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ψ-pseudomonotone generalized strong vector variational inequalities with application

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Abstract. In this paper, we establish an existence result of the solution for an generalized strong vector variational inequality already considered in the literature and as applications we obtain a new coincidence point theorem in Hilbert spaces.

Keywords: generalized strong vector variational inequality; ψ-psedomonotone; coincidence point; nonexpansive mappings

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1. INTRODUCTION

Let X and Y be two real Banach spaces, $K \subseteq X$ be a nonempty, closed and convex set, and $C \subseteq Y$ be a closed, convex and pointed cone with apex at origin. Recall that C is said to be a closed, convex and pointed cone with its apex at the origin iff C is closed and the following conditions hold:

(i) $\lambda C \subseteq C, \ \forall \ \lambda > 0$;
(ii) $C + C \subseteq C$;
(iii) $C \cap (-C) = \{0\}$.

Given C in Y, we can define relations " $\leq_C$ " and " $\nleq_C$ " as follows:

$$x \leq_C y \Leftrightarrow y - x \in C \text{ and } x \nleq_C y \Leftrightarrow y - x \notin C$$

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If " $\leq_C$ " is a partial order, then $(Y, \leq_C)$ is called a Banach space ordered by C. Let $L(X,Y)$ denote the space of all continuous linear maps from X into Y. Let $T : K \to L(X,Y)$ be a nonlinear mapping and let $\psi : K \times K \to X$ be a map. The strong vector variational inequality (for short, SVVI) consists in finding a vector $x \in K$, such that

$$\langle Tx, y - x \rangle \nleq_{C \setminus \{0\}} 0, \ \forall \ y \in K,$$

where $a \nleq_{C\setminus\{0\}} b$ means $b - a \notin C \setminus \{0\}$. The generalized strong vector variational inequality (for short, GSVVI) consists in finding a vector $x \in K$, such that

$$\langle Tx, \psi(y,x) \rangle \nleq_{C \setminus \{0\}} 0, \ \forall \ y \in K.$$

A variational inequality is a general formulation that encompasses a wide variety of mathematical problems, including nonlinear equations, optimization problems, complementarity problems, and fixed point problems, among others. However, the scalar-valued variational inequality has many restrictions in application. To overcome this problem, Giannessi [7] introduced the vector variational inequality in a finite dimensional Euclidean space. Since then, many researchers have studied vector variational inequalities in a general setting; see, for example, [1,3,5,6,9-13] and references therein.

This paper is organized as follows: In section 2, we prove the solvability for the GSVVI with monotonicity by using the Ky Fan inequality lemma. In section 3, we apply our existence result to establish an coincidence point result involving nonexpansive operators.

2. Existence of Solutions

Let $f : K \to X$ be a mapping. We say that f is affine if for each $x_1, ..., x_n \in K$ and $t_1, ..., t_n \in [0,1]$ with $\sum_{i=1}^n t_i = 1$, we have

$$f(\sum_{i=1}^n t_i x_i) = \sum_{i=1}^n t_i f(x_i).$$

Throughout the paper, let Ψ denote the set of all maps $\psi : K \times K \to X$ which are affine in each of its variables, $\psi(x,.)$ is continuous and $\psi(x,x) = 0$ for each $x \in K$. For example, if $\psi(y,x) = g(y) - g(x)$ for $x, y \in K$, where $g : K \to X$ is a continuous affine map, then $\psi \in \Psi$. Let $\psi \in \Psi$. A mapping $T : K \to L(X,Y)$ is said to be ψ-hemicontinuous if for any $x, y \in K$, the mapping $t \rightarrowtail \langle T(ty + (1-t)x), \psi(y,x) \rangle$ is continuous at 0^+. If $\psi(y,x) = y - x$ for each $x, y \in K$, then we say that T is hemicontinuous briefly. Let $\psi \in \Psi$. A mapping $T : K \to L(X,Y)$ is said to be ψ-psedomonotone if for any $x, y \in K$,

$$\langle Tx, \psi(y,x) \rangle \nleq_{C\setminus\{0\}} 0 \Rightarrow \langle Ty, \psi(y,x) \rangle \geq_C 0.$$

If $\psi(y,x) = y - x$ for each $x, y \in K$, then we say that T is psedomonotone briefly.

Example 2.1. Let $X = Y = \mathbb{R}$, $K = C = [0, \infty)$ and let $\psi(y,x) = y^2 - x^2$ for each $x, y \in K$. Let $T : K \to \mathbb{R}$ be a non-decreasing function. Then T is ψ-psedomonotone.

Let H be a Hilbert space, $K \subseteq H$ and let $f : K \to H$ and $g : K \to H$ are two mappings. We say that f is g-nonexpansive if

$$\|f(x) - f(y)\| \leq \|g(x) - g(y)\|, \ \forall\ x, y \in K.$$

In the case that $g = I$, the identity mapping, f is called nonexpansive.

Example 2.2. Let $X = H$ be a Hilbert space, $K \subseteq H$, $Y = \mathbb{R}$ and let $C = [0, \infty)$. Let $f : K \to H$ and $g : K \to H$ are mappings such that f is g-nonexpansive. Let $\psi(y,x) = g(y) - g(x)$ and let $Tx = g(x) - f(x)$ for each $x, y \in K$. Now we show that T is ψ-psedomonotone. To show the claim, notice that since f is g-nonexpansive then by Cauchy-Schwartz inequality, we have

$$\begin{aligned}\langle f(y) - f(x), g(y) - g(x)\rangle &\leq \|f(y) - f(x)\|\|g(y) - g(x)\| \\ &\leq \|g(y) - g(x)\|^2 = \langle g(y) - g(x), g(y) - g(x)\rangle,\end{aligned}$$

for each $x, y \in K$. Thus for each $x, y \in K$

$$\langle (g(y) - g(x)) - (f(y) - f(x)), g(y) - g(x)\rangle \geq 0.$$

Then

$$\langle T(y) - T(x), \psi(y,x)\rangle \geq 0,$$

and so

$$\langle T(y), \psi(y,x)\rangle \geq \langle T(x), \psi(y,x)\rangle \geq 0,$$

for each $x, y \in K$.

Lemma 2.3. *([4]) Let M be a nonempty closed and convex subset of a Hausdorff topological space and $F : M \to 2^M$ be a multivalued mapping. Suppose that for any finite set $\{x_1, ..., x_n\} \subseteq M$, one has $co\{x_1, ..., x_n\} \subseteq \cup_{i=1}^{n} F(x_i)$ (i.e. F is a KKM mapping) and $F(x)$ is closed for each $x \in M$ and compact for some $x \in M$. Then $\cap_{x \in M} F(x) \neq \emptyset$.*

Lemma 2.4. *Let K be a nonempty closed convex subset of a real Banach space X and Y be a real Banach space ordered by a pointed closed convex cone C with its apex at the origin and int $C \neq \emptyset$. Let $T : K \to L(X, Y)$ be a hemicontinuous ψ-psedomonotone mapping. Then for any given $x_0 \in K$,*

$$\langle Tx_0, \psi(y, x_0)\rangle \not\leq_{C\setminus\{0\}} 0 \ \forall\ y \in K$$

if and only if

$$\langle Ty, \psi(y, x_0)\rangle \geq_C 0 \ \forall\ y \in K.$$

Proof. Let $x_0 \in K$ such that

$$\langle Tx_0, \psi(y, x_0)\rangle \nleq_{C\setminus\{0\}} 0 \ \forall\ y \in K.$$

Since T is ψ-psedomonotone then

$$\langle Ty, \psi(y, x_0)\rangle \geq_C 0 \ \forall\ y \in K.$$

Conversly, suppose that

$$\langle Ty, \psi(y, x_0)\rangle \geq_C 0 \ \forall\ y \in K.$$

Then, we have (note that K is convex)

$$\langle T(ty + (1-t)x_0), \psi(ty + (1-t)x_0, x_0)\rangle \geq_C 0 \ \forall\ y \in K,\ t \in [0, 1].$$

Since ψ is affine with respect to the first argument and $\psi(x_0, x_0) = 0$, then we get

$$\langle T(ty + (1-t)x_0), \psi(ty + (1-t)x_0, x_0)\rangle =$$

$$t\langle T(ty + (1-t)x_0), \psi(y, x_0)\rangle \geq_C 0 \ \forall\ y \in K,\ t \in [0, 1].$$

The ψ-hemicontinuity of T implies that

$$\langle Tx_0, \psi(y, x_0)\rangle \geq_C 0 \ \forall\ y \in K.$$

Hence

$$\langle Tx_0, \psi(y, x_0)\rangle \nleq_{C\setminus\{0\}} 0 \ \forall\ y \in K.$$

□

The following theorem generalize Theorem 2.3 in [5].

Theorem 2.5. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X and Y be a real Banach space ordered by a pointed closed convex cone C with its apex at the origin and int $C \neq \emptyset$. Let $T : K \to L(X, Y)$ be a ψ-hemicontinuous ψ-psedomonotone mapping. Then GSVVI is solvable.*

Proof. Define $F, G : K \to 2^K$ by

$$F(v) = \{u \in K : \langle Tu, \psi(v, u)\rangle \nleq_{C\setminus\{0\}} 0\},\ \forall\ v \in K$$

and

$$G(v) = \{u \in K : \langle Tv, \psi(v, u)\rangle \geq_C 0\},\ \forall\ v \in K.$$

Since $\psi(v, v) = 0$ then $v \in F(v) \cap G(v)$ and so both $F(v)$ and $G(v)$ are nonempty. Since K is bounded and $G(v) \subseteq K$ then $G(v)$ is bounded. Since ψ is affine with respect to the second argument, we have that $G(v)$ is convex. The closedness of $G(v)$ follows from the continuity of $\psi(v, .)$. So, $G(v)$ is bounded, convex and closed in K. Since T is ψ-psedomonotone and ψ-hemicontinuous, it follows from Lemma 2.4 that

$$F(v) \subseteq G(v) \text{ and } \cap_{v\in K} F(v) = \cap_{v\in K} G(v).$$

Now we prove that F is a KKM map. Indeed, assume that F is not a KKM mapping. Then there exist $x, x_1, ..., x_n \in K$, $t_1 \geq 0, ..., t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ and $x = \sum_{i=1}^n t_i x_i$ such that

$$x \notin F(x_i),\ i = 1, ..., n.$$

That is

$$\langle T(x), \psi(x_i, x)\rangle \leq_{C\setminus\{0\}} 0,\ i = 1, ..., n.$$

It follows that

$$0 = \langle Tx, \psi(x, x)\rangle = \sum_{i=1}^n \langle Tx, \psi(\sum_{i=1}^n t_i x_i, x)\rangle$$

$$= \sum_{i=1}^n t_i \langle Tx, \psi(x_i, x)\rangle \leq_{C\setminus\{0\}} 0,$$

a contradiction. Hence F is a KKM mapping and so is G (note that $F(v) \subseteq G(v)$ for each $v \in K$). Since X is reflexive, $G(v)$ is weakly compact. It follows from Lemma 2.3 that

$$\cap_{v\in K} F(v) = \cap_{v\in K} G(v) \neq \emptyset.$$

This implies that there exists $x_0 \in \cap_{v\in K} F(v) \subseteq K$ such that

$$\langle Tx_0, \psi(y, x_0)\rangle \not\leq_{C\setminus\{0\}} 0,\ \forall\ y \in K.$$

□

3. A coincidence point result

In this section we apply the existence results that have obtained in previous section, to establish a coincidence point result involving nonexpansive operators. As particular case we obtain a fixed point theorem for nonexpansive mappings which is due to Browder [2] and Göhde [8]. Everywhere in the sequel H denotes a real Hilbert space, identified with its dual. Our first coincidence point result is stated as follows.

Theorem 3.1. *Let $K \subseteq H$ be convex and weakly compact. Let $f : K \to H$ and $g : K \to H$ are mappings such that f is g-nonexpansive and g is a continuous affine mapping. Assume that $f(K) \subseteq g(K)$. Then there exists $x_0 \in K$ such that $f(x_0) = g(x_0)$.*

Proof. Consider the operator $T : K \to H$; $T(x) = g(x) - f(x)$ for each $x \in K$. Since g is continuous and f is g-nonexpansive then f is continuous and so T is hemicontinuous. Let $\psi(y, x) = g(y) - g(x)$ for each $x, y \in K$. Since g is affine then ψ is affine with respect to both of its variables. From Example 2.2 we have that T is ψ-psedomonotone.

Hence, the conditions of Theorem 2.5 are satisfied. Thus there exists $x_0 \in K$ such that for each $y \in K$

$$0 \leq \langle T(x_0), \psi(y, x_0) \rangle = \langle (g(x_0) - f(x_0), g(y) - g(x_0) \rangle.$$

Since $f(K) \subseteq g(K)$ let $y \in K$ such that $g(y) = f(x_0)$. Then we have $\langle g(x_0) - f(x_0), f(x_0) - g(x_0) \rangle$ or equivalently $-\|g(x_0) - f(x_0)\| \geq 0$ which shows that $f(x_0) = g(x_0)$. □

Corollary 3.2. *([2], [8]) Let $K \subseteq H$ be convex and weakly compact, and let $f : K \to K$ be a nonexpansive mapping. Then there exists $x_0 \in K$ such that $f(x_0) = x_0$.*

Now we pose the following question.

Problem 3.3. Does the conclusion of Theorem 3.1 hold in the setting of uniformly convex Banach spaces?

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