta\varphi|^{p(t)}\, dt - \frac{\beta}{2}\left(\int_\Gamma \frac{1}{p(t)} |\eta\Delta\varphi|^{p(t)}\, dt\right)^2 \\
- \lambda \int_\Gamma \frac{1}{p(t)} |\eta\varphi|^{p(t)}\, dx - \int_\Gamma G(t, \eta\varphi)\, dt \\
\leq \alpha \int_\Gamma \frac{1}{p(t)} |\eta\Delta\varphi|^{p(t)}\, dx - \frac{\beta}{2}\left(\int_\Gamma \frac{1}{p(t)} |\eta\Delta\varphi|^{p(t)}\, dt\right)^2 \\
- \lambda \int_\Gamma \frac{1}{p(t)} |\eta\varphi|^{p(t)}\, dt - M \int_\Gamma |\eta\varphi|^{\theta}\, dt + C_M|\Gamma| \\
\leq \frac{\alpha\eta^{p^+}}{p^-} \int_\Gamma |\Delta\varphi|^{p(t)}\, dt - \frac{\beta\eta^{2p^-}}{2(p^+)^2}\left(\int_\Gamma |\Delta\varphi|^{p(t)}\, dt\right)^2 \\
- \frac{\lambda}{p^+}\eta^{p^-} \int_\Gamma |\varphi|^{p(t)}\, dx - M\eta^{\theta} \int_\Gamma |\varphi|^{\theta}\, dt + C_M|\Gamma|.
\end{aligned}
$$

Since $\theta > 2p^- > p^+ > p^-$, we have $J(\eta\varphi) \to -\infty$ as $t \to +\infty$. So, choosing $e = \eta\varphi$ with $\eta > 1$ large enough, we obtain $\|e\| > \rho$ and $J(\eta\varphi) < 0$. □

By Lemmas 3.3, 3.4 and the fact that $J(0) = 0$, J satisfies the Mountain Pass Theorem. Therefore, problem (1.1) has indeed a nontrivial weak solution.

References

[1] G. A. Afrouzi, M. Mirzapour, N.T. Chung, Existence and multiplicity of solutions for Kirchhoff type problems involving $p(\cdot)$-Biharmonic operators, *Z. Anal. Anwend.* **33***(2014), 289-303.*

[2] Z .E. Allali, M. K. Hamdani, S. Taarabti, Three solutions to a Neumann boundary value problem driven by p(x)-biharmonic operator, *J. Elliptic Parabol. Equ. (2024), https://doi.org/10.1007/s41808-023-00257-1.*

[3] A. Ayoujil, A. R. El Amrouss, On the spectrum of a fourth order elliptic equation with variable exponents, *Nonlinear Anal.* **71***(2009), 4916-4926.*

[4] A. El Amrouss, F. Moradi and M. Moussaoui, Existence of solutions for fourth-order PDEs with variable exponents, *Electron. J. Differ. Equ.,* **153***(2009), 1-13.*

[5] X. L. Fan, X. Fan, A Knobloch-type result for $p(t)$ Laplacian systems, *J. Math. Anal. Appl.* **282***(2003), 453-464.*

[6] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263***(2001), 424-446.*

[7] M. KL. Hamdani, A. Harrabi, F. Mtiri, D. D. Repovs, Existence and multiplicity results for a new $p(x)$-Kirchhoff problem, *Nonlinear Anal.* **90***(2020), 111598.*

[8] A. Khaleghi, A. Razani, Solutions to a $(p(x); q(x))$-biharmonic elliptic problem on a bounded domain, *Bound. Value Probl.* **53** *(2023), https://doi.org/10.1186/s13661-023-01741-2.*

[9] A. Khaleghi, A. Razani, F. Safari, Three Weak Solutions for a Class of $p(x)$-Kirchhoff Type Biharmonic Problems, *Lobachevskii J. Math.* **44** *(2023), 5298–5305, https://doi.org/10.1134/S199508022312020X.*

[10] X. Qian, W. Chao, Existence of positive solutions for a nonlocal problems with indefinite nonlinearity, *Bound. Value Probl.* **40***(2020), https://doi.org/10.1186/s13661-020-01343-2.*

[11] G. Yin, J. Liu, Existence and multiplicity of nontrivial solutions for a nonlocal problem, *Bound. Value Probl.* **26***(2015), https://doi.org/10.1186/s13661-015-0284-x.*

[12] A. Zang, Y. Fu, Interpolation inequalities for derivatives in variable exponent Lebesgue- Sobolev spaces, *Nonlinear Anal. T.M.A.* **69** *(2008), 3629-3636.*

[13] Z. Zhang, Y. Song, High perturbation of a new Kirchhoff problem involving the p-Laplace operator, *Bound. Value Probl.* **98***(2021), https://doi.org/10.1186/s13661-021-01566-x.*