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Concircular Vector Fields on Lightlike Submanifolds

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Abstract. Concircular vector fields on lightlike submanifolds are investigated. With the aid of this review, some relations on Ricci soliton lightlike submanifolds containing concircular vector fields are obtained.

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1. Introduction

One of the most effective ways to characterize a Riemannian manifold is to examine the geometrical characteristics belonging to several type of particular vector fields. With this examination, various basic properties and relations provided by Riemannian manifolds can be obtained. These

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particular vector fields are geodesic, Killing, concurrent, concircular, recurrent and torse-forming vector fields, etc.

Concircular vector fields were firstly established by A. Fialkow in [14]. A smooth vector field v on a Riemannian manifold is defined a concircular vector field if there subsists a smooth function φ satisfying

$$\widetilde{\nabla}_X v = \varphi X \tag{1.1}$$

for each tangent vector field X. Here, $\widetilde{\nabla}$ indicates the Levi-Civita connection. If $\varphi = 1$, then v is said to be concurrent.

There exist remarkable applications of concircular vector fields to Ricci solitons in the literature. Some results dealing with concircular vector fields and their practices to Ricci solitons are obtained on Riemannian manifolds in [4, 6, 7, 8, 17, 21], on space-time in [18, 19], on various contact space forms in [1, 15, 16, 20], on Euclidean space [5], etc.

The primary aim of this study is to maintain this investigation on lightlike submanifolds. For this purpose, the followings are obtained:

(1) Some identities and examples of Ricci soliton lightlike submanifolds admitting a concircular vector field are presented.
(2) With the aim of the rigged metric defined on lightlike submanifolds, it is proved that every Ricci soliton lightlike submanifold whose potential vector field is concircular belongs to the radical space.

2. Preliminaries

Let $(\widetilde{M}, \widetilde{h})$ be an $(\widetilde{m}+\widetilde{n})-$dimensional semi-Riemannian manifold and (M, h) be an $\widetilde{n}-$ dimensional submanifold having the induced metric h from $\widetilde{h}$. If h is degenerate on M, then (M, h) is said to be a lightlike submanifold of $(\widetilde{M}, \widetilde{h})$. In this circumstance, the radical distribution at $p \in M$ is given by

$$\mathrm{Rad}T_pM = \{\xi_p \in T_pM : h_p(\xi, X) = 0 \text{ for any } X \in \Gamma(TM)\}.$$

Assume that $\mathrm{S}(TM)$ is a complementary vector bundle of $\mathrm{Rad}(TM)$. Then we write

$$TM = \mathrm{Rad}(TM) \oplus_{orth} \mathrm{S}(TM). \tag{2.1}$$

Here, $\oplus_{orth}$ indicates the orthogonal direct sum. The distribution $\mathrm{S}(TM)$ is defined a screen distribution (M, h). We pay attention that $\mathrm{S}(TM)$ is a non-degenerate distribution and it is not single. Hence, a lightlike submanifold is mostly indicated by the triple $(M, h, \mathrm{S}(TM))$.

Now, let rank(Rad(TM)) $= r$, $r > 0$, rank(S(TM)) $= n$ such that $\widetilde{n} = r + n$, $\widetilde{m} = \widetilde{n} + m + r$. In the circumstances, one of the undernamed situations occurs:

i) $(M, h, \mathrm{S}(TM))$ is a $r-$lightlike submanifold if $1 \leq r < min\{n + r, m + r\}$.
ii) $(M, h, \mathrm{S}(TM))$ is a coisotropic lightlike submanifold if $m = 0$.
iii) $(M, h, \mathrm{S}(TM))$ is a isotropic lightlike submanifold if $n = 0$.
iv) $(M, h, \mathrm{S}(TM))$ is a totally lightlike submanifold if $m = n = 0$ [12, 13].

Let $(M, h, \mathrm{S}(TM))$ be a $r-$lightlike submanifold. Then, we can consider the following quasi-orthonormal basis on $\Gamma(T\widetilde{M})$:

$$\{\xi_1, \ldots, \xi_r, e_1, \ldots, e_n, N_1, \ldots, N_r, u_1, \ldots, u_m\}$$

such that

$$\widetilde{h}(N_i, \xi_l) = \delta_{il}, \quad \widetilde{h}(N_i, N_l) = \widetilde{h}(N_i, u_\alpha) = \widetilde{h}(\xi_i, u_\alpha) = 0 \tag{2.2}$$

for any $i, l \in \{1, \ldots, r\}$, $\alpha \in \{1, \ldots, m\}$ and

$$\widetilde{h}(N_i, e_j) = \widetilde{h}(u_\alpha, e_j) = 0 \tag{2.3}$$

for any $j \in \{1, \ldots, n\}$. Here, δ_{il} denotes the Kronecker delta function. For an $r-$lightlike submanifold, we get the following subspaces:

$$\mathrm{Rad}(TM) = Span\{\xi_1, \ldots, \xi_r\}, \quad \mathrm{ltr}(TM) = Span\{N_1, \ldots, N_r\},$$
$$\mathrm{S}(TM) = Span\{e_1, \ldots, e_n\}, \quad \mathrm{S}(TM^\perp) = Span\{u_1, \ldots, u_m\}.$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection on $(\widetilde{M}, \widetilde{h})$. The Gauss and Weingarten formulae are expresses as

$$\widetilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + II(X_1, X_2),$$
$$= \nabla_{X_1} X_2 + \sum_{l=1}^{r} B^l(X_1, X_2) N_l + \sum_{\alpha=1}^{m} D^\alpha(X_1, X_2) u_\alpha, \tag{2.4}$$

$$\widetilde{\nabla}_{X_1} N_k = -A_{N_k} X_1 + \sum_{l=1}^{r} \gamma_{kl}(X_1) N_l + \sum_{\alpha=1}^{m} \gamma'_{k\alpha}(X_1) u_\alpha, \tag{2.5}$$

$$\widetilde{\nabla}_{X_1} u_\beta = -A_{u_\beta} X_1 + \sum_{l=1}^{r} \theta_{\beta l}(X_1) N_l + \sum_{\alpha=1}^{m} \theta'_{\beta\alpha}(X_1) u_\alpha \tag{2.6}$$

for any $X_1 \in \Gamma(TM)$, $k \in \{1, \ldots, r\}$ and $\beta \in \{1, \ldots, m\}$. Here, ∇ is the induced connection on $\Gamma(TM)$, A_{N_k} and A_{u_β} are the shape operators of M, γ, γ', θ and θ' are $1-$forms, II is a second fundamental form.

Let P be the projection of $\Gamma(TM)$ onto $\Gamma(\mathrm{S}(TM))$. The Gauss and Weingarten formulae on $\mathrm{S}(TM)$ are demonstrated with

$$\begin{aligned}\nabla_{X_1} PX_2 &= \nabla^*_{X_1} PX_2 + II^*(X_1, X_2)\\ &= \nabla^*_{X_1} PX_2 + \sum_{l=1}^{r} C^l(X_1, PX_2)\xi_l, \end{aligned} \tag{2.7}$$

$$\nabla_{X_1}\xi_k = -A^*_{\xi_k} X_1 - \sum_{l=1}^{r} \gamma_{kl}(X_1)\xi_l, \tag{2.8}$$

where ∇^* is the induced connection on $\Gamma(\mathrm{S}(TM))$, A_{ξ_k} is the local shape operator of M for each $k \in \{1, \dots, r\}$ and II^* is the second fundamental form. An $r-$lightlike submanifold is defined

i) totally geodesic if $II = 0$,
ii) $\mathrm{S}(TM)-$geodesic if $II^* = 0$,
iii) totally umbilical if there subsists a smooth transversal vector field $H \in \mathrm{ltr}(TM)$ such that $II(X, Y) = \widetilde{h}(X, Y)H$ is held,
iv) $\mathrm{S}(TM)-$totally umbilical if there subsists a smooth transversal vector field $H^* \in \mathrm{ltr}(TM)$ such that $II^*(X, Y) = \widetilde{g}(X, Y)H^*$ is held [10].

Now, let us symbolize the Riemannian curvature tensors of M and $\widetilde{M}$ by R and $\widetilde{R}$, consecutively. The Gauss equation for $(M, h, \mathrm{S}(TM))$ is demonstrated by the undermentioned equality:

$$\begin{aligned}\widetilde{h}(\widetilde{R}(X_1, X_2)PX_3, PX_4) = h(R(X_1, X_2)PX_3, PX_4)&\\ + \sum_{l=1}^{r} B^l(X_1, PX_3)C^l(X_2, PX_4)&\\ - \sum_{l=1}^{r} B^l(X_2, PX_3)C^l(X_1, PX_4)&\\ \sum_{\alpha=1}^{m} [D^\alpha(X_1, PX_3)D^\alpha(X_2, PX_4)&\\ -D^\alpha(X_2, PX_3)D^\alpha(X_1, PX_4)]\,.&\end{aligned} \tag{2.9}$$

Considering the fact that II^* is not symmetric in (2.9), it follows that the sectional curvature map is not needed to be symmetric on a lightlike submanifold. From (2.4)-(2.8), we find the following equalities:

$$B^l(X_1, Y) = h(A^*_{\xi_l} X_1, X_2), \tag{2.10}$$

$$C^l(X_1, Y) = h(A_{N_l} X_1, X_2), \tag{2.11}$$

$$\varepsilon_\alpha D^\alpha(X_1, X_2) = h(A_{u_\alpha} X_1, X_2) - \theta_{\alpha l}(X_1)\eta_\alpha(X_2), \tag{2.12}$$

where $\varepsilon_\alpha = \widetilde{h}(u_\alpha, u_\alpha) = \mp 1$, $\eta_\alpha(X_2) = \widetilde{h}(X_2, N_\alpha)$, $i, l \in \{1, \dots, r\}$ and $\alpha \in \{1, \dots, m\}$ [11]. Handling the truth that $\widetilde{\nabla}$ is a metric connection, we find

$$\begin{aligned} X_3 h(X_1, X_2) &= \widetilde{h}(\widetilde{\nabla}_{X_3} X_1, X_2 + \widetilde{h}(\widetilde{\nabla}_{X_3} X_2, X_1) \\ &= h(\nabla_{X_3} X_1, X_2) + h(\nabla_{X_3} X_2, X_1) + \sum_{l=1}^{r} B^l(X_1, X_3)\eta_l(X_2) \\ &\quad + \sum_{l=1}^{r} B^l(X_2, X_3)\eta_l(X_1). \end{aligned} \tag{2.13}$$

Also, the Lie derivative of g is demonstrated by

$$\begin{aligned} (L_{X_3} h)(X_1, X_2) = X_3 h(X_1, X_2) - h(\nabla_{X_3} X_1, X_2) - h(\nabla_{X_3} X_2, X_1) \\ + h(\nabla_{X_1} X_3, X_2) + h(\nabla_{X_2} X_3, X_1). \end{aligned} \tag{2.14}$$

Putting (2.13) in (2.14), we find

$$\begin{aligned} (L_{X_3} h)(X_1, X_2) = \sum_{l=1}^{r} B^l(X_2, X_3)\eta_l(X_1) + h(\nabla_{X_1} X_3, X_2) \\ + h(\nabla_{X_2} X_3, X_1). \end{aligned} \tag{2.15}$$

From (2.14), it is easy to see that the induced connection ∇ is not a metric connection.

Now, let $\{e_1, \dots, e_n\}$ be an orthonormal frame field of $S(TM)$. The induced Ricci type tensor, denoted by $R^{(0,2)}$, is expressed as

$$R^{(0,2)}(X_1, X_2) = \sum_{j=1}^{n} h(R(X_1, e_j)e_j, X_1) + \sum_{j=1}^{n} \widetilde{h}(R(\xi_l, X_1)X_2, N_\alpha). \tag{2.16}$$

Putting (2.9) in (2.16), it can be obtained that
$R^{(0,2)}(X_1, X_2) \neq R^{(0,2)}(X_2, X_1)$ [9].

Theorem 2.1. [9] *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold. Then, the undermentioned situations are equivalent:*

i) *$S(TM)$ is integrable.*
ii) *II^* is symmetric on $S(TM)$.*
iii) *A_N is self-adjoint.*

In view of Theorem 2.1, if $S(TM)$ is integrable, then $R^{(0,2)}$ is symmetric. We pay attention that $R^{(0,2)}$ is said to be Ricci tensor when it is symmetric.

3. Ricci soliton lightlike submanifolds

Let ν be a vector field on $\Gamma(T\widetilde{M})$. Thus, we may put

$$\nu = \nu^{\top} + \sum_{l=1}^{r} f_l N_l + \sum_{\alpha=1}^{m} \rho_\alpha u_\alpha, \tag{3.1}$$

where f_l and ρ_α are smooth functions for $l \in \{1, \ldots, r\}$ and $\alpha \in \{1, \ldots, m\}$. In view of (3.1), we find

$$f_l = \widetilde{h}(\nu, \xi_l) \ \text{ and } \ \rho_\alpha = \widetilde{h}(\nu, u_\alpha). \tag{3.2}$$

Proposition 3.1. *Let $(M, g, S(TM))$ be an $r-$lightlike submanifold. If the vector field ν is a concircular, then the undermentioned relations are held:*

$$\nabla_X \nu^{\top} = \varphi X + \sum_{l=1}^{r} f_l A_{N_l} X + \sum_{\alpha=1}^{m} \rho_\alpha A_{u_\alpha} X, \tag{3.3}$$

$$B^l(X, \nu^{\top}) = -\widetilde{\nabla}_X f_l - \sum_{s=1}^{r} f_l \gamma_{ls}(X) - \sum_{\alpha=1}^{m} \rho_\alpha \theta'_{\alpha l}(X), \tag{3.4}$$

$$D^\alpha(X, \nu^{\top}) = -\widetilde{\nabla}_X \rho_\alpha - \sum_{\beta=1}^{m} \rho_\alpha \theta_{\alpha\beta}(X) - \sum_{l=1}^{r} f_l \gamma'_{l\alpha}(X). \tag{3.5}$$

Proof. Under the supposition, if the vector field ν is a concircular, we put

$$\widetilde{\nabla}_X \nu = \varphi X = \widetilde{\nabla}_X \nu^{\top} + \widetilde{\nabla}_X (\sum_{l=1}^{r} f_l N_l) + \widetilde{\nabla}_X (\sum_{\alpha=1}^{m} \rho_\alpha u_\alpha),$$

which implies that

$$\begin{aligned}
\varphi X = {} & \nabla_X \nu^{\top} + \sum_{l=1}^{r} B^l(X, \nu^{\top}) N_l + \sum_{\alpha=1}^{m} D^\alpha(X, \nu^{\top}) u_\alpha \\
& + \sum_{l=1}^{r} (\widetilde{\nabla}_X f_l) N_l + \sum_{\alpha=1}^{m} (\widetilde{\nabla}_X \rho_\alpha) u_\alpha - \sum_{l=1}^{r} f_l A_{N_l} X \\
& + \sum_{l=1}^{r} \sum_{s=1}^{r} f_l \gamma_{ls}(X) N_s + \sum_{l=1}^{r} f_l \sum_{\alpha=1}^{m} \gamma'_{l\alpha}(X) u_\alpha \\
& - \sum_{\alpha=1}^{m} \rho_\alpha A_{u_\alpha} X + \sum_{\alpha=1}^{m} \rho_\alpha \sum_{l=1}^{r} \theta_{\alpha l}(X) N_l + \sum_{\alpha=1}^{m} \rho_\alpha \sum_{\beta=1}^{m} \theta'_{\alpha\beta}(X) u_\beta.
\end{aligned}$$

Taking into account the tangential and transversal components in last equality, we achieve (3.3), (3.4) and (3.5) immediately. $\square$

For a special case $m = 0$, we write

$$\nu = \nu^{\top} + \sum_{l=1}^{r} f_l N_l. \tag{3.6}$$

Then we find

Proposition 3.2. *Let $(M, h, S(TM))$ be a coisotropic lightlike submanifold. If the vector field ν is a concircular, then the undermentioned relations are held:*

$$\nabla_X \nu^{\top} = \varphi X + \sum_{l=1}^{r} f_l A_{N_l} X, \tag{3.7}$$

$$B^l(X, \nu^{\top}) = -\widetilde{\nabla}_X f_l - \sum_{s=1}^{r} f_l \gamma_{ls}(X). \tag{3.8}$$

For a special case $\nu = \nu^{\top}$, we have the followings:

Proposition 3.3. *For any $r-$lightlike submanifold $(M, h, S(TM))$ containing a concircular vector field $\nu = \nu^{\top}$, we have*

$$\nabla_X \nu^{\top} = \varphi X \tag{3.9}$$

and

$$B^l(X, \nu^{\top}) = D^{\alpha}(X, \nu^{\top}) = 0, \tag{3.10}$$

where $l \in \{1, \ldots, r\}$ and $\alpha \in \{1, \ldots, m\}$.

Corollary 3.4. *If $\nu = \nu^T$ is a concircular vector field on $(M, h, S(TM))$, v is also concircular in regard to the induced connection ∇.*

Proposition 3.5. *Let $(M, h, S(TM))$ be a coisotropic lightlike submanifold containing a concircular vector field $\nu = \nu^{\top}$. Then, we get*

$$\nabla_X \nu^{\top} = \varphi X \quad \text{and} \quad B^l(X, \nu^{\top}) = 0 \tag{3.11}$$

for each $l \in \{1, \ldots, r\}$.

From (2.13), (3.10) and (3.11), we find

Theorem 3.6. *Let $(M, h, S(TM))$ be an $r-$lightlike or coisotropic lightlike submanifold containing a concircular vector field $\nu = \nu^{\top}$. Then g is parallel in regard to $\nu^{\top}$.*

Example 3.7. Let $\mathbb{R}^8_4$ be the Euclidean 8-space with the signature $(-, -, +, +, -, -, +, +)$. Take into account a submanifold M of $\mathbb{R}^8_4$ defined by

$$x_3 = \sqrt{x_1^2 + x_2^2}, \quad x_4 = \sqrt{x_5^2 + x_6^2}, \quad x_7 = x_8 = 0.$$

Then we find $(M, h, S(TM))$ is a 2-lightlike submanifold of $\mathbb{R}^8_4$ that satisfies

$$\begin{aligned}
\mathrm{Rad}(TM) &= \mathrm{Span}\,\{\xi_1 = x_1\partial x_1 + x_2\partial x_2 + x_3\partial x_3, \\
&\quad \xi_2 = x_4\partial x_4 + x_5\partial x_5 + x_6\partial x_6\}\,, \\
\mathrm{S}(TM) &= \mathrm{Span}\{X_1 = x_3\partial x_1 + x_1\partial x_3,\ X_2 = x_5\partial x_4 + x_4\partial x_5\}, \\
\mathrm{ltr}(TM) &= \mathrm{Span}\{N_1 = \frac{1}{2x_3^2}(-x_1\partial x_1 - x_2\partial x_2 + x_3\partial x_3), \\
&\quad N_2 = \frac{1}{2x_4^2}(x_4\partial x_4 - x_5\partial x_5 - x_6\partial x_6)\}\}, \\
\mathrm{S}(TM^\perp) &= \mathrm{Span}\{u_1 = \partial x_7,\ u_2 = \partial x_8\},
\end{aligned}$$

where $\{\partial x_i\}_{i\in\{1,\dots,8\}}$ is the standard basis on $\mathbb{R}^8_4$. If we put $v = \xi_1 + \xi_2$ then we obtain that vector field v is a concircular on $\Gamma(TM)$ with $\varphi = 1$.

Now, we remember the description of Ricci soliton lightlike submanifolds.

Definition 3.8. Let $(M, h, \mathrm{S}(TM))$ be a lightlike submanifold such that $\mathrm{S}(TM)$ is integrable. Then $(M, h, \mathrm{S}(TM))$ is said to be a Ricci soliton if the undermentioned equality holds for any $X_1, X_2 \in \Gamma(TM)$:

$$(L_\nu h)(X_1, X_2) + 2R^{(0,2)}(X_1, X_2) = 2\lambda h(X_1, X_2), \tag{3.12}$$

where λ is a constant and ν is said to be the potential vector field.

A Ricci soliton lightlike submanifold is said to be shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$.

We remark that the Ricci soliton equation vanishes its geometrical meaning when $R^{(0,2)}$ is not symmetric. Considering Theorem 2.1, we investigate the Ricci soliton equation on lightlike submanifolds whose screen distribution is integrable throughout this paper.

Proposition 3.9. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold containing a concircular vector field ν. Then we possess the undermentioned relation:*

$$\begin{aligned}
(L_{\nu^\top} h)(X_1, X_2) &= \sum_{l=1}^{r} B^l(X_1, \nu^\top)\eta_l(X_2) + \sum_{l=1}^{r} B^l X_2, \nu^\top)\eta_l(X_1) \\
&\quad + 2h(\varphi X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2) \\
&\quad + 2\sum_{\alpha=1}^{m} \rho_\alpha h(A_{u_\alpha} X_1, X_2).
\end{aligned} \tag{3.13}$$

If the vector field ν is a concurrent, then we possess

$$(L_{\nu^\top} h)(X_1, X_2) = \sum_{l=1}^{r} B^l(X_1, \nu^\top)\eta_l(X_2) + \sum_{l=1}^{r} B^l X_2, \nu^\top)\eta_l(X_1) + 2h(X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2) + 2\sum_{\alpha=1}^{m} \rho_\alpha h(A_{u_\alpha} X_1, X_2). \tag{3.14}$$

Proof. Using (3.3) in (2.15), the proof of (3.13) is straightforward. If ν is concurrent, putting $\varphi = 1$ in (3.13), we obtain (3.14). □

For a special case $m = 0$, we find

Proposition 3.10. *Let $(M, h, S(TM))$ be a coisotropic submanifold admitting a concircular vector field ν. Then we possess the undermentioned relation:*

$$(L_{\nu^\top} h)(X_1, X_2) = \sum_{l=1}^{r} B^l(X_1, \nu^\top)\eta_l(X_2) + \sum_{l=1}^{r} B^l(X_2, \nu^\top)\eta_l(X_1) + 2h(\varphi X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2). \tag{3.15}$$

If the vector field ν is a concurrent, then we possess

$$(L_{\nu^\top} h)(X_1, X_2) = \sum_{l=1}^{r} B^l(X_1, \nu^\top)\eta_l(X_2) + \sum_{l=1}^{r} B^l(X_2, \nu^\top)\eta_l(X_1) + 2h(X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2). \tag{3.16}$$

In view (3.10) and (3.13), we find

Proposition 3.11. *Let $(M, h, S(TM))$ be an $r-$dimensional lightlike submanifold containing a concircular vector field $\nu = \nu^\top$. Then we possess*

$$(L_{\nu^\top} h)(X_1, X_2) = 2h(\varphi X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2) + 2\sum_{\alpha=1}^{m} \rho_\alpha h(A_{u_\alpha} X_1, X_2). \tag{3.17}$$

If the vector field ν is a concurrent, then we possess

$$(L_{\nu^\top} h)(X_1, X_2) = 2h(X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2) + 2\sum_{\alpha=1}^{m} \rho_\alpha g(A_{u_\alpha} X_1, X_2). \tag{3.18}$$

Proposition 3.12. *Let $(M, h, S(TM))$ be a coisotropic lightlike submanifold containing a concircular vector field $\nu = \nu^\top$. Then we possess*

$$(L_{\nu^\top} h)(X_1, X_2) = 2h(\varphi X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2). \tag{3.19}$$

If the vector field ν is a concurrent, then we possess

$$(L_{\nu^\top} h)(X_1, X_2) = 2h(X_1, X_2) + 2\sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2). \tag{3.20}$$

Based on (3.12) and (3.13), we find

Theorem 3.13. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold containing a concircular vector field ν. Then, $(M, h, S(TM))$ is a Ricci soliton having the potential vector $\nu^\top$ if and only if*

$$R^{(0,2)}(X_1, X_2) = -\frac{1}{2}\sum_{l=1}^{r} B^l(X_1, \nu^\top)\eta_l(X_2) - \frac{1}{2}\sum_{l=1}^{r} B^l(X_2, \nu^\top)\eta_l(X_1) + (\lambda - \varphi)h(X_1, X_2) - \sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2) - \sum_{\alpha=1}^{m} \rho_\alpha h(A_{u_\alpha} X_1, X_2). \tag{3.21}$$

is satisfied.

Based on (3.12) and (3.15), we possess

Theorem 3.14. *Let $(M, h, S(TM))$ be a coisotropic lightlike submanifold containing vector field ν. Then, $(M, h, S(TM))$ is a Ricci soliton having the potential vector field $\nu^\top$ if and only if*

$$R^{(0,2)}(X_1, X_2) = -\frac{1}{2}\sum_{l=1}^{r} B^l(X_1, \nu^\top)\eta_l(X_2) - \frac{1}{2}\sum_{l=1}^{r} B^l(X_2, \nu^\top)\eta_l(X_1) + (\lambda - \varphi)h(X_1, X_2) - \sum_{l=1}^{r} f_l h(A_{N_l} X_1, X_2). \tag{3.22}$$

is satisfied.

Based on (3.12) and (3.17), we possess

Theorem 3.15. *Let $(M, h, S(TM))$ be an $r-$dimensional lightlike submanifold containing a concircular vector field $\nu = \nu^{\top}$. Then, $(M, h, S(TM))$ is a Ricci soliton having the potential vector field ν if and only if*

$$R^{(0,2)}(X_1, X_2) = (\lambda - \varphi)h(X_1, X_2) \tag{3.23}$$

is satisfied.

Based on (3.12) and (3.19), we possess

Theorem 3.16. *Let $(M, h, S(TM))$ be a coisotropic lightlike submanifold admitting a concircular vector field $\nu = \nu^{\top}$. $(M, h, S(TM))$ is a Ricci soliton having the potential vector field ν if and only if*

$$R^{(0,2)}(X_1, X_2) = (\lambda - \varphi)h(X_1, X_2) \tag{3.24}$$

is satisfied.

As a result of Theorem 3.15 and Theorem 3.16, we find

Corollary 3.17. *Every r-lightlike or coisotropic lightlike submanifold containing a concircular vector field $\nu = \nu^{\top}$ is an Einstein manifold.*

4. Some characterizations on lightlike submanifolds involving a rigged metric

First, we recall the notions of rigging vector fields and rigged metrics on lightlike hypersurfaces:

Definition 4.1. [2, 3] Let $(M, h, \mathrm{S}(TM))$ be a lightlike hypersurface and $\omega_p \notin T_pM$. The tangent vector ω_p is called a rigging vector field at $p \in M$ if there subsists a $1-$form η that satisfies

$$\eta(X_1) = \widetilde{h}(X_1, \omega) \tag{4.1}$$

for any $X_1 \in \Gamma(TM)$.

Now, let us choose N_p as a rigging vector field. In the circumstances, we can determine a $(0,2)-$type tensor $\overline{h}$ such that

$$\overline{h}(X_1, X_2) = h(X_1, X_2) + \eta(X_1)\eta(X_2) \tag{4.2}$$

is satisfied. Then the tensor field $\overline{h}$ is called a rigged metric on $(M, h, \mathrm{S(TM)})$.

The concept of rigged metrics could be given on $r-$lightlike submanifolds as follows:

Definition 4.2. Let $(M, h, \mathrm{S}(TM))$ be a $r-$lightlike submanifold. A metric $\overline{h}$ on $(M, h, \mathrm{S}(TM))$ satisfying

$$\overline{h}(X_1, X_2) = h(X_1, X_2) + \sum_{l=1}^{r} \eta_l(X_1)\eta_l(X_2) \tag{4.3}$$

is called a rigged metric.

Considering (4.3), we find

$$\overline{h}(\xi_l, \xi_s) = \delta_{ls}, \quad \overline{h}(\xi_l, X_1) = \eta_l(X_1), \quad \forall X_1 \in \Gamma(TM), \tag{4.4}$$

and

$$\overline{h}(X_1, X_2) = h(X_1, X_2), \quad \forall X_1, X_2 \in \Gamma(\mathrm{S}(TM)). \tag{4.5}$$

Proposition 4.3. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold containing a concircular vector field ν. Then we have the undermentioned equality:*

$$X\overline{h}(\nu, \nu) = 2h(\varphi X, \nu) + 2\sum_{l=1}^{r}[\eta_l^2(\nu) - h(A_N X, \nu)\eta(\nu)] \tag{4.6}$$

for any $X \in \Gamma(TM)$.

Proof. Based on (4.3), we can write

$$X\overline{h}(\nu, \nu) = X\widetilde{h}(\nu, \nu) + X\left(\sum_{l=1}^{r} \eta_l^2(\nu)\right). \tag{4.7}$$

Putting (2.4) and (2.5) in (4.7), we arrive at (4.6). □

Theorem 4.4. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold containing a concircular vector field ν that satisfies $\nu \in \Gamma(S(TM))$. Then ν is not a constant velocity vector in regard to $\overline{h}$.*

Proof. Assume that ν is a constant velocity vector in regard to $\overline{h}$. In this case, we find

$$h(\varphi X, \nu) = 0$$

for any $X \in \Gamma(\mathrm{S}(TM))$. The last statement shows that ν belongs to $\Gamma(\mathrm{Rad}(TM))$. This result contradicts the fact that $\nu \in \Gamma(\mathrm{S}(TM))$. □

Proposition 4.5. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold containing a concircular vector field ν that satisfies $\nu = \nu^{\top}$. Then we possess*

$$\begin{aligned} X_2\overline{h}(X_1, \nu) &= h(\nabla^*_{X_2}X_1, \nu) + h(X_1, \varphi X_2) + \sum_{l=1}^{r} B^l(X_2, X_1)\eta_l(\nu) \\ &+ \sum_{l=1}^{r}\sum_{s=1}^{r}[\gamma_{ls}(X_2)\eta_l(X_1)\eta_l(\nu) + \varphi\eta_l(X_2) \\ &- h(A_{N_l}X_2, \nu) + \gamma_{ls}(X_2)\eta_l(\nu)]\eta_l(X_1) \end{aligned} \tag{4.8}$$

for any $X_1, X_2 \in \Gamma(TM)$.

Proof. Based on (4.3), we get

$$X_2\overline{h}(X_1,\nu) = X_2 h(X_1,\nu) + X_2\left[\sum_{l=1}^{r}\eta_l(X_1)\eta_l(\eta)\right]. \tag{4.9}$$

If we put (2.4) and (2.5) in (4.9), we find

$$\begin{aligned} X_2\overline{h}(X_1,\nu) = h(\nabla_{X_2}X_1,\nu) &+ \sum_{l=1}^{r} B^l(X_2,X_1)\eta_l(\nu) + h(X_1,\widetilde{\nabla}_{X_2}\nu) \\ &+ \sum_{l=1}^{r}[h(\nabla_{X_2}X_1,N_l) - h(A_{N_l}X_2,X_1) \\ &+ \sum_{s=1}^{r}\gamma_{ls}(X_2)\eta_l(X_1)]\eta_l(\nu) + \sum_{l=1}^{r}[h(\widetilde{\nabla}_{X_2}\nu,N_l) \\ &- h(A_{N_l}X_2,\nu) + \sum_{s=1}^{r}\gamma_{ls}(X_2)\eta_l(\nu)]\eta_l(X_1). \end{aligned} \tag{4.10}$$

Handling the truth that the vector field ν is a concircular and (2.7),(2.8) in (4.10), we get (4.8). □

Theorem 4.6. *Let* $(M,h,S(TM))$ *be an* $r-$*lightlike submanifold containing a concircular vector field* ν *that satisfies* $\nu=\nu^\top$*. Then* $S(TM)$ *is integrable if and only if* $\nu^\top$ *belongs to* $\Gamma(Rad(TM))$*.*

Proof. Assume that $\nu \in \Gamma(\mathrm{Rad}(TM))$ and $X_1, X_2 \in \Gamma(\mathrm{S}(TM))$. In this circumstance, we find based on (4.10) that

$$X_2\overline{h}(X_1,\nu) = h(\nabla^*_{X_2}X_1,\nu) + h(X_1,\varphi X_2) + \sum_{l=1}^{r} B^l(X_2,X_1)\eta_l(\nu). \tag{4.11}$$

Changing X_1 and X_2 roles in (4.11), we immediately get

$$X_1\overline{h}(X_2,\nu) = h(\nabla^*_{X_2}X_1,\nu) + h(X_2,\varphi X_1) + \sum_{l=1}^{r} B^l(X_1,X_2)\eta_l(\nu). \tag{4.12}$$

Subtracting (4.11) and (4.12) side by side, we obtain

$$\overline{h}([X_2,X_1],\nu) = X_2\overline{h}(X_1,\nu) - X_1\overline{h}(X_2,\nu). \tag{4.13}$$

Using the fact that $\nu \in \Gamma(\mathrm{Rad}(TM))$ and $X_1, X_2 \in \Gamma(\mathrm{S}(TM))$, we find

$$\overline{h}([X_2,X_1],\nu) = 0. \tag{4.14}$$

In view of (4.14), we see that $[X_2,X_1] \in \Gamma(\mathrm{S}(TM))$ for each $X_1, X_2 \in \Gamma(\mathrm{S}(TM))$. This fact required that $\mathrm{S}(TM)$ is integrable. □

Now, we suppose that $\mathrm{S}(TM)$ is integrable. For any $X_1, X_2 \in \Gamma(\mathrm{S}(TM))$, we get from (4.13) that the equality

$$X_2\overline{h}(X_1,\nu) = X_1\overline{h}(X_2,\nu) \tag{4.15}$$

is satisfied.

Since the equation (4.15) is satisfied for any $X_1, X_2 \in \Gamma(\mathrm{S}(TM))$, we can choose $X_2 = \lambda X_1$, where λ is a smooth function. Placing this fact in (4.15), we find

$$\lambda X_1\overline{h}(X_1,\nu) = X_1\overline{h}(\lambda X_1,\nu),$$

which shows that λ is a constant function or $\nu \in \Gamma(\mathrm{Rad}(TM))$. Since λ should not be a constant function for each writing $X_2 = \lambda X_1$ in $\Gamma(\mathrm{S}(TM))$, we achieve that $\nu \in \Gamma(\mathrm{Rad}(TM))$. This completes the proving.

Theorem 4.7. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold involving a concircular vector field $\nu = \nu^\top$. If $(M, h, S(TM))$ is a Ricci soliton having the potential vector field ν, then ν lies on $\Gamma(Rad(TM))$.*

Proof. If we consider (3.12), we see that $R^{(0,2)}$ is symmetric. Therefore, if $(M, h, S(TM))$ is a Ricci soliton, then from Theorem 2.1, we obtain that $\mathrm{S}(TM)$ is integrable. Considering Theorem 4.6, the proof is straightforward. □

Proposition 4.8. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold involving a concircular vector field $\nu = \nu^\top$. Then we possess*

$$h(A_{N_k}Y,\nu) = Y\overline{h}(\xi_k,\nu) - \varphi\eta_k(Y) - \sum_{l=1}^{r}\gamma_{kl}(Y)\eta_l(\nu) \tag{4.16}$$

for $k \in \{1, \dots, r\}$.

Proof. From (4.3) and (4.4), we put

$$\begin{aligned} Y\overline{h}(\xi_k,\nu) &= Y\left[\sum_{l=1}^{r}\eta_l(\xi_k)\eta_l(\nu)\right] \\ &= Y\widetilde{h}(\nu, N_k) \\ &= \widetilde{h}(\widetilde{\nabla}_Y\nu, N_k) + \widetilde{h}(\nu, \widetilde{\nabla}_Y N_k) \\ &= \varphi\eta_k(Y) - h(A_{N_k}Y,\nu) + \sum_{l=1}^{r}\gamma_{kl}\eta_l(\nu), \end{aligned}$$

which is equivalent to (4.16). □

For a special case $\nu = \xi_k$ and $\nu = \xi_\alpha$, where $k, \alpha \in \{1, \dots, r\}$ and $k \neq \alpha$, we find

Proposition 4.9. *Let $(M, h, S(TM))$ be an $r-$lightlike submanifold involving a concircular vector field $\nu \in \Gamma(Rad(TM))$. Then we possess*

i) *if $\nu = \xi_k$ then $\gamma_{kk} = -1$.*
ii) *if $\nu = \xi_\alpha$ then $\gamma_{k\alpha} = 0$ for $k \neq \alpha$.*

Example 4.10. Discuss a submanifold in $\mathbb{R}^6_3$ stated by

$$x_4 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad x_3 = (x_5^2 + x_6^2)^{\frac{1}{2}}, \quad x_1, x_2, x_3, x_4 > 0.$$

Hence, we possess

$$\begin{aligned} \mathrm{Rad}(TM) &= Span\{\xi_1 = x_1\partial x_1 + x_2\partial x_2 + x_4\partial x_4, \\ &\qquad \xi_2 = x_3\partial x_3 + x_5\partial x_5 + x_6\partial x_6\}, \\ \mathrm{S}(TM) &= Span\{X_1 = x_4\partial x_1 + x_1\partial x_4, X_2 = x_3\partial x_5 + x_5\partial x_3\}, \\ \mathrm{ltr}(TM) &= Span\{N_1 = \frac{1}{2x_2^2}(x_1\partial x_1 - x_2\partial x_2 + x_3\partial x_3), \\ &\qquad N_2 = \frac{1}{2x_5^2}(x_3\partial x_3 - x_5\partial x_5 + x_6\partial x_6)\}, \end{aligned}$$

where $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6\}$ is the standard frame field on $\mathbb{R}^6_3$. For this reason, $(M, h, \mathrm{S}(TM))$ is a coisotropic lightlike submanifold and we obtain

$$\begin{array}{llll} \widetilde{\nabla}_{X_1}\xi_1 = X_1, & \widetilde{\nabla}_{X_1}\xi_2 = 0, & \widetilde{\nabla}_{X_2}\xi_1 = 0, & \widetilde{\nabla}_{X_2}\xi_2 = X_2, \\ \widetilde{\nabla}_{\xi_1}\xi_1 = \xi_1, & \widetilde{\nabla}_{\xi_1}\xi_2 = \widetilde{\nabla}_{\xi_2}\xi_1 = 0, & \widetilde{\nabla}_{\xi_2}\xi_2 = \xi_2, & \\ \widetilde{\nabla}_{\xi_1}N_1 = N_1, & \widetilde{\nabla}_{\xi_1}N_2 = 0, & \widetilde{\nabla}_{\xi_2}N_1 = 0, & \widetilde{\nabla}_{\xi_2}N_2 = N_2. \end{array}$$

By a straightforward computation, we find $R^{(0,2)}(X, Y) = 0$. If we consider $v = \varphi(\xi_1 + \xi_2)$, where φ is a smooth function and $v = v^T$, then we see that $(M, h, \mathrm{S}(TM))$ is a Ricci soliton having the potential vector field v such that $\lambda = 1$. Thus, the submanifold is an example of shrinking Ricci soliton.

We pay attention that the submanifold given in Example 4.10 holds the claim of Theorem 4.7.

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