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A Class of Common Fixed Point Results in Cone Metric Spaces

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Abstract. In this paper, we introduce the concept of $\alpha-\psi-f$-contractive mappings and get some new common fixed point results, in particular, generalized Lipschitz condition for such mappings in cone metric spaces over Banach algebras. Also, we support our results by some examples.

Keywords: Cone metric space, Contractive mapping, Weakly compatible, Common fixed point.

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1. Introduction

Fixed point theory is one of the most powerful tools in nonlinear analysis. It is a rich, interesting and exciting branch of mathematics. Fixed point theory is a beautiful mixture of analysis, topology and geometry. It is an interdisciplinary branch of mathematics which can be applied in various disciplines of mathematics and mathematical sciences such as economics, optimization theory, approximate theory, game theory, integral equations, differential equations, operator theory, etc. Fixed point

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theory has been developed through different spaces such as topological spaces, metric spaces, fuzzy metric spaces, etc. The Banach contraction principle [1] is the simplest and one of the most versatile elementary results in fixed point theory, which states that every self contraction mapping on a complete metric space has a unique fixed point. This principle has many applications and was extended by several authors (see [1, 3, 6, 8]).

Cone metric spaces were introduced by Huang and Zhang as a generalization of metric spaces in [2]. The distance $d(x, y)$ of two elements x and y in a cone metric space X is defined to be a vector in an ordered Banach space $\mathcal{A}$. Moreover, they proved the Banach contraction principle in the setting of cone metric spaces with the assumption that the cone is normal. Later, the assumption of normality of cone was removed by Rezapour and Hamlbarani [5]. Some authors have proved the existence and uniqueness of the fixed point in cone metric spaces [2, 4, 5, 9].

In this paper, we introduce the concept of $\alpha-\psi-f$-contractive mappings and present some common fixed point results for such mappings in cone metric spaces over Banach algebras. Moreover, we support our results by some examples.

2. Preliminaries

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [7].

The following proposition is well known (see [7]).

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e-x$ is invertible. Actually,

$$(e-x)^{-1}=\sum_{i=0}^{\infty} x^i.$$

Remark 2.2. If $r(k)<1$, then $\|k^n\|\to 0\ \ (n\to\infty)$.

Now let us recall the concept of cone for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone of $\mathcal{A}$ if

(1) P is non-empty closed and $\{\theta, e\}\subset P$;
(2) $\alpha P+\beta P\subset P$ for all non-negative real numbers α, β;
(3) $P^2=PP\subset P$;
(4) $P\cap(-P)=\{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P\subset\mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x\preceq y$ if and only if $y-x\in P$. $x\prec y$ will stand for $x\preceq y$ and $x\neq y$, while $x\ll y$ will stand for $y-x\in \text{int}P$, where $\text{int}P$ denotes the interior of P. If $\text{int}P\neq\emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $M>0$ such that, for all $x, y\in\mathcal{A}$,

$$\theta\preceq x\preceq y\Rightarrow \|x\|\leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of P [2].

Definition 2.3. [9] Let X be a non-empty set. Suppose that the mapping $d: X\times X\to\mathcal{A}$ satisfies

(1) $\theta\preceq d(x,y)$ for all $x,y\in X$ and $d(x,y)=\theta$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,y)\preceq d(x,z)+d(z,x)$ for all $x,y,z\in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 2.4. [9] Let (X,d) be a cone metric space over a Banach algebra $\mathcal{A}, x\in X$ and let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ converges to x whenever for each $c\in\mathcal{A}$ with $\theta\ll c$ there is a natural number N such that $d(x_n,x)\ll c$ for all $n\geq N$. We denote this by $\lim\limits_{n\to\infty} x_n=x$ or $x_n\to x$.
(2) $\{x_n\}$ is a Cauchy sequence whenever for each $c\in\mathcal{A}$ with $\theta\ll c$ there is a natural number N such that $d(x_n,x_m)\ll c$ for all $n,m\geq N$.
(3) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Remark 2.5. The cone metric is not continuous in the general case, i.e., from $x_n \to x$, $y_n \to y$ as $n \to \infty$ it need not follow that $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. However, if (X, d) is a cone metric space with a normal cone P, then the cone metric d is continuous (see Lemma 5 in [2]).

Definition 2.6. Let (X, d) be a cone metric space and $\alpha : X \times X \to [0, \infty)$, be a mapping. X is α-regular, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n (see [8]).

Lemma 2.7. [3] *Let (X, d) be a cone metric space. Then the following properties hold:*

(PT1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(PT2) If $u \ll v$ and $v \preceq w$, then $u \ll w$.
(PT3) If $u \ll v$ and $v \ll w$, then $u \ll w$.
(PT4) If $\theta \preceq u \ll c$ for each $c \in \text{int}P$, then $u = \theta$.
(PT5) If $c \in \text{int}P$ and $\{a_n\}$ is a sequence in $\mathcal{A}$ such that $\theta \preceq a_n$ for all $n \in \mathbb{N}$ and $a_n \to \theta$ as $n \to \infty$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $a_n \ll c$.

In the following, we always suppose that Ψ be a family of functions $\psi : P \to P$ such that

(i) $\psi(\theta) = \theta$ and $\theta \prec \psi(t) \prec t$ for $t \in P - \{\theta\}$,
(ii) $\psi(t) \ll t$ for all $\theta \ll t$,
(iii) $\lim_{n\to\infty} \psi^n(t) = \theta$ for every $t \in P - \{\theta\}$,
(iv) ψ is a strictly increasing function, i.e., $\psi(a) \prec \psi(b)$ whenever $a \prec b$.

Definition 2.8. Let f and T be self mappings of a non-empty set X. If $w = fx = Tx$ for some $x \in X$, then x is called a coincidence point of f and T, and w is called a point of coincidence of f and T. If $w = x$, then x is called a common fixed point of f and T.

Definition 2.9. [6] Let X be a non-empty set and T, $f : X \to X$. The mapings T, f are said to be weakly compatible if they commute at their coincidence points (i.e., $Tfx = fTx$ whenever $Tx = fx$).

Definition 2.10. [6] Let X be a non-empty set, T, $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is $f - \alpha$-admissible if, for all $x, y \in X$ such that $\alpha(fx, fy) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$. If f is the identity mapping, then T is called α-admissible.

3. Main results

In this section, we shall prove some common fixed point results in the setting of cone metric spaces over Banach algebras.

We begin this section with defining the concept of $\alpha-\psi-f$-contractive mappings.

Definition 3.1. Let (X,d) be a cone metric space and $T:X\to X$ be a given mapping. T is said to be an $\alpha-\psi-f$-contractive mapping if there exist three functions $f:X\to X$, $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(fx,fy)d(Tx,Ty)\preceq\psi(d(fx,fy)),$$

for all $x,y\in X$.

Remark 3.2. We note that Definition 3.1 is a generalization of Definition 2.1 in [8], in the setting of cone metric space with a Banach algebra.

Example 3.3. Let $\mathcal{A}=\mathbb{R}^2$, $P=\{(x,y)\in\mathcal{A}:x,y\geq 0\}$, $X=\mathbb{R}$. We define $d:X\times X\to\mathcal{A}$ by

$$d(x,y)=(|x-y|,\alpha|x-y|),\quad \text{where } \alpha\geq 0.$$

Then (X,d) is a cone metric space. Now, we consider $\psi:P\to P$ by

$$\psi(t)=\frac{1}{2}(t_1,t_2),\quad \text{for }\ t=(t_1,t_2)\in P.$$

Clearly, $\psi\in\Psi$. Let us define $T:X\to X$ and $f:X\to X$ by

$$T(x)=\begin{cases}\dfrac{x}{5} & \text{if } x\in[0,1],\\ 0 & \text{otherwise}\end{cases}\quad \text{and}\quad f(x)=\begin{cases}\dfrac{x}{2} & \text{if } x\in[0,1],\\ 0 & \text{otherwise.}\end{cases}$$

Assume also that the mapping $\alpha:X\times X\to[0,\infty)$ is defined by

$$\alpha(x,y)=\begin{cases}1 & \text{if } x,y\in[0,1],\\ 0 & \text{otherwise.}\end{cases}$$

Clearly,

$$\alpha(fx,fy)d(Tx,Ty)\preceq\psi(d(fx,fy)),$$

i.e., T is an $\alpha-\psi-f$-contractive mapping.

Theorem 3.4. *Let (X,d) be a cone metric space with a normal cone P and $T:X\to X$ be an $\alpha-\psi-f$-contractive mapping satisfying the following conditions:*

(i) T is $f-\alpha$-admissible;
(ii) there exists $x_0\in X$ such that $\alpha(fx_0,Tx_0)\geq 1$;
(iii) X is α-regular;
(iv) either $\alpha(fu,fv)\geq 1$ or $\alpha(fv,fu)\geq 1$ whenerver $fu=Tu$ and $fv=Tv$ for all $u,v\in X$.

If $TX \subseteq fX$ and fX is a complete subspace of X, then f and T have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$. Since $TX \subseteq fX$, we define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n, \quad \text{for all } n \in \mathbb{N}.$$

If $fx_{n+1} = fx_n$ for some $n \in \mathbb{N}$, then trivially y_n is a point of coincidence of f and T. Assume that $fx_{n+1} \neq fx_n$ for all $n \in \mathbb{N}$.

By condition (ii), we have $\alpha(fx_0, Tx_0) = \alpha(fx_0, fx_1) \geq 1$. Since T is $f-\alpha$-admissible, we obtain

$$\alpha(Tx_0, Tx_1) = \alpha(fx_1, fx_2) \geq 1.$$

By induction, we get

$$\alpha(fx_n, fx_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Now, we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq \alpha(fx_{n-1}, fx_n) d(Tx_{n-1}, Tx_n) \\ &\preceq \psi(d(fx_{n-1}, fx_n)) \\ &\vdots \\ &\preceq \psi^n(d(fx_0, fx_1)) \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\psi \in \Psi$, we have $\psi^n(d(fx_0, fx_1)) \to \theta$ as $n \to \infty$. Fix $c \in \mathcal{A}$ such that $\theta \ll c$. Using the property of ψ, we yet $\theta \ll c - \psi(c)$. By (PT5), there exists $N \in \mathbb{N}$ such that $\psi^n(d(fx_0, fx_1)) \ll c - \psi(c)$ and $\psi^n(d(fx_0, fx_1)) \ll c$ for $n \geq N$. Therefore, $d(fx_n, fx_{n+1}) \ll c - \psi(c)$ and $d(fx_n, fx_{n+1}) \ll c$ for $n \geq N$. For fixed $m > n \geq N$, we have

$$\begin{aligned} d(fx_n, fx_{n+2}) &\preceq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) \\ &\ll c - \psi(c) + \psi(d(fx_n, fx_{n+1})) \\ &\ll c - \psi(c) + \psi(c) = c, \\ d(fx_n, fx_{n+3}) &\preceq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+3}) \\ &\ll c - \psi(c) + \psi(d(fx_n, fx_{n+2})) \\ &\ll c - \psi(c) + \psi(c) = c, \\ &\vdots \\ d(fx_n, fx_m) &\preceq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_m) \\ &\ll c - \psi(c) + \psi(d(fx_n, fx_{m-1})) \\ &\ll c - \psi(c) + \psi(c) = c. \end{aligned}$$

This shows that $\{fx_n\}$ is a cauchy sequence. By the completeness of fX, there exists $z \in fX$ such that $fx_n \to z$. Let $x \in X$ be such that $fx = z$.

Since X is α-regular, we observe that

$$\begin{aligned} d(fx_n, Tx) &= d(Tx_{n-1}, Tx) \\ &\preceq \alpha(fx_{n-1}, fx)d(Tx_{n-1}, Tx) \\ &\preceq \psi(d(fx_{n-1}, fx) \\ &\preceq d(fx_{n-1}, fx). \end{aligned}$$

By making $n \to \infty$, we have $d(fx, Tx) \preceq \theta$. Therefore, $d(fx, Tx) = \theta$, that is $z = fx = Tx$. Hence z is a point of coincidence of f and T.

For uniqueness of the point of coincidence of f and T, let $y \neq z$ be another point of coincidence of f and T. Then

$$y = fu = Tu \quad \text{and} \quad z = fv = Tv \quad \text{for some } u, v \in X.$$

By condition (iv), we have

$$\begin{aligned} d(y, z) &= d(fu, fv) \\ &= d(Tu, Tv) \\ &\preceq \alpha(fu, fv)d(Tu, Tv) \\ &\preceq \psi(d(fufv)) \\ &= \psi(d(y, z)) \end{aligned}$$

The property of ψ, implies that $d(y, z) = \theta$, that is, $y = z$. Let z be the point of coincidence of f and T. Then $z = f(x) = T(x)$ for some $x \in X$. Moreover, if T and f are weakly compatible, we have

$$Tz = Tfx = fTx = fz = u \ \ (say).$$

Thus u is a point of coincidence of T and f. Therefore $u = z$ by uniqueness. Hence $Tz = fz = z$. So z is the unique common fixed point of T and f. □

Form Theorem 3.4, if we choose $f = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 3.5. *Let (X, d) be a complete cone metric space with a normal cone P and $T : X \to X$ be an $\alpha - \psi$-contractive mapping, i.e., T satifies the following condition:*

$$\alpha(x, y)d(Tx, Ty) \preceq \psi(d(x, y)), \quad \textit{for all } x, y \in X.$$

Assume also that the following conditions hold:

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) X is α-regular;

(iv) either $\alpha(u,v) \geq 1$ *or* $\alpha(v,u) \geq 1$ *whenever* $u = Tu$ *and* $v = Tv$ *for all* $u, v \in X$.

Then T *has a unique fixed point.*

From Theorem 3.4, if the function $\alpha : X \times X \to [0, \infty)$ is such that $\alpha(x,y) = 1$ for all $x, y \in X$, we get the following corollary.

Corollary 3.6. *Let* (X, d) *be a cone metric space with a normal cone* P *and* $T : X \to X$ *be an* $\psi - f$*-contractive mapping, i.e.,* T *satisfies the following condition*

$$d(Tx, Ty) \preceq \psi(d(fx, fy)), \quad \text{for all}\, x, y \in X.$$

If $TX \subseteq fX$ *and* fX *is a complete subspace of* X*, then* T *and* f *have a unique point of coincidence in* X*. Moreover, if* T *and* f *are weakly compatible, then* T *and* f *have a unique common fixed point.*

Now, we give an example to support Corollary 3.6.

Example 3.7. Let $X = [0, 1]$, $\mathcal{A} = \mathbb{R}^2$, with usual norm, be a Banach algebra, $P = \{(x, y) \in \mathcal{A} : x, y \geq 0\}$ be a normal cone and the partial ordering $\preceq$ with respect to the cone P be the usual partial ordering in $\mathcal{A}$. Define

$$\begin{aligned} &d : X \times X \to \mathcal{A} \quad \text{by} \\ &d(x, y) = (|x - y|, |x - y|), \quad \text{for all} \;\; x, y \in X. \end{aligned}$$

Then (X, d) is a complete cone metric space with $d(x, y) \in \text{int}P$, for all $x, y \in X$ with $x \neq y$. Define

$$\psi : \text{int}P \cup \{\theta\} \to \text{int}P \cup \{\theta\} \text{ by}$$
$$\psi(t) = \begin{cases} \dfrac{1}{2}(t_1, t_1), \text{ for } t = (t_1, t_2) \in \text{int}P \cup \{\theta\} \;\text{ with } t_1 \leq t_2; \\ \dfrac{1}{2}(t_2, t_2), \text{ for } t = (t_1, t_2) \in \text{int}P \cup \{\theta\} \;\text{ with } t_1 > t_2. \end{cases}$$

By the definition of Ψ, we have $\psi \in \Psi$. Let us define $f : X \to X$ and $T : X \to X$ by

$$fx = \frac{x}{2} \quad \text{and} \quad Tx = \frac{x^2}{8}, \quad \text{for all } x \in X.$$

Without loss of generality, take $x, y \in X$ with $x > y$. Now, we have

$$\begin{aligned} d(Tx, Ty) &= d\left(\frac{x^2}{8}, \frac{y^2}{8}\right) \\ &= \left(\frac{x^2}{8} - \frac{y^2}{8}, \frac{x^2}{8} - \frac{y^2}{8}\right) \\ &= \left(\frac{(x-y)(x+y)}{8}, \frac{(x-y)(x+y)}{8}\right) \\ &\preceq \left(\frac{2(x-y)}{8}, \frac{2(x-y)}{8}\right) \\ &= \left(\frac{x-y}{4}, \frac{x-y}{4}\right) \\ &= \psi(d(fx, fy)). \end{aligned}$$

Hence, $0 \in X$ is the unique common fixed point of f and T.

Remark 3.8. In Definition 3.1, if we take $\psi(t) = kt$, for all $t \in P$ and $k \in P$ with $r(k) < 1$, then we get the following result from Theorem 3.4.

Theorem 3.9. *Let (X, d) be a cone metric space and P be a normal cone with a normal constant M. Assume that the mapping $T : X \to X$ satisfies the following condition*

$$\alpha(fx, fy)d(Tx, Ty) \preceq kd(fx, fy), \quad \textit{for all}\, x, y \in X, \tag{3.1}$$

where $k \in P$ with $r(k) < 1$. Assume also that the following conditions hold:

(i) T is $f - \alpha$ admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$;
(iii) X is α-regular;
(iv) either $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$ for all $u, v \in X$.

If $TX \subseteq fX$ and fX is a complete subspace of X, then f and T have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n, \qquad n \in \mathbb{N}.$$

As in the proof of Theorem 3.4, we obtain

$$\alpha(fx_n, fx_{n+1}) \geq 1 \quad \text{for all}\, n \in \mathbb{N}.$$

Now, we observe that

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq \alpha(fx_{n-1}, fx_n) d(Tx_{n-1}, Tx_n) \\ &\preceq k d(fx_{n-1}, fx_n) \\ &\vdots \\ &\preceq k^n d(fx_0, fx_1). \end{aligned}$$

Hence, for $m > n$ we have

$$\begin{aligned} d(fx_n, fx_m) &\preceq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \cdots + d(fx_{m-1}, fx_m) \\ &\preceq k^n d(fx_0, fx_1) + k^{n+1} d(fx_0, fx_1) + \cdots + k^{m-1} d(fx_0, fx_1) \\ &\preceq \left(\sum_{i=0}^{\infty} k^i \right) k^n d(fx_0, fx_1) \\ &= (e-k)^{-1} k^n d(fx_0, fx_1). \end{aligned}$$

Since $r(k) < 1$, by Remark 2.2, we have $\|k^n\| \to 0$ as $n \to \infty$. Since P is normal with a normal constant M, we get

$$\big\|d(x_n, x_m)\big\| \leq M \big\|(e-k)^{-1}\big\| \big\|k^n\big\| \big\|d(fx_0, fx_1)\big\| \to 0 \quad \text{as } n \to \infty.$$

This implies that $\{fx_n\}$ is a cauchy sequence. Since fX is a complete subspace of X, there exists $z \in fX$ such that $fx_n \to z$. Let $x \in X$ be such that $fx = z$. Since X is α-regular, we observe that

$$\begin{aligned} d(fx_n, Tx) &= d(Tx_{n-1}, Tx) \\ &\preceq \alpha(fx_{n-1}, fx) d(Tx_{n-1}, Tx) \\ &\preceq k d(fx_{n-1}, fx) \end{aligned}$$

By making $n \to \infty$, we have $d(fx, Tx) \preceq \theta$. Therefore $fx = Tx = z$. Hence z is a point of coincidence of f and T. Let $u \neq z$ be another point of coincidence of f and T. Then

$$z = ft = Tt \quad \text{and} \quad u = fy = Ty \quad \text{for some} \quad y, t \in X.$$

By condition (iv), we have

$$\begin{aligned} d(ft, fy) &= d(Tt, Ty) \\ &\preceq \alpha(ft, fy) d(Tt, Ty) \\ &\preceq k d(ft, fy) \end{aligned}$$

Hence,

$$d(z, u) \preceq k d(z, u).$$

That is,

$$(e-k)d(z,u) \preceq \theta$$

Multiplying both sides above by

$$(e-k)^{-1} = \sum_{i=0}^{\infty} k^i \succeq \theta.$$

We get $d(z,u) \preceq \theta$. Thus $d(z,u) = \theta$, which implies that $z = u$. Moreover, if T, f are weakly compatible, then as in the proof of Theorem 3.4, z is the unique common fixed point of T and f. □

Remark 3.10. In condition (3.1) of Theorem 3.9, if we put $f = I_X$, the identity mapping and $\alpha(x,y) = 1$ for all $x, y \in X$, then the following condition is called the generalized Lipschitz condition in the setting of cone metric space with a Banach algebra,

$$d(Tx,Ty) \preceq kd(x,y), \quad \text{for all } x, y \in X,$$

where $k \in P$ with $r(k) < 1$.

Corollary 3.11. *Let (X,d) be a complete cone metric space and P be a normal cone with a normal constant M. Suppose that the mapping $T : X \to X$ satisfies the generalized Lipschitz condition. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.*

Here, we give an example which illustrate Corollary 3.11.

Example 3.12. Let $\mathcal{A} = \mathbb{R}^2$. For each $(x_1, x_2) \in \mathcal{A}$, $\|(x_1, x_2)\| = |x_1| + |x_2|$. The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1).$$

Then $\mathcal{A}$ is a Banach algebra with unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$. Then P is normal with a normal constant $M = 1$. Let $X = \mathbb{R}^2$ and the metric d be defined by

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = (|x_1 - y_1|, |x_2 - y_2|) \in P.$$

Then (X,d) is a complete cone metric space. Now, we define the mapping $T : X \to X$ by

$$T(x_1, x_2) = \left(\frac{x_1}{2}, \frac{x_2}{2} + \alpha x_1\right),$$

where α can be any large positive real number. Now, we have

$$\begin{aligned} d(T(x_1,x_2),T(y_1,y_2)) &= \left(\frac{1}{2}|x_1-y_1|, \left(\frac{x_2}{2}+\alpha x_1 - \frac{y_2}{2} - \alpha y_1\right|\right) \\ &\preceq \left(\frac{1}{2}|x_1-y_1|, \frac{1}{2}|x_2-y_2| + \alpha|x_1-y_1|\right) \\ &= \left(\frac{1}{2},\alpha\right)(|x_1-y_1|,|x_2-y_2|) \\ &= \left(\frac{1}{2},\alpha\right) d((x_1,x_2),(y_1,y_2)) \end{aligned}$$

Moreover, $r\left(\left(\frac{1}{2},\alpha\right)\right) < 1$, (see Example 2.1 of [4]). Thus all the conditions of Corollary 3.11, are fulfilled. Hence, T has a unique fixed point in X.

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