Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/CJMS.2024.26868.1685
Caspian J Math Sci. **13**(1)(2024), 155-170 (Research Article)

Positive solutions for the quasilinear system by using of weak sub-super solutions method and energy functional

G.Karamali [1]
[1] Department of Basic Sciences, Shahid Sattari Aeronautical University of Science and Technology, South Mehrabad, Tehran, Iran.

Abstract. In this paper we consider the quasilinear system

$$\begin{cases} -\Delta_p u = \lambda g(x) f_1(u,v) + \mu h_1(u) \ in \ x \in \Omega \\ \\ -\Delta_q v = \lambda g(x) f_2(u,v) + \mu h_2(v) \ in \ x \in \Omega \\ \\ u(x) = v(x) = 0 \ on \ x \in \partial\Omega \ , \end{cases}$$

Where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter and g is C^1 sign-changing function that may be negative near the boundary and h_1, h_2, f_1, f_2 are nondecreasing functions and satisfy in additional conditions that We shall express in the following. Additionally, we using weak subsolution and supersolution methods and introduce the energy functional association to our problem, and subsequently analyze the existence of a solution within the framework of the energy functional we will discuss in context the existence of solutions for said problem.

Keywords: Positive solutions; quasilinear elliptic system; supersolution and subsolution; energy functional.

2000 Mathematics subject classification: $35J60, 35B30, 35B40$.

[1]Corresponding author: rezakaramali918@gmail.com
Received: 12 March 2024
Revised: 17 April 2024
Accepted: 02 May 2024
How to Cite: Karamali, Gholamreza. Positive Solutions for Quasilinear System by Using of Weak Sub-Super Solutions Method and Energy Function, Casp.J. Math. Sci.,**13**(1)(2024), 155-170.

1. **Introduction**

In almost all physical and engineering problems, an unknown function to be appears in an equation and must be determined. More generally, not only there is an unknown, but also its various derivatives in different combinations do appear in function the equations as well leading to a partial differential equation (PDE). So the mathematical problem is to find the unknown function from the equation and study its properties. This will not be an easy task at all as the various physical phenomena are diverse in nature and this diversity is reflected in the equations as well. For the same reason, obtaining very general results is quite hard, in fact it is almost impossible. Partial differential equations ($PDEs$) are central to mathematics be it pure or applied. That fact is a consequence of the interplay between PDEs and their real-world applications. This is in keeping with the philosophy that the basic simple structure of many PDEs enables knowledge-holders to make a quantitative model of almost any continuous process occurring around them. More exactly, PDEs arise in mathematical models whose dependent variables vary continuously as functions of several independent variables, usually space and time. Sub- and supersolutions have played an important role in the study of nonlinear boundary value problems for elliptic partial differential equations for a long time. While some of the underlying principles are already present in the Perron process for obtaining harmonic functions satisfying (in a generalized sense) given boundary data [12]. Scorza-Dragoni s paper [13] was one of the earliest works, where the existence of an ordered pair of solutions of differential inequalities was used to establish the existence of a solution of a given boundary value problem for a nonlinear second order ordinary differential equation. This was followed by some fundamental work of Nagumo [16] which inspired much work on such problems subject to Dirichlet boundary conditions for both ordinary and partial differential equations during the decade of the sixties [14, 15].

In section 2 we express preliminaries about the quasilinear system of boundary value the problem, in section 3, we express the main results and some theorem about the problem and then in section 4, we use weak subsolution and supersolution methods and introduce the energy functional association to our problem, and subsequently analyze the existence of a solution within the framework of the energy functional for said problem.

2. **Preliminaries**

In this paper we consider the quasilinear system of boundary value problem

$$\begin{cases} -\Delta_p u = \lambda g(x) f_1(u,v) + \mu h_1(u) \ in \ x \in \Omega \\ -\Delta_q v = \lambda g(x) f_2(u,v) + \mu h_2(v) \ in \ x \in \Omega \ (1) \\ u(x) = v(x) = 0 \ on \ x \in \partial\Omega \ , \end{cases}$$

Where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta_z u = div(|\nabla u|^{z-2}\nabla u)$ that is called z-Laplacian operator for $z > 1$, $\lambda > 0$ is a parameter.

In [1, 8] the authors considered the existence of positive solutions for the p-Laplacian system with large λ

$$\begin{cases} -\Delta_p u = \lambda f(v) \ in \ x \in \Omega \\ -\Delta_q v = \lambda g(u) \ in \ x \in \Omega \ (2) \\ u(x) = v(x) = 0 \ on \ x \in \partial\Omega \ . \end{cases}$$

such that $f(z), g(z)$ are the increasing function in $[0, \infty)$ and satisfy in additional conditions. Casheng Chen studied the existence of a positive weak solution for the quasilinear elliptic system with each $\lambda > 0$ see [9, 10]. In [5] the authors considered the system:

$$\begin{cases} -\Delta_p u = \lambda f(u,v) \ in \ x \in \Omega \\ -\Delta_q v = \lambda g(u,v) \ in \ x \in \Omega \ (3) \\ u(x) = v(x) = 0 \ on \ x \in \partial\Omega \ . \end{cases}$$

They discussed the existence of a large positive solution for λ large when

$$\lim_{x\to\infty} \frac{f(M(g(x,x))^{\frac{1}{x^{q-1}}}}{x^{p-1}} = 0$$

for every $M > 0$ and $\lim_{x\to\infty} \frac{g(x,x)}{x^{p-1}} = 0$. In this paper, we shall prove upon conditions on functions that our problem admits a positive weak solution with each $\lambda, \mu > 0$. To end this, we use the method of sub-super solutions, see [4, 6, 7].

In this paper, we focus on sign-changing weight functions g and also do not assume any sign condition on $h_1(0)$ or $h_2(0)$.

To precisely state our theorem we first consider the eigenvalue problem

$$\begin{cases} -\Delta_z u = \lambda |u|^{z-2} u \ in \ x \in \Omega \\ \\ u = 0 \ on \ x \in \partial\Omega \ (z = p, q). \ (4) \end{cases}$$

We make the following assumptions:

$(A.1)$ $h_1, h_2 \in C^1(\bar{\Omega})$ are nondecreasing functions,

$$\lim_{x\to\infty} h_1(x) = \lim_{x\to\infty} h_2(x) = \infty,$$

and

$$\lim_{x\to\infty} \frac{h_1(M(h_2(x))^{\frac{1}{x^{q-1}}}}{x^{p-1}} = 0,$$

for all $M > 0$.

$(A.2)$ $f_1, f_2 \in C^1(\bar{\Omega} \times \bar{\Omega})$ are nonnegative, nondecreasing functions and

$$\lim_{u\to\infty} \frac{f_1(u,.)}{u^{p-1}} = 0, \lim_{v\to\infty} \frac{f_2(.,v)}{v^{q-1}} = 0.$$

Definition 2.1. A pair of nonnegative functions (ψ_1, ψ_2) and (φ_1, φ_2) are called a weak subsolution and supersolution of our problem (1), if they satisfy $\psi_i(x) \leq \varphi_i(x)$ in Ω for $i = 1, 2$ and

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w_1 dx \leq \lambda \int_\Omega g(x) f_1(\psi_1, \psi_2) w_1 dx + \mu \int_\Omega h_1(\psi_1) w_1 dx \ (5)$$

$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w_2 dx \leq \lambda \int_\Omega g(x) f_2(\psi_1, \psi_2) w_2 dx + \mu \int_\Omega h_2(\psi_2) w_2 dx \ (6)$$

$$\int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1.\nabla w_1 dx \geq \lambda \int_\Omega g(x) f_1(\varphi_1, \varphi_2) w_1 dx + \mu \int_\Omega h_1(\varphi_1) w_1 dx \ (7)$$

$$\int_\Omega |\nabla\varphi_2|^{q-2}\nabla\varphi_2.\nabla w_2 dx \geq \lambda \int_\Omega g(x) f_2(\varphi_1, \varphi_2) w_2 dx + \mu \int_\Omega h_2(\varphi_2) w_2 dx \ (8)$$

for $w_1, w_2 \in W = \{\phi \in C_0^\infty(\bar{\Omega}) : \phi \geq 0 \ in \ \Omega\}$.

Here, we assume that the weight g takes nonnegative values in Ω but requires g to be strictly positive in $\Omega - \partial\bar{\Omega}_\delta$ then, there exist positive β, η constants such that $g(x) \geq -\beta$ on Ω_δ and $g(x) \geq \eta$ on $\Omega - \partial\bar{\Omega}_\delta$. Let $t_0 > 0$ be such that $\eta f_1(t,.) + h_1(t) > 0$ and $\eta f_2(.,t) + h_2(t) > 0$ for any $t > t_0$ and $f_{01} = \max\{0, -f_1(0,.)\}$ and $f_{02} = \max\{0, -f_2(.,0)\}$. for $\gamma > \frac{t_0}{\alpha}$ we define

$$A(\gamma) = \min\{\frac{m\gamma^p}{\beta f_1((\frac{p}{p-1})\gamma^{\frac{p}{p-1}},.) + f_{01}}, \frac{m\gamma^q}{\beta f_2(.,(\frac{q}{q-1})\gamma^{\frac{q}{q-1}}) + f_{02}}\},$$

$$B(\gamma) = \max\{\frac{\gamma^p\lambda_1}{\eta f_1((\frac{p}{p-1})\gamma^{\frac{p}{p-1}}\alpha,.) + h_1((\frac{q}{q-1})\gamma^{\frac{q}{q-1}}\alpha)}, \frac{\gamma^q\lambda_2}{\eta f_2(.,(\frac{q}{q-1})\gamma^{\frac{q}{q-1}}\alpha) + h_2((\frac{p}{p-1})\gamma^{\frac{p}{p-1}}\alpha)}\},$$

where $\alpha = \min\{\sigma_1^{\frac{p}{p-1}}, \sigma_2^{\frac{q}{q-1}}\}$.

Let $W^{1,p}(\Omega)$ be the Sobolev space and $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Definition 2.2. Let $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ we define the energy functional $\Phi : X \to R$ associated to problem (1) as follows:

$$\Phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \int_\Omega [\lambda g(x) f(u,v) + \mu h(u)].$$

then ϕ is C^1 functional whose critical points are solutions to problem (1).

3. Main result

Theorem 3.1. *Suppose that (A.1)-(A.2) hold and* $G = \{t > \frac{t_0}{\alpha} : B(t) \leq A(t)\} \neq \emptyset$. *Let* $T = \bigcup_{t\in G}[B(t), A(t)]$, *for* $(\lambda + \mu) \in T$ *then the quasilinear system of boundary value problem (1) has at least one positive solution.*

Proof. Let λ_1, λ_2 be the respective first eigenvalues of Δ_p, Δ_q with Dirichlet boundary conditions and Φ_1, Φ_2 the corresponding eigenfunctions with

$$\Phi_1, \Phi_2 > 0,$$

and $\|\Phi_1\|_\infty = \|\Phi_2\|_\infty = 1$. Since $|\nabla\Phi_i| \neq 0$ on $\partial\Omega$ and also $\Phi_i = 0$ on $\partial\Omega$ for $i = 1, 2$, there exist $\delta \geq 0$ and $\sigma_1, \sigma_2 \in (0, 1]$ and $m > 0$ for which

$$\begin{cases} |\nabla\Phi_1|^p - \lambda_1\Phi_1^p \geq m \ on \ \partial\bar{\Omega}_\delta \ (9) \\ \\ \Phi_1 \geq \sigma_1 \ on \ \Omega - \partial\bar{\Omega}_\delta \end{cases},$$

$$\begin{cases} |\nabla\Phi_2|^q - \lambda_2\Phi_2^q \geq m \ on \ \partial\bar{\Omega}_\delta \ (10) \\ \\ \Phi_2 \geq \sigma_2 \ on \ \Omega - \partial\bar{\Omega}_\delta \end{cases},$$

Where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$.

Let $\lambda+\mu \in T$ and $\gamma > \frac{t_0}{\alpha}$ be such that $\lambda+\mu \in [B(t), A(t)]$. We will prove that

$$(\psi_1, \psi_2) = [((\lambda+\mu)^{\frac{1}{p-1}}/k)\gamma^{\frac{1}{p-1}}\Phi_1^k, ((\lambda+\mu)^{\frac{1}{q-1}}/l)\gamma^{\frac{1}{q-1}}\Phi_2^l]$$

is a sub-solution, where $k = \frac{p}{p-1}$ and $l = \frac{q}{q-1}$, then it follows from (9)

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w_1 dx = \gamma^p(\lambda+\mu)\int_\Omega \Phi_1|\nabla\Phi_1|^{p-1}\nabla\Phi_1.\nabla w_1 dx =$$

$$\gamma^p(\lambda+\mu)\int_\Omega [|\nabla\Phi_1|^{p-}\nabla\Phi_1.\nabla(\Phi_1 w_1) - |\nabla\Phi_1|^p w_1]dx =$$

$$\gamma^p(\lambda+\mu)\int_\Omega (\lambda_1\Phi_1^p - |\nabla\Phi_1|^p)w_1 dx.$$

Similarly, we have from (10) that

$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w_2 dx = \gamma^q(\lambda+\mu)\int_\Omega (\lambda_2\Phi_2^q - |\nabla\Phi_2|^q)w_2 dx.$$

(a) If $x \in \bar{\Omega_\delta}$, since $(\lambda+\mu) \le A(\gamma)$ so $(\lambda+\mu) \le \frac{m\gamma^p}{\beta f_1((\frac{p}{p-1})\gamma^{\frac{p}{p-1}},.)+f_{01}}$,

then $-m\gamma^p \le (\lambda+\mu)[-\frac{m\gamma^p}{\beta f_1((\frac{p}{p-1})\gamma^{\frac{p}{p-1}},.)-f_{01}}]$ and

$$\int_{\bar{\Omega_\delta}} |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w_1 dx = \gamma^p(\lambda+\mu)\int_{\bar{\Omega}} \Phi_1|\nabla\Phi_1|^{p-2}\nabla\Phi_1.\nabla w_1 dx \le$$

$$\gamma^p(\lambda+\mu)\int_{\bar{\Omega_\delta}} (-m)w_1 dx \le (\lambda+\mu)\int_{\bar{\Omega_\delta}} [-\frac{m\gamma^p}{\beta f_1((\frac{p}{p-1})\gamma^{\frac{p}{p-1}},.)+f_{01}}]w_1 dx \le$$

$$\lambda\int_{\bar{\Omega_\delta}} g(x)f_1(\psi_1,\psi_2)w_1 dx + \mu\int_{\bar{\Omega_\delta}} h_1(\psi_1)w_1 dx.$$

(b) If $x \in \Omega - \bar{\Omega_\delta}$, since $(\lambda+\mu) \ge B(\gamma)$

so

$$(\lambda+\mu) \ge \frac{\gamma^p\lambda_1}{\eta f_1((\frac{p}{p-1})\gamma^{\frac{p}{p-1}}\alpha,.) + h_1((\frac{q}{q-1})\gamma^{\frac{q}{q-1}}\alpha)} \ ,$$

then

$$\int\limits_{\Omega-\bar{\Omega}_\delta} |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w_1 dx = \gamma^p(\lambda+\mu)\int\limits_{\Omega-\bar{\Omega}_\delta} \Phi_1|\nabla\Phi_1|^{p-2}\nabla\Phi_1.\nabla w_1 dx \leq$$

$$\gamma^p(\lambda+\mu)\int\limits_{\Omega-\bar{\Omega}_\delta}\lambda_1 w_1 dx \leq \gamma^p(\lambda+\mu)\int\limits_{\Omega-\bar{\Omega}_\delta}\frac{\gamma^p\lambda_1}{\eta f_1((\frac{p}{p-1})\gamma^{\frac{p}{p-1}}\alpha,.)+h_1((\frac{q}{q-1})\gamma^{\frac{q}{q-1}}\alpha)} w_1 dx \leq$$

$$\lambda\int\limits_{\Omega-\bar{\Omega}_\delta} g(x)f_1(\psi_1,\psi_2)w_1 dx + \mu\int\limits_{\Omega-\bar{\Omega}_\delta} h_1(\psi_1)w_1 dx.$$

Now for ψ_2

(a) If $x\in\bar{\Omega}_\delta$ then $\lambda+\mu\leq A(\gamma)$, therefore

$$\lambda+\mu\leq\frac{m\gamma^q}{\beta f_2(.,(\frac{q}{q-1})\gamma^{\frac{q}{q-1}})+f_{02}},$$

then

$$-m\gamma^q\leq(\lambda+\mu)[-\beta f_2(.,\frac{q}{q-1}\gamma^{\frac{1}{q-1}})-f_{02}],$$

so

$$\int\limits_{\bar{\Omega}_\delta}|\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w_2 dx=\gamma^q(\lambda+\mu)\int\limits_{\bar{\Omega}}\Phi_2|\nabla\Phi_2|^{q-2}\nabla\Phi_2.\nabla w_2 dx\leq$$

$$\gamma^q(\lambda+\mu)\int\limits_{\bar{\Omega}_\delta}(-m)w_2 dx\leq(\lambda+\mu)\int\limits_{\bar{\Omega}_\delta}[-\frac{m\gamma^q}{\beta f_2(.,(\frac{q}{q-1})\gamma^{\frac{q}{q-1}})+f_{02}}]w_2 dx\leq$$

$$\lambda\int\limits_{\bar{\Omega}_\delta}g(x)f_2(\psi_1,\psi_2)w_2 dx+\mu\int\limits_{\bar{\Omega}_\delta}h_2(\psi_2)w_2 dx.$$

(b) If $x\in\Omega-\bar{\Omega}_\delta$ then $(\lambda+\mu)\geq B(\gamma)$ thus

$$(\lambda+\mu)\geq\frac{\gamma^q\lambda_2}{\eta f_2(.,(\frac{q}{q-1})\gamma^{\frac{q}{q-1}}\alpha)+h_2((\frac{p}{p-1})\gamma^{\frac{p}{p-1}}\alpha)}\ .$$

Therefore

$$\int_{\Omega-\bar{\Omega_\delta}} |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w_2 dx = \gamma^q(\lambda+\mu)\int_{\Omega-\bar{\Omega_\delta}} \Phi_2|\nabla\Phi_2|^{q-2}\nabla\Phi_2.\nabla w_2 dx \leq$$

$$\gamma^q(\lambda+\mu)\int_{\Omega-\bar{\Omega_\delta}} \lambda_2 w_2 dx \leq \gamma^q(\lambda+\mu)\int_{\Omega-\bar{\Omega_\delta}} \frac{\gamma^q\lambda_2}{\eta f_2(.,(\frac{q}{q-1})\gamma^{\frac{q}{q-1}}\alpha)+h_2((\frac{p}{p-1})\gamma^{\frac{p}{p-1}}\alpha)} w_1 dx \leq$$

$$\lambda\int_{\Omega-\bar{\Omega_\delta}} g(x)f_2(\psi_1,\psi_2)w_2 dx + \mu\int_{\Omega-\bar{\Omega}_\delta} h_2(\psi_2)w_2 dx.$$

Therefore (ψ_1,ψ_2) is a sub-solution of our problem.

Next, we construct a supersolution (φ_1,φ_2) of our problem. Let $e_1(x), e_2(x)$ be the positive solutions of the following problems, respectively.

$$\begin{cases} -\Delta_p e_1 = 1 \ in \ x\in\Omega, \ e_1(x)=0 \ on \ x\in\partial\Omega \\ -\Delta_q e_2 = 1 \ in \ x\in\Omega, \ e_2(x)=0 \ on \ x\in\partial\Omega \ . \ (11) \end{cases}$$

Now, we prove by demonstrating that for a sufficiently large value of the constant c. $(\varphi_1,\varphi_2) = (\frac{c}{\|e_p\|}(\lambda+\mu)^{\frac{1}{p-1}}e_p, [2f_2(.,c(\lambda+\mu)^{\frac{1}{p-1}})]^{\frac{1}{q-1}}(\lambda+\mu)^{\frac{1}{q-1}}e_q)$ is a super-solution, where (e_p,e_q) is the unique positive solution of

$$\begin{cases} -\Delta_p e_1 = 1 \ in \ x\in\Omega, \ e_1(x)=0 \ on \ x\in\partial\Omega \\ -\Delta_q e_2 = 1 \ in \ x\in\Omega, \ e_2(x)=0 \ on \ x\in\partial\Omega \ . \ (12) \end{cases}$$

By $(A.1)$ and choosing c sufficiently large value we have

$$\frac{\|e_p\|_\infty[\|g\|_\infty f_1(c(\lambda+\mu)^{\frac{1}{p-1}},.)+h_1([2f_2(.,c(\lambda+\mu)^{\frac{1}{p-1}})]^{\frac{1}{q-1}}(\lambda+\mu)^{\frac{1}{q-1}}e_q)]}{c^{p-1}} \leq 1 \ , \ (13)$$

then

$$\int\limits_{\Omega} |\nabla\varphi_1|^{p-2}\nabla\varphi_1.\nabla w_1 dx = (\lambda+\mu)(\frac{c}{\|e_p\|_\infty})^{p-1}\int\limits_{\Omega}|\nabla e_p|^{p-2}\nabla e_p.\nabla w_1 dx =$$

$$(\lambda+\mu)(\frac{c}{\|e_p\|_\infty})^{p-1}\int\limits_{\Omega} w_1 dx \geq$$

$$(\lambda+\mu)\int\limits_{\Omega}[\|g\|_\infty f_1(c(\lambda+\mu)^{\frac{1}{p-1}},.)+h_1([2f_2(.,c(\lambda+\mu)^{\frac{1}{p-1}})])]w_1 dx \geq$$

$$(\lambda+\mu)\int\limits_{\Omega}[g(x)f_1(c(\lambda+\mu)^{\frac{1}{p-1}}\frac{e_p}{\|e_p\|_\infty},.)+h_2(\varphi_2)]w_2 dx,$$

and similarly by c sufficiently large value and assumptions

$$\frac{\|g\|_\infty f_2(.,[2f_2(.,c(\lambda+\mu)^{\frac{1}{p-1}})]^{\frac{1}{q-1}}(\lambda+\mu)^{\frac{1}{q-1}}e_q)}{h_2(c(\lambda+\mu)^{\frac{1}{p-1}})} \leq 1\ ,\ (14)$$

then

$$\int\limits_{\Omega}|\nabla\varphi_2|^{q-2}\nabla\varphi_2.w_2 dx = 2(\lambda+\mu)h_2(c(\lambda+\mu)^{\frac{1}{p-1}})\int\limits_{\Omega} w_2 dx \geq$$

$$(\lambda+\mu)\int\limits_{\Omega}|g|_\infty f_2(.,2(\lambda+\mu)h_2(c(\lambda+\mu)^{\frac{1}{p-1}})+h_2(c(\lambda+\mu)^{\frac{1}{p-1}}))w_2 dx \geq$$

$$\lambda\int\limits_{\Omega}(g(x)f_2(\varphi_1,\varphi_2)+\mu\int\limits_{\Omega}h_2(\varphi_2))w_2 dx.$$

Theorem 3.2. *Assume that $u\mapsto f_1(u,v), u\mapsto f_2(u,v)$ are nondecreasing for a.e $u>0$ and $v\mapsto f_1(u,v), v\mapsto f_2(u,v)$ are nondecreasing for a.e $v>0$ and $f_1(0,v)=f_2(u,0)=0$ for a.e $u,v>0$, and there exists $C>0$ such that $|f_1(u,v)|\leq C(1+|u|^{p-1})$ and $|f_2(u,v)|\leq C(1+|v|^{q-1})$. Also, suppose that problem (1) admits a supersolution $(\bar{u},\bar{v})$ and a subsolution (u,v) with $u\leq\bar{u}$ and $v\leq\bar{v}$ in Ω, then problem (1) has a solution (u_0,v_0) with $u_0,v_0>0$ in Ω.*

Proof. For a functions $f_i : \Omega \times \Omega \to R$ for $i = 1, 2$ we define

$$F_i(r,t) = \begin{cases} f_i(r,t)\ u \le r \le \bar{u},\ v \le t \le \bar{v} \\ f_i(u,t)\ r \le u,\ v \le t \le \bar{v} \\ f_i(r,v)\ u \le r \le \bar{u},\ t \le v \\ f_i(u,\bar{v})\ u \le r,\ t \le v \\ f_i(\bar{u},t)\ r \le \bar{u},\ v \le t \le \bar{v} \\ f_i(\bar{u},\bar{v})\ \bar{u} \le r,\ \bar{v} \le t \\ f_i(u,\bar{v})\ r \le u,\ \bar{v} \le t \\ f_i(\bar{u},v)\ \bar{u} < r, t < v \\ f_i(r,\bar{v})\ u \le r \le \bar{u},\ \bar{v} < t \end{cases}$$

We will construct two sequences $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$, $n \in \mathbb{N}$ for the following problem

$$\begin{cases} -\Delta_p u = \lambda g(x) F_1(u,\bar{v}) + \mu h(u)\ in\ \Omega \\ u = o\ on\ \partial\Omega.\ (2) \end{cases}$$

The energy functional associated with the above system is

$$\Phi(u,v) = \frac{1}{p}\int |\nabla u|^p - \int_\Omega \int_{[0,u]} [\lambda g(x) F_1(r,\bar{v}) + \mu h(u)] dr dx,$$

Which is bounded of below in $W_0^{1,p}(\Omega)$. Therefore the infimum of $\Phi(.,.)$ is achieved at some point $u_1 \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$ which is a solution of problem (2). We claim that $u(x) \le u_1(x) \le \bar{u}(x),\ \forall x \in \Omega$. Indeed assume that the set $A := \{x \in \Omega\ :\ u_1(x) \le u(x)\}$ is nonempty. Since it must have positive measures and

$$-\Delta_p u_1 = \lambda g(x) f_1(u_1,\bar{v}) + \mu h(u_1)\ in\ A$$

while

$$-\Delta_p u \le \lambda g(x) f_1(u,\bar{v}) + \mu h(u)\ in\ A.$$

multiplying two above relations with $u - u_1$ and integrating over A we get

$$\int_A |\nabla u_1|^{p-2}\nabla u_1 \nabla(u - u_1) =$$
$$\int_A [\lambda g(x) f_1(u_1, \bar{v}) + \mu h(u_1)](u - u_1)$$

and

$$\int_A |\nabla u|^{p-2}\nabla u \nabla(u - u_1) \leq$$
$$\int_A [\lambda g(x) f_1(u, \bar{v}) + \mu h(u)](u - u_1)$$

which combined yield

$$\int_A [|\nabla u|^{p-2}\nabla u - |\nabla u_1|^{p-2}\nabla u_1]\nabla(u - u_1) < 0,$$

so contradicting the strong monotonicity of the $-\Delta_p$ operator. Therefore A is empty. Similarly, if we set $B : \{x \in \Omega \ \bar{u}(x) \leq u_1(x)\}$ then we can prove that B is empty. Thus $u_1(x) \leq \bar{u}(x)$.

Now we consider the problem

$$\begin{cases} -\Delta_q v = \lambda g(x) F_2(u_1, v) + \mu h(u_1) \ in \ \Omega \\ \\ v = o \ on \ \Omega. \ (3) \end{cases}$$

Working as problem (2) we can show that it admits a solution $v_1 \in W_0^{1,q}(\Omega) \cap C^1(\bar{\Omega})$ with $v(x) \leq v_1(x) \leq \bar{v}(x)$. Assuming that $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$ for $n = 1, 2, ..., k-1$ have been defined, we let $u_k \in W_0^{1,p}(\Omega)$ be a solution of problem (2) with v_{k-1} in the place of $\bar{v}$ and $v_k \in W_0^{1,q}(\Omega)$ be a solution of problem (3) with u_k in the place of u_1. Since F_i for $i = 1, 2$ are bounded, the sequences $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$ are bounded, therefore $u_n \to u_0$ weakly in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ weakly in $W_0^{1,q}(\Omega)$. By the continuity of $f_1(.,.)$ and $f_2(.,.)$ and Sobolove embedding, we easily deduce that (u_0, v_0) is a solution of the system

$$\begin{cases} -\Delta_p u = \lambda g(x) F_1(u, \bar{v}) + \mu h(u) \ in \ \Omega \\ u = o \ on \ \Omega, \\ -\Delta_q v = \lambda g(x) F_2(u_1, v) + \mu h(u_1) \ in \ \Omega \\ v = o \ on \ \Omega, \end{cases}$$

while

$$u(x) \leq u_0(x) \leq \bar{u}(x),$$
$$v(x) \leq v_0(x) \leq \bar{v}(x),$$

for all $x \in \Omega$. Therefore

$$F_1(u_0, v_0) = f_1(u_0, v_0), \ F_2(u_0, v_0) = f_2(u_0, v_0).$$

Consequently, (u_0, v_0) is a critical point of $\Phi(.,.)$ and therefore a solution of problem (1). By hypotheses we have

$$-\Delta_p u < f_1(u, v) \leq f_1(u_0, v_0) = -\Delta_p u_0 \ in \ \Omega$$

and so by the strong comparison principle in [11] we deduce that $0 \leq u < u_0 \ in \ \Omega$. By similar way we can show that $v_0 > 0 \ in \ \Omega$.

In the next section, we use weak subsolution and supersolution methods and analyze the existence of a solution within the framework of the energy functional for said problem.

4. L-sub and L-super solutions

Let $W^{1,p}(\Omega)$ be the Sobolev space and $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The norms in $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are defined by

$$\|f\|_{L^p(\Omega)} = (\int_\Omega |f|^p dx)^{\frac{1}{p}} \ and \ \|f\|_{W^{1,p}(\Omega)} = \sum_{|\beta| \leq 1} \|D^\beta f\|_{L^p(\Omega)}.$$

Definition 4.1. A function (u, v) is said to be a weak L-subsolution of equation (1) in Ω

$$u = \max\{u_i \ : \ u_i \ is \ a \ weak \ subsolution \ of \ (1)\}$$

and $u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$v = \max\{v_i \ : \ v_i \ is \ a \ weak \ subsolution \ of \ (1)\}$$

and $v_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Definition 4.2. A function (u, v) is said to be a weak L-supersolution of equation (1) in Ω

$$u = \max\{u_i \ : \ u_i \ is \ a \ weak \ supersolution \ of \ (1)\}$$

and $u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$v = \max\{v_i \ : \ v_i \ is \ a \ weak \ supersolution \ of \ (1)\}$$

and $v_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.3. *Let $(\hat{u}, \hat{v})$ and $(\check{u}, \check{v})$ be respectively weak L-subsolution and weak L-supersolution of problem (1) in Ω such that $\hat{u} \le \check{u}$ and $\hat{v} \le \check{v}$ a.e in Ω. Suppose that there exist two constants $\alpha, \beta \in (0,]$ and functions $f, g \in L^1(\Omega)$ such that*

$$\begin{cases} |B_1(x, u, v)| \le j_p(|u|)(|f(x)| + |u|^{p-\alpha}) \\ \\ |B_2(x, u, v)| \le j_q(|v|)(|g(x)| + |v|^{q-\beta}) \\ \end{cases}$$

for all $x \in \Omega$ and $(u, v) \in \mathbb{R} \times \mathbb{R}$ where $j : [0, \infty) \to [0, \infty)$ is a nondecreasing function and

$$B_1(x, u, v) = \lambda g(x) f_1(u, v) + \mu h_1(u)$$

and

$$B_2(x, u, v) = \lambda g(x) f_2(u, v) + \mu h_2(v).$$

Then the problem (1) has a positive solution (u, v) such that $\hat{u} \le u \le \check{u}$ and $\hat{v} \le v \le \check{v}$ a.e in Ω.

Proof. Since $(\hat{u}, \hat{v})$ and $(\check{u}, \check{v})$ are weak $L-$subsolution and weak $L-$supersolution therefore

$$\hat{u} = \max\{\hat{u}_i \ : \ \hat{u}_i \ is \ weak \ subsolution \ \forall\, i = 1, .., n \ \hat{u}_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)\},$$

$$\hat{v} = \max\{\hat{v}_i \ : \ \hat{v}_i \ is \ weak \ subsolution \ \forall\, i = 1, .., n \ \hat{v}_i \in W^{1,q}(\Omega) \cap L^\infty(\Omega)\}$$

and

$$\check{u} = \min\{\check{u}_i \ : \ \check{u}_i \ is \ weak \ supersolution \ \forall\, i = 1, ..., n \ \check{u}_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega),$$

$$\check{v} = \min\{\check{v}_i \ : \ \check{v}_i \ is \ weak \ supersolution \ \forall\, i = 1, ..., n \ \check{v}_i \in W^{1,q}(\Omega) \cap L^\infty(\Omega).$$

For any $i = 1, ..., n$ and every $\varphi \in C_0^\infty(\Omega)$ we have

$$(15) \quad \int_\Omega |\nabla \hat{u}_i|^{p-2} \nabla \hat{u}_i . \nabla \varphi \le \int_\Omega B_1(x, \hat{u}_i, v)\varphi, \ \int_\Omega |\nabla \hat{v}_i|^{q-2} \nabla \hat{v}_i . \nabla \varphi \le \int_\Omega B_1(x, u, \hat{v}_i)\varphi$$

and

$$(16) \quad \int_\Omega |\nabla \check{u}_i|^{p-2} \nabla \check{u}_i . \nabla \varphi \ge \int_\Omega B_2(x, \check{u}_i, v)\varphi, \ \int_\Omega |\nabla \check{v}_i|^{q-2} \nabla \check{v}_i . \nabla \varphi \ge \int_\Omega B_2(x, u, \check{v}_i)\varphi.$$

For functions $F_i : \Omega \times \mathbb{R} \times \mathbb{R}$ for all $i = 1, ..., n$ we define

$$\tilde{F_i}(x,s,t) = \begin{cases} f_i(x,s,t) \ \hat{u}_i(x) \leq s \leq \check{u}_i(x), \ \hat{v}_i(x) \leq t \leq \check{v}_i(x) \\ f_i(x,\hat{u}_i(x),t) \ s < \hat{u}_i(x), \ \hat{v}_i(x) \leq t \leq \check{v}_i(x) \\ f_i(x,s,\hat{v}_i(x)) \ \hat{u}_i(x) \leq s \leq \check{u}_i(x), \ t < \hat{v}_i(x) \\ f_i(x,\hat{u}_i(x),\hat{v}_i(x)) \ s < \hat{u}_i(x), \ t < \hat{v}_i(x) \\ f_i(x,\check{u}_i(x),t) \ \check{u}_i(x) < s, \ \hat{v}_i(x) \leq t \leq \check{v}_i(x) \\ f_i(x,\check{u}_i(x),\check{v}_i(x)) \ \check{u}_i(x) < s, \ \check{v}_i(x) < t \\ f_i(x,\hat{u}_i(x),\check{v}_i(x)) \ \hat{u}_i(x) > s, \ \check{v}_i(x) > t \\ f_i(x,\check{u}_i(x),\hat{v}_i(x)) \ \check{u}_i(x) < s, \ t < \hat{v}_i(x) \\ f_i(x,s,\check{v}_i(x)) \ \hat{u}_i(x) \leq s \leq \check{u}_i(x), \ \check{v}_i(x) < t \end{cases}$$

We will consider two sequences in product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ as follows: we take $(u_n, v_n) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and consider the following problem

$$\begin{cases} -\Delta_p u = B_1(x,u,\check{v}) \ in \ \Omega \\ u = 0 \ on \ \partial\Omega. \ (17) \end{cases}$$

The Euler-Lagrange functional associated with the system (17) is

$$\breve{\Phi}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega \int_{[0,u]} \tilde{B}_1(x,s,\check{v}) ds dx$$

which is bounded from below, weakly lower semi-continuous and coercive in $W_0^{1,p}(\Omega)$. Therefore, the $\inf \breve{\Phi}(.)$ is achieved at some point $u_1 \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ which is a solution of problem (17). We claim that

$$\hat{u}_i(x) \leq \hat{u}(x) \leq u_1(x) \leq \check{u}(x) \leq \check{u}_i(x) \ \forall \ x \in \Omega.$$

Indeed, assume that the set

$$A = \{x \in \Omega \ : \ \hat{u}_i(x) > u_1(x)\}$$

is nonempty. Since it is an open set, the problem has a positive measure and

$$(18) \ -\Delta_p u_1 = B_1(x,u_1,\check{v}) \ in \ A$$

while

$$(19)\ \ -\Delta_p \hat{u}_i < B(x, \hat{u}_i, \check{v}\ in\ A\ \forall\ i = 1, ...n\ .$$

By the relations $(15), (16)$ and multiplying $(18), (19)$ with $\hat{u}_i - u_1$ and integrating over A we obtain

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla(\hat{u}_i - u_1)dx = \int_\Omega B_1(x, u_1, \check{v})(\hat{u}_i - u_1)dx$$

and

$$\int_\Omega |\nabla \hat{u}_i|^{p-2} \hat{u}_i \nabla(\hat{u}_i - u_1)dx = \int_\Omega B_1(x, \hat{u}_i, \check{v})(\hat{u}_i - u_1)dx\ \forall\ i = 1, .., n\ .$$

Therefore, combined yield

$$\int_\Omega [|\nabla \hat{u}_i|^{p-2} \nabla \hat{u}_i - |\nabla u_1|^{p-2} \nabla u_1] \nabla(\hat{u}_i - u_1) < 0$$

contradicting the strong monotonicity of the $-\Delta_p$ operator. Thus A is empty. Similarly, $u_1(x) \leq \check{u}_i(x)$ for any $x \in \Omega$. Consider the following problem

$$\begin{cases} -\Delta_q v = B_2(x, u_1, v)\ in\ \Omega \\ \\ v = 0\ on\ \partial\Omega.\ (20) \end{cases}$$

Working as in (17) we can show that it admits a solution $v_1 \in W_0^{1,q}(\Omega) \cap C^1(\bar{\Omega})$ with

$$\hat{v}_i(x) \leq \hat{v}(x) \leq v_1(x) \leq \check{v}(x) \leq \check{v}_i(x)\ a.e.\ in\ \Omega.$$

Assuming now that $(u_n, v_n) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)\ n = 1, ..., k-1$ have been defined, we let $u_k \in W_0^{1,p}(\Omega)$ be a solution of (17)with v_{k-1} in the place of $\check{v}_i$ and $v_k \in W_0^{1,p}(\Omega)$ be a solution of (20) with u_k in the place of u_1. Since $\tilde{b}_1(x, s, t)$ and $\tilde{b}_2(x, s, t)$ are bounded, the sequences $u_n \in W_0^{1,p}(\Omega)$ and $v_n \in W_0^{1,q}(\Omega)$ are also bonded, thus $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and $v_n \to v$ weakly in $W_0^{1,q}(\Omega)$. By the continuity of $B_1(x, ., .)$ and $B_2(x., .,)$ the sobolve embedding we easily deduce that (u, v) is solution of the system

$$\begin{cases} -\Delta_p v = B_1(x, u, v)\ in\ \Omega \\ \\ -\Delta_q v = B_2(x, u, v)\ in\ \Omega \\ \\ v = u = 0\ on\ \partial\Omega. \end{cases}$$

while

$$\hat{u}_i(x) \le \hat{u}(x) \le u_1(x) \le \check{u}(x) \le \check{u}_i(x)\ \forall\ x \in \Omega$$

and

$$\hat{v}_i(x) \le \hat{v}(x) \le v_1(x) \le \check{v}(x) \le \check{v}_i(x)\ a.e.\ in\ \Omega.$$

Therefore $\tilde{B}_1(x,u,v) = B_1(x,u,v)$ and $\tilde{B}_2(x.,u,v) = B_2(x,u,v)$. Consequently, (u,v) is a critical point of $\phi(.,.)$ and hence a solution of problem (1). By assumption

$$-\Delta_P \hat{u} < B_1(x,\hat{u},\hat{v}) \le B_1(x,u,v) = -\Delta_p u\ in\ \Omega$$

and so by the strong comparison principle in [11] we deduce that

$$0 \le \hat{u} < u\ in\ \Omega.$$

Similarly we can obtain $v > 0\ in\ \Omega$.

References

[1] D.D Hai , R. Shivaji, An existence result on positive solutions for a class of p-Laplacian systems. Nonl. Anal.**56**,(2004)1007–1010.

[2] J. Ali, R. Shivaji, Existence results for classes of Laplacian systems with sign-changing weight . Appl. Math. Lett. **20**,(2007)558–562.

[3] J. Ali, R. Shivaji, Positive solutions for a class of p-Laplacian systems with multiple parameters. J. Math. Anal. Appl. **335**,(2007)1019–1037.

[4] C. Maya, S. Oruganti and R. Shivaji, Positive solutions for classes of p-Laplacian Equations .Diff. Int. Eqns. **(16)** no(6),(2003)757–768.

[5] C. H. Chen, On positive weak solutions for a class of quasilinear elliptic systems. Nonl Anal. **62**,(2005) 751–756.

[6] T.Kura, The weak supersolution-subsolution method for second order quasilinear elliptic equations . Hiroshima Math. J. **19**,(1989) 1–36.

[7] D. A. Kandilakis, M. Magiropoulos, A subsolution-supersolution method for quasilinear systems. Elec. J.Diff. Eqns. **97**,(2005) 1–5.

[8] M. Chhetri, D. D. Hai and R. Shivaji, On positive solutions for a class of p-Laplacian semipositone systems. Dis. Con. Dyn. Syst. **9** no(4),(2003) 1063–1071.

[9] P. Dravak, J. Hernandez, Existence uniqueness of positive solution for some qusilinear elliptic problems. Nonl. Anal. **44**,(2001) 189–204.

[10] L. C. Evens, Partial Differential Equation (Graduate Studies in MathISSN 1065-7339, vol. 19). Amer. Math. Soc. **1998**.

[11] M. Guedda and L. Veron, Quasilinear elliptic equations involving critical Sobolev exponents, Nonlin. Anal. TMA **13**, no 8,(1989) 879-902.

[12] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer Verlag, New York, 2001.

[13] G. Scorza Dragoni, Il problema dei valori ai limiti studiato in grande per le equazioni differenziali del secondo ordine, Math. Ann.**105**, (1931) 133-143.

[14] K. Akô, On the Dirichlet problem for quasi-linear elliptic differential equations of the second order, J. Math. Soc. Japan **13**, (1961) 45-62.

[15] L. Jackson and K. Schrader, Comparison principles for nonlinear differential equations, J. Differential Equations **3**, (1967) 248 255.

[16] M. Nagumo,*Ü*ber das Randwertproblem der nichtlinearen gewöhnlichen Differentialgleichung zweiter Ordnung, Proc. Phys. Math. Soc. Japan **24**,(1942),845 851.