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Application of Daubechies wavelets for solving Kuramoto-Sivashinsky type equations

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Abstract. We show how Daubechies wavelets are used to solve Kuramoto-Sivashinsky type equations with periodic boundary condition. Wavelet bases are used for numerical solution of the Kuramoto-Sivashinsky type equations by Galerkin method. The numerical results in comparison with the exact solution prove the efficiency and accuracy of our method.

Keywords: Daubechies wavelets; Connection coefficients; Kuramoto-Sivashinsky type equations

2000 Mathematics subject classification: 65T60, 42C40

1. INTRODUCTION

In this work, we consider the Kuramoto-Sivashinsky (KS) type equations

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \quad a \leq x \leq b,\, t \geq 0 \tag{1.1}$$

where α, β and γ are real constants. KS equation was derived from Kuramoto in order to study dissipative structure of reaction-diffusion[12]. This equation was originally derived from the context of plasma instabilities, flame front propagation, and phase turbulence in reaction-diffusion system [16]. It is one of the simplest partial differential equations which is capable of exhibiting chaotic behavior. The chaotic behavior typically

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occurs when Eq.(1.1) is integrated over finite x-domain with a periodic boundary condition.

Equation (1.1) is called Burgers equation if $\beta = \gamma = 0$. This equation is often used to test numerical methods because an analytical expression for its solution is available for different sets of boundary and initial conditions. However, some researchers call it KdV-Burgers-Kuramoto (KBK) equation[10, 14].

The KS equation has been studied numerically by many researchers, see [9, 11, 13, 22]. Also many methods have been developed to construct the exact solutions of KS equation. For more details readers are referred to [20, 23].

Wavelet analysis is a numerical concept which allows one to represent a function in terms of a set of basis functions, called wavelets, which are localized both in location and scale. In wavelet applications to the solution of partial differential equations the most frequently used wavelets are those with compact support introduced by Daubechies [6]. Several studies explored the sage of Daubechies wavelets to solve partial differential equations such as [3, 8, 17, 18, 21].

The goal of this project is to find the numerical solution of Kuramoto-Sivashinsky type equations by using Daubechies scaling functions as a spatial approximation for derivatives of u. Many different methods could be applied for time discretization most of which are based on a Taylor series expansion in time such as backward Euler, Crank-Nicolson or Leap frog methods. Here, we introduce a three-step method based on a Taylor series proposed in [17] as follow

$$\begin{cases} u(x, t + \frac{\Delta t}{3}) \simeq u(x,t) + \dfrac{\Delta t}{3}\dfrac{\partial u}{\partial t}(x,t) \\ u(x, t + \frac{\Delta t}{2}) \simeq u(x,t) + \dfrac{\Delta t}{2}\dfrac{\partial u}{\partial t}(x, t + \frac{\Delta t}{3}) \\ u(x, t + \Delta t) \simeq u(x,t) + \Delta t\dfrac{\partial u}{\partial t}(x, t + \frac{\Delta t}{2}). \end{cases} \tag{1.2}$$

So this scheme involves neither complicated expression nor higher order derivatives.

The organization of the paper is as follows. In Section 2, as a background, fundamental properties of Daubechies wavelet functions and connection coefficients are described. In Section 3, we show how wavelets are used to solve KS type equations with periodic boundary condition. Some noteworthy numerical examples are presented in Section 4. Finally, Section 5 provides conclusions of the study.

2. Daubechies wavelets

2.1. **Basic properties.** In this section, the basic properties of Daubechies scaling functions and wavelet functions are reviewed. Before that we mention a basic definition in wavelet analysis which is called multiresolution analysis and Daubechies wavelets conform to the properties of multiresolution analysis. For details see [1].

A multiresolution analysis of $L^2(\mathbb{R})$ is defiend as a sequence of closed subspaces [6] if the following conditions hold:

(1) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset L^2(\mathbb{R})$
(2) $\overline{\cup_j V_j} = L^2(\mathbb{R})$
(3) $\cap_j V_j = \{0\}$
(4) $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$
(5) there exists a $\phi(x) \in V_0$ such that $\{\phi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis in V_0.

Daubechies wavelets have the remarkable properties that they are closely supported, orthogonal under translation and dilation and continuous. To define Daubechies wavelets, consider the two functions $\phi(x)$, the scaling function, and $\psi(x)$, the wavelet function. The scaling function is the solution of the dilation equation

$$\phi(x) = \sqrt{2} \sum_{k=0}^{N-1} a_k \phi(2x-k)$$

where N is an even positive integer and is called the wavelet genus and $\phi(x)$ is normalized such that: $\int_{-\infty}^{\infty} \phi(x)dx = 1$. The wavelet $\psi(x)$ is defined in terms of the scaling function

$$\psi(x) = \sqrt{2} \sum_{k=0}^{N-1} b_k \phi(2x-k)$$

where $b_k = (-1)^k a_{N-1-k}$ for $k = 0, 1, \cdots, N-1$. The scaling function is uniquely characterized by the numbers $a_0, a_1, \cdots, a_{N-1}$ which are called filter coefficients. This choice of filter coefficients implies that

$$supp(\phi) = supp(\psi) = [0, N-1].$$

The translates of the scaling function and wavelet define orthogonal subspaces

$$V_j = span\{\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k) : k \in \mathbb{Z}\}$$

and

$$W_j = span\{\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k) : k \in \mathbb{Z}\}$$

such that W_j is the orthogonal complement of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \bigoplus W_j. \tag{2.1}$$

Relation (2.1) implies that

$$V_0 \subset V_1 \subset \cdots \subset V_{j+1}.$$

Now consider two spaces V_{J_0} and V_J, where $J > J_0$. Applying (2.1) recursively we find that

$$V_J = V_{J_0} \bigoplus \Big(\bigoplus_{j=J_0}^{J-1} W_j \Big). \tag{2.2}$$

Since we have defined V_J for $J < 0$, we can keep going the decomposition in (2.2) for $J_0 \longrightarrow -\infty$ and $J \longrightarrow \infty$ and obtain [1]

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

According to the multiresolution analysis definition (part 1 and 2), any function $f \in L^2(\mathbb{R})$ can be expanded in terms of scaling functions. $P_{V_j} f$, projection of f into the subspace V_j, has the following expansion

$$(P_{V_j} f)(x) = \sum_{l=-\infty}^{\infty} c_{j,l}\phi_{j,l}(x)$$

where

$$c_{j,l} = \langle f, \phi_{j,l} \rangle \tag{2.3}$$

and $(P_{V_j} f) \longrightarrow f$ as $j \longrightarrow \infty$ [7].

In addition, if the projection of f onto the subspace W_j is denoted by P_{W_j}, then from Eq. (2.2), it follows

$$P_{V_J} f = P_{V_{J_0}} f + \sum_{j=J_0}^{J-1} P_{W_j} f, \quad J > J_0. \tag{2.4}$$

The decomposition (2.4) is orthogonal, as, by construction [8],

$$\begin{aligned} \langle \phi_{j,k}, \phi_{j,l} \rangle &= \delta_{k,l}, \\ \langle \psi_{j,k}, \psi_{i,l} \rangle &= \delta_{j,i}\delta_{k,l}, \\ \langle \psi_{j,k}, \phi_{i,l} \rangle &= 0, \quad j \geq i. \end{aligned} \tag{2.5}$$

Another important property of the Daubechies scaling functions is its ability to represent polynomials exactly up to degree $\frac{N}{2} - 1$. More precisely, it is required that [5]

$$x^p = \sum_{k=-\infty}^{\infty} M_k^p \phi(x-k), \quad x \in \mathbb{R} \ \text{ and } \quad p = 0, 1, \cdots, \frac{N}{2} - 1 \tag{2.6}$$

where

$$M_k^p = \int_{-\infty}^{\infty} x^p \phi(x-k)dx, \quad k \in \mathbb{Z} \text{ and } \quad p = 0, 1, \cdots, \frac{N}{2} - 1$$

M_k^p denotes the p^{th} moment of $\phi(x-k)$ and Eq. (2.6) is called moment equation. This property has been used numerously in computing connection coefficients.

Lemma 2.1. *Let $f \in L^2(\mathbb{R})$ be the periodic function with period p. Then periodicity in f induces periodicity in the wavelet coefficients in (2.3) with period 2^j, i.e.,*

$$c_{j,l+2^j p} = c_{j,l}.$$

Hence there are only 2^j distinct periodized wavelets coefficients. The proof of this lemma is found in [15].

2.2. **Connection coefficients.** In general form, an n-term connection coefficient is defined as

$$\Gamma^{d_1,d_2,\cdots,d_n}_{j,l_1,l_2,\cdots,l_n} = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \phi^{(d_i)}_{j,l_i}(x)dx.$$

Since the Daubechies wavelet functions can not be represented in closed form for $N > 2$, analytic calculation of the integrals is not an option. Latto, Resnikoff and Tenenbaum described an exact method for evaluating connection coefficients [4]. In this paper two and three term connection coefficients are used such that, respectively, have representation as follow

$$\Gamma^{0,d}_{0,0,l} = \Gamma^d_l = \int_{-\infty}^{\infty} \phi(x)\phi^{(d)}_l(x)dx, \quad l \in [2-N\,, N-2] \tag{2.7}$$

and

$$\Gamma^{d_1,d_2}_{0,0,l,m} = \Gamma^{d_1,d_2}_{l,m} = \int_{-\infty}^{\infty} \phi(x)\phi^{(d_1)}_l(x)\phi^{(d_2)}_m(x), \tag{2.8}$$

where

$$l = 2-N, \cdots, N-2,\ m = max(2-N, 2-N+l), \cdots, min(N-2, N-2+l).$$

Since Daubechies wavelets have compact support, there are finite nonzero terms for both two and three term connection coefficients. So the shift parameters, l and m, in (2.7) and (2.8) are restricted. Connection coefficients, their properties and computational algorithms are described in [4, 5].

3. Numerical solutions of the KS type equations

3.1. **Discretization in time.** Consider Eq. (1.1) with initial value

$$u(x,0) = u_0(x)$$

and periodic boundary condition

$$u(x,t) = u(x+L,t).$$

Assume that $n \geq 0$, Δt denote the time step such that $t_n = n\Delta t$ and $u(x,t_n) = u^n$, the numerical method begins with discretization of the time by the three-step method as in Eq. (1.2),

$$\begin{aligned}
u^{n+\frac{1}{3}} &= u^n + \frac{\Delta t}{3}[-u^n u_x^n - \alpha u_{xx}^n - \beta u_{xxx}^n - \gamma u_{xxxx}^n],\\
u^{n+\frac{1}{2}} &= u^n + \frac{\Delta t}{2}[-u^{n+\frac{1}{3}} u_x^{n+\frac{1}{3}} - \alpha u_{xx}^{n+\frac{1}{3}} - \beta u_{xxx}^{n+\frac{1}{3}} - \gamma u_{xxxx}^{n+\frac{1}{3}}],\\
u^{n+1} &= u^n + \Delta t[-u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} - \alpha u_{xx}^{n+\frac{1}{2}} - \beta u_{xxx}^{n+\frac{1}{2}} - \gamma u_{xxxx}^{n+\frac{1}{2}}].
\end{aligned} \tag{3.1}$$

3.2. **Discretization in space.** After time discretization, the spatial derivatives of u is approximated by Daubechies scaling functions. Actually Galerkin method utilizes in this part. In Galerkin method a finite number of functions called basis functions are chosen to approximate the exact solution. Here Galerkin bases are constructed from Daubechies functions. For more details about Galerkin method see [19].

Let the solution $u_J(x,t_n)$ of the problem be approximated by its J^{th} level wavelet series at time t_n, i.e.,

$$u_J(x,t_n) = \sum_{k=-\infty}^{\infty} (c_u)_{J,k}(t_n)\phi_{J,k}(x) = \sum_{k=-\infty}^{\infty} (c_u)_{J,k}^n \phi_{J,k}(x). \tag{3.2}$$

Substituting the wavelet series approximation u_J in Eq. (3.2) with its necessary spatial derivatives for u, u_x, u_{xx},u_{xxx} and u_{xxxx} in first equation of (3.1), yields

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} (c_u)_{J,k}^{n+\frac{1}{3}} \phi_{J,k}(x) = &\sum_{k=-\infty}^{\infty} (c_u)_{J,k}^n \phi_{J,k}(x)\\
&+ \frac{\Delta t}{3}\Big(- \sum_{k=-\infty}^{\infty} (c_u)_{J,k}^n \phi_{J,k}(x) \sum_{m=-\infty}^{\infty} (c_u)_{J,m}^n \phi_{J,m}^{(1)}(x)\\
&- \alpha \sum_{k=-\infty}^{\infty} (c_u)_{J,k}^n \phi_{J,k}^{(2)}(x) - \beta \sum_{k=-\infty}^{\infty} (c_u)_{J,k}^n \phi_{J,k}^{(3)}(x)\\
&- \gamma \sum_{k=-\infty}^{\infty} (c_u)_{J,k}^n \phi_{J,k}^{(4)}(x)\Big)
\end{aligned} \tag{3.3}$$

For completing Galerkin method, take the inner product from both sides of Eq. (3.3) with $\{\phi_{J,l}(x)\}_{l=0}^{2^J-1}$

$$
\begin{aligned}
(c_u)_{J,l}^{n+\frac{1}{3}} = (c_u)_{J,l}^{n} + \frac{\Delta t}{3}\Big(&- \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(c_u)_{J,k}^{n}(c_u)_{J,m}^{n}\Gamma_{J,l,k,m}^{0,0,1} \\
&- \alpha \sum_{k=-\infty}^{\infty}(c_u)_{J,k}^{n}\Gamma_{J,l,k}^{0,2} - \beta \sum_{k=-\infty}^{\infty}(c_u)_{J,k}^{n}\Gamma_{J,l,k}^{0,3} \\
&- \gamma \sum_{k=-\infty}^{\infty}(c_u)_{J,k}^{n}\Gamma_{J,l,k}^{0,4}\Big). \qquad (3.4)
\end{aligned}
$$

Using the change of variable $(2^j x - l) \to x$, $n_1 = k - l$ and $n_2 = m - l$ in Eq. (3.4) and by using the properties of connection coefficients, this equation can be written

$$
\begin{aligned}
(c_u)_{J,l}^{n+\frac{1}{3}} = (c_u)_{J,l}^{n} + \frac{\Delta t}{3}\Big(&- 2^{\frac{3J}{2}} \sum_{n_1=2-N}^{N-2}\sum_{n_2=\eta_1}^{\eta_2}(c_u)_{J,n_1+l}^{n}(c_u)_{J,n_2+l}^{n}\Gamma_{n_1,n_2}^{0,0,1} \\
&- 2^{2J}\alpha \sum_{n_1=2-N}^{N-2}(c_u)_{J,n_1+l}^{n}\Gamma_{n_1}^{0,2} \\
&- 2^{3J}\beta \sum_{n_1=2-N}^{N-2}(c_u)_{J,n_1+l}^{n}\Gamma_{n_1}^{0,3} \\
&- 2^{4J}\gamma \sum_{n_1=2-N}^{N-2}(c_u)_{J,n_1+l}^{n}\Gamma_{n_1}^{0,4}\Big) \qquad (3.5)
\end{aligned}
$$

where

$$\eta_1 = \max(2-N\,, n_1+2-N)$$

and

$$\eta_2 = \min(N-2\,, N-2+n_1).$$

A matrix vector form of Eq. (3.5) is

$$\boldsymbol{c}_u^{n+\frac{1}{3}} = \boldsymbol{c}_u^{n} + \frac{\Delta t}{3}\big(-(\boldsymbol{c}_u^{n})^T\boldsymbol{H}(\boldsymbol{c}_u^{n}) - \alpha\boldsymbol{D}^{(2)}\boldsymbol{c}_u^{n} - \beta\boldsymbol{D}^{(3)}\boldsymbol{c}_u^{n} - \gamma\boldsymbol{D}^{(4)}\boldsymbol{c}_u^{n}\big) \quad (3.6)$$

where

$$
\begin{aligned}
&[\boldsymbol{H}]_{<n_1+l>_{2^J},<n_2+l>_{2^J}} = 2^{\frac{3J}{2}}\Gamma_{n_1,n_2}^{0,0,1}, && l = 0,1,\cdots,2^J-1, \\
&[\boldsymbol{D}^d]_{l,<n_1+l>_{2^J}} = 2^{Jd}\Gamma_{n_1}^{0,d}, && n_1 = 2-N,\cdots,N-2, \\
&[\boldsymbol{c}_u^n] = [(c_u)_{J,0}^n\,,\cdots,(c_u)_{J,2^J-1}^n]
\end{aligned}
$$

and

$$n_2 = \max(2-N\,, n_1+2-N),\cdots,\min(N-2\,, N-2+n_1).$$

Here $< n >_p$ is the modulus operator [2] which is defined as

$$< n >_p = n - p\lfloor \frac{n}{p} \rfloor, \quad n, p \in \mathbb{Z}.$$

We can obtain vector coefficients $\boldsymbol{c}_u^{n+\frac{1}{3}}$ of the approximate solution from Eq. (3.6). To determine the vector coefficients $\boldsymbol{c}_u^{n+\frac{1}{2}}, \boldsymbol{c}_u^{n+1}$ the same manner should be taken for the second and third equations in Eq. (3.1). The solution $\boldsymbol{c}_u^n$ gives the coefficients in the approximation $u_J(x, t_n)$ of $u(x, t_n)$.

4. Numerical examples

Exercise 4.1. In this example, we consider the KS equation with periodic boundary condition, represented by $\alpha = \gamma = 1$ and $\beta = 4$. The exact solution is [22]

$$u(x,t) = c_0 + 9 - 15\Big(\tanh\theta + \tanh^2\theta - \tanh^3\theta\Big);$$

with $\theta = k(x - x_0 - ct)$. This solution is evaluated at $t = 0$, as the initial condition. The L_∞ error is obtained in Table (1) for the presented method in different values of J and time with $\Delta t = 0.002$.

Table 1. *Accuracy for Example 4.1*

$c = 6,\ k = \frac{1}{2},\ x_0 = -10,\ N = 6,\ [a,b] = [-100, 100]$		
J	t	L_∞
4	0.02	0.0856
	0.14	0.4316
	0.26	0.5351
5	0.02	0.0006
	0.14	0.0031
	0.26	0.0043
6	0.02	3.2×10^{-5}
	0.14	1.7×10^{-4}
	0.26	2.4×10^{-4}

Exercise 4.2. Consider Eq.(1.1) with $\alpha = \gamma = 0$ and $\beta = 0.0013$. The exact solution is [18]

$$u(x,t) = 3c\ \mathrm{sech}^2\Big(\sqrt{\frac{c}{4\beta}}(x - ct)\Big).$$

The L_∞ and L_2 errors are obtained in Table (2) for the presented method in different values of t and J.

TABLE 2. *Accuracy for Example 4.2*

$c = \frac{1}{3}$, $a = -2$, $b = 2$, $\Delta t = .001$, $N = 6$.			
J	t	L_∞	L_2
4	0.02	10^{-4}	2×10^{-4}
	0.14	8×10^{-4}	18×10^{-4}
	0.26	11×10^{-4}	26×10^{-4}
5	0.02	3.74×10^{-5}	7.16×10^{-5}
	0.14	2.75×10^{-4}	5.104×10^{-4}
	0.26	5.2630×10^{-4}	9.461×10^{-4}

5. CONCLUSIONS

In this paper we presented a numerical scheme for solving the Kuramoto-Sivashinsky type equations. The method employed to find the numerical solutions of these equations is based on the Daubechies scaling functions. This method was applied on two test problems from the literature. The computational results are found to be in good agreement with the exact solutions.

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