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0N REMOVABLE CYCLES IN GRAPHS AND DIGRAPHS

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ABSTRACT. In this paper we define the removable cycle that, if $\Im$ is a class of graphs, $G \in \Im$, the cycle C in G is called removable if $G - E(C) \in \Im$. The removable cycles in Eulerian graphs have been studied. We characterize Eulerian graphs which contain two edge-disjoint removable cycles, and the necessary and sufficient conditions for Eulerian graph to have removable cycles have been introduced. Further, the even and odd removable cycles in Eulerian graphs have also been studied. The necessary and sufficient conditions for regular graphs (digraphs) to have a removable cycles have been characterized. We also define, the removable cycle class.

Keywords : Removable cycle, Connected graph, Eulerian graph.

1. INTRODUCTION

A spanning subgraph of G is a subgraph with vertex set $V(G)$. A factor of a graph G is a spanning subgraph of G. A $k-$factor is a spanning $k-$regular subgraph.

Theorem 1.1. $[2]$*:(Peterson) Every regular graph with positive even degree has a* $2-$*factor.*

Theorem 1.2. $[2]$*: A connected graph G is Euler if and only if the degree of every vertex is even.*

Y. M. BORSE and B.N. Waphare [9], called a cycle C of a graph G removable if $G - E(C)$ is connected. In this paper we defined the removable cycle as following:

Definition 1.3. Let $\Im$ be a class of graphs (digraphs), $G \in \Im$, the cycle C in G is called removable if $G - E(C) \in \Im$.

Also we defined the concept of removable cycle class.

Definition 1.4. Let $\Im$ be a class of graphs, if there is a removable cycle for every graph G in $\Im$, then $\Im$ is called removable cycle class.

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In this paper, a graph may have multiple edges but not loop . $\delta(G)$ denote the minimum degree of G.

Hobbs [1] conjectured that every $2-$connected simple graph of minimum degree at least 4 has a cycle C such that $G-E(C)$ is $2-$connected. This conjecture immediately follows from the following theorem, which can be derived from an older result of Mader[7].

Theorem 1.5. [7]. *If G is an $n-$connected simple graph of minimum degree at least $n+2$, then G contains a cycle C such that $G-E(C)$ is $n-$connected.*

From Theorem (1.5), we can see that the class of $n-$connected simple graphs of minimum degree at least n+2, is removable cycle class.

Prompted by Hobb's conjecture, Jackson[1], independently, proved the following strengthened of Theorem (1.5) in the $2-$connected case: If G is a $2-$connected simple graph of minimum degree at least 4 and $e \in E(G)$, then G contains a cycle C such that $G-E(C)$ is $2-$connected and $e \notin E(G)$. Lemos and Oxley [4] generalized Jackson's result and strengthened Theorem (1.5), further in the $2-$connected case as follows:

Theorem 1.6. [4]. *Let G be a simple $2-$connected graph and let H be a subgraph of G such that H is either $2-$connected or isomorphic to K_2. Suppose that $d_G(v) \geq 4$ for all v in $V(G) \setminus V(H)$ and H^c contains a cycle. Then there exists a cycle C in H^c such that $G-E(C)$ is $2-$connected.*

Y.M.Borse[9] proved a result for connected graphs which is analogous to Theorem (1.6).

In the following theorem, Y.M.Borse obtained the sufficient condition for existence a removable cycle in a connected graph which avoids edges of two connected subgraphs.

Theorem 1.7. [9]. *Let G be a connected graph and let H_1 and H_2 be subgraphs of G. Suppose that there exists at least two cycles in $(H_1 \cup H_2)^c$ and further, no two edges of $(H_1 \cup H_2)^c$ forms an edge cut of G. Then there exists a cycle C in $(H_1 \cup H_2)^c$ such that $G-E(C)$ is connected.*

Further, he proved the following results.

Theorem 1.8. [9]. *Let G be a $3-$edge-connected graph and let x and y be distinct edges of G. Suppose that $|X| \geq 4$ for every edge-cut X of G with $y \in X$. Then G has edge- disjoint cycles C_x and C_y containing x and y, respectively, such that both $G-E(C_x)$ and $G-E(C_y)$ are connected.*

The above theorem is analogous to the following result of Lemos and Oxley [5].

Theorem 1.9. [5]. *Let G be a $2-$connected graph. Suppose that $G-X$ is a $2-$connected graph for every $2-$subset X of $E(G)$, and let x and y be distinct edges of G. Then G has edge- disjoint cycles C_x and C_y containing x and y, respectively, such that both $G-E(C_x)$ and $G-E(C_y)$ are $2-$connected.*

Theorem(1.7) is best possible as the complete graph K_4 is $3-$edge-connected and it does not contain two edge-disjoint cycles.

Sinclair [6] proved the following result for removable even cycles.

Theorem 1.10. [6]. *Let G be a connected graph with $\delta(G) \geq 3$ and let H be a connected subgraph of G such that there exists an even cycle in H^c. Then there exists an even cycle C in H^c such that $G - E(C)$ is either connected or contains precisely two components one of which is isomorphic to K_2.*

Y.M.Borse [9] strengthened Theorem (1.10) by replacing " $\delta(G) \geq 3$ " by " $d_G(v) \geq 3$ " for all $v \in V(G) \backslash V(H)$ in the hypothesis of the theorem. He also obtain the following forbidden characterizations.

Theorem 1.11. *Let G be a connected graph with $\delta(G) \geq 4$ not isomorphic to K_5. Then there exist edge-disjoint even cycles C_1 and C_2 in G such that both $G - E(C_1)$ and $G - E(C_2)$ are connected.*

Theorem 1.12. *Let G be a connected graph with $\delta(G) \geq 3$, and let D_3 represent a multiple $3-$edge. If G is not isomorphic to $K_4, K_{3,3}, D_3$ then there exists edge-disjoint cycles C_1 and C_2 in G such that both $G - E(C_1)$ and $G - E(C_2)$ are connected.*

The following theorem is analogous to Theorem (1.7) for connected graphs.

Theorem 1.13. *Let G be a connected graph and let H be a connected subgraph of G such that there exists a cycle in H^c. Suppose that $d_G(v) \geq 3$ for all $v \in V(G) \backslash V(H)$. Then there exists a cycle C in H^c such that $G - E(C)$ is connected.*

It follows from the above theorem that every connected graph G of minimum degree at least 3 contains a cycle C such that $G - E(C)$ is connected.

Further, we see from the above theorem that, the class of connected graphs of minimum degree at least 3, is removable cycle class.

Lemma 1.14. *Let G be a graph of minimum degree at least 3. Then there exist two edge-disjoint cycles in G if and only if G is not isomorphic to any of the graphs $K_4, K_{3,3}$ and a $3-$regular graph on two vertices.*

Corollary 1.15. *Let G be a connected graph of minimum degree at least 4. Then there exist edge-disjoint cycles C_1 and C_2 in G such that both $G - E(C_1)$ and $G - E(C_2)$ are connected.*

Theorem 1.16. *Let G be an Eulerian $3-$edge-connected graph such that it has at least four mutually edge-disjoint cycles. Then there exists three mutually edge-disjoint cycles C_1, C_2 and C_3 such that $G - E(C_i)$ is connected for $i = 1, 2, 3$.*

Theorem 1.17. *Let G be a connected graph and let H be a connected subgraph of G such that H^c contains an odd cycle. Suppose that $G - X$ is connected for every $X \subseteq E(H^c)$ with $|X| \leq 2$. Then there exists an odd cycle C in H^c such that $G - E(C)$ is connected.*

It follows from the above theorem that if G is a $3-$edge-connected non-bipartite graph, then there exists an odd cycle C in G such that $G - E(C)$ is connected.

2. REMOVABLE CYCLES IN EULERIAN GRAPHS.

In this section we characterize the removable cycles in Eulerian graphs.
In the following result we proved that the Eulerian $n-$connected simple graph has a removable cycle.

Theorem 2.1. *Let G be an Eulerian $n-$connected simple graph of minimum degree at least $n+2$. Then G has a removable cycle.*

Proof. Let G be an Eulerian graph. By Theorem (1.2) , G is a connected and every vertex in G is even. To prove that G contains a removable cycle, we must prove:
i) G contains a cycle C such that $G-E(C)$ is connected,
ii) The degree of every vertex in $G-E(C)$ is even.

As G is $n-$connected simple graph of minimum degree at least $n+2$, by Theorem (1.5) , G contains a cycle C such that $G-E(C)$ is connected, and (i) holds.

To establish (ii), consider the cycle C in G. As the cycle C enters and exists the vertex one time, then removing the edges of C from G reduce the degree of every vertex in C by 2. As every vertex in G is even and the minimum degree of G at least 4, then every vertex in $G-E(C)$ is even, and (ii) holds. □

Corollary 2.2. *Let G be an Eulerian $2-$connected simple graph of minimum degree at least 4, then G contains a removable cycle.*

We prove the following result for existence two edge-disjoint even removable cycles in Eulerian graphs.

Theorem 2.3. *Let G be an Eulerian graph with $\delta(G) \geq 4$ not isomorphic to K_5. Then G contain two edge-disjoint removable even cycles.*

Proof. As G is Eulerian. By Theorem (1.2) , G is connected and the degree of every vertex in G is even. If G is isomorphic to K_5, it is easy to see that K_5 does not contain two edge-disjoint even cycles. Then G is connected graph with $\delta(G) \geq 4$ not isomorphic to K_5 , by applying Theorem (1.11), there exist two edge-disjoint even cycles C_1 and C_2 in G such that both $G-E(C_1)$ and $G-E(C_2)$ are connected. We prove that each of $G-E(C_1)$ and $G-E(C_2)$ is Eulerian.

For $G-E(C_1)$. From Theorem (1.11), $G-E(C_1)$ is connected. To prove that each vertex in $G-E(C_1)$ has even degree. As removing the edges of the cycle C_1 in G reduce the degree of each vertex of C_1 by 2, and each vertex in G has even degree. Thus each vertex in C_1 after removing the edges of C_1 has even degree in G. Hence every vertex in $G-E(C_1)$ has even degree. Therefore, $G-E(C_1)$ is Eulerian and C_1 is removable cycle. Similarly we prove that $G-E(C_2)$ is Eulerian and C_2 is removable. □

Here we characterized the necessary and sufficient conditions for Eulerian graph to have edge-disjoint removable cycles.

Theorem 2.4. *Let G be an Eulerian graph with $\delta(G) > 3$, and let D_3 represent a multiple $3-$edge. Then G contain two edge-disjoint removable cycles if and only if G is not isomorphic to $K_4, K_{3,3}, D_3$.*

Proof. Suppose that G is Eulerian graph which contain two edge-disjoint removable cycles. As G is Eulerian, by Theorem (1.2), G is connected and the degree of every vertex of G is even. It is clear that the degree of every vertex in each of $K_4, K_{3,3}, D_3$ is odd. Thus G is not isomorphic to any of $K_4, K_{3,3}, D_3$.

Conversely, suppose that G is not isomorphic to $K_4, K_{3,3}, D_3$. Then by Theorem (1.12), there exist edge-disjoint cycles C_1 and C_2 in G. Such that both $G-E(C_1)$ and $G-E(C_2)$ are connected. As the degree of every vertex in G is even, then removing the edges of any cycle C_1 or C_2 in G reduce the degree of every vertex in C_1 or C_2 by 2. That is every vertex in $G-E(C_1)$ or $G-E(C_2)$ has even degree. Thus both $G-E(C_1)$ and $G-E(C_2)$ are Eulerian. Hence C_1 and C_2 are edge-disjoint removable cycles in G. □

We prove the following strengthened of Theorem (2.4) for Eulerian graphs.

Theorem 2.5. *Let G be an Eulerian graph with minimum degree at least 4. Then G contain edge-disjoint cycles C_1 and C_2 which are both removable.*

Proof. Let G be an Eulerian graph, then by Theorem (1.2), G is connected and every vertex in G has even degree. Since G is connected with minimum degree at least 4, then by Corollary (1.15) there exist two cycles C_1 and C_2 in G such that both $G-E(C_1)$ and $G-E(C_2)$ are connected. By similar argument to that in Theorem (2.4) we prove that both of $G-E(C_1)$ and $G-E(C_2)$ are Eulerian and each of C_1 and C_2 is removable cycle in G. □

We also introduce the following result regarding Eulerian graphs.

Theorem 2.6. *Let G be an Eulerian $3-$edge-connected graph such that it has at least four mutually edge-disjoint cycles. Then G contain three mutually edge-disjoint removable cycles.*

Proof. The proof follows from Theorem (1.16) and the fact, that removing the edges of any cycle from an Eulerian graph preserve the even degree for the vertices of G. □

Now, we prove the sufficient conditions for existence of removable odd cycles in Eulerian graphs.

Theorem 2.7. *Let G be an Eulerian $3-$edge-connected non-bipartite graph. Then G contains an odd removable cycle.*

Proof. Let G be an Eulerian $3-$edge-connected graph. If G is bipartite, then G does not contain an odd cycle. Suppose that G is not bipartite. By Theorem (1.17), G contains an odd cycle C such that $G-E(C)$ is connected.

Now , as G is Eulerian, every vertex in G has even degree. It is clear that removing the edges of the cycle C in G reduce the degree of every vertex in C by 2. That is every vertex in $G-E(C)$ has even degree. Thus $G-E(C)$ is Eulerian and C is removable odd cycle in G. □

3. REMOVABLE CYCLES IN REGULAR GRAPHS AND DIGRAPHS.

In this section we characterized the necessary and sufficient conditions for existing a removable cycles in regular graphs and regular digraphs.

Proposition 3.1. *The connected $r-$regular graph G has a removable cycle if and only if it contains a Hamiltonian cycle.*

Proof. Let G be a connected $r-$regular graph.
Suppose that G has a removable cycle C. By Definition (1.3), $G - E(C)$ is regular. Suppose that C is not Hamiltonian cycle. Then there exists a vertex v in G such that v is not in C. As the cycle enters and exits each vertex one time, thus removing $E(C)$ from G reduce the regularity degree of G by 2. Then the degree of every vertex in $G - E(C)$ is $r - 2$. Since v is not in C, the degree of v is r. But $G - E(C)$ is regular, thus $r - 2 = r$ which is a contradiction. Hence v is in C, and every vertex in G is in C. Therefore C is a Hamiltonian cycle.

Conversely, let C be a Hamiltonian cycle in G. That is C pass through all the vertices of G. Then removing the edges of C reduce the degree of every vertex of G by 2. Thus every vertex in $G-E(C)$ has a degree $r-2$. That means $G-E(C)$ is $(r-2)-$regular graphs. Hence C is removable cycle. □

Also we get the following result

Theorem 3.2. *Let G be an $r-$regular graph, where r is a positive integer even number. Then either G contains one removable cycle or there exists k disjoint cycles $C_1, C_2, \cdots, C_k$ in G such that $\cup_{i=1}^{k} C_i$ is removable.*

Proof. Let G be an $r-$regular graph, where r is even. Then by Theorem (1.1), G has a $2-$factor. That is G contains a spanning $2-$regular subgraph H. It is clear that either H is isomorphic to a Hamiltonian cycle or H is isomorphic to a union of distinct $2-$regular cycles $C_1, C_2, \cdots, C_k$. If H is isomorphic to a Hamiltonian cycle C, then G must be connected, and by Proposition (3.1), C is removable cycle. If H is isomorphic to a union of distinct $2-$regular cycles $C_1, C_2, \cdots, C_k$. As H is a spanning $2-$regular subgraph then every vertex in G is belongs to exactly one cycle of $C_1, C_2, \cdots, C_k$. Then removing the edges of the cycles $\cup_{i=1}^{k} C_i$ reduce the degree of every vertex in G by 2. That is the degree of every vertex in $G-E(\cup_{i=1}^{k} C_i)$ is $r-2$. Hence $G-E(\cup_{i=1}^{k} C_i)$ is $(r - 2)-$ regular graph, and $\cup_{i=1}^{k} C_i$ is removable. □

Now we characterize the removable cycles in the regular digraphs.

Theorem 3.3. *Let D be a digraph, D contains a removable directed cycle if and only if D is strongly connected.*

Proof. Let D be an $r-$regular digraph with n vertices.
Suppose that D contains a removable directed cycle. By Definition (1.3), there exists a directed cycle C in D such that $D - E(C)$ is regular digraph. Suppose that C is not Hamiltonian directed cycle. Then there exists a vertex v in D such that v is not in C. As the cycle enters and exits each vertex one time, then removing $E(C)$ from D reduce

the indegree and the outdegree of each vertex in D by 1. That is each of the indegree and the outdegree of every vertex in $D - E(C)$ is $r - 1$. Since v is not in C, then $id(v) = od(v) = r$. But $D - E(C)$ is regular, then $r = r - 1$ which is a contradiction. Hence v is in C, and every vertex in D is in C. Thus C is Hamiltonian directed cycle. Now let $C = v_1\, v_2\, \cdots\, v_n\, v_1$, then given any $v_i,\, v_j$ in the vertex set of D, if $i > j$ then $v_j\, v_{j+1}\, \cdots\, v_i$ is a directed path P_1 from v_j to v_i while $v_i\, v_{i+1}\, \cdots v_{n-1}\, v_n\, v_1\, \cdots\, v_{j-1}\, v_j$ is a directed path P_2 from v_i to v_j. Thus each vertex is reachable from any other vertex and so D is strongly connected.

Conversely, suppose that D is strongly connected . Then for any pair of vertices u and v in D there is a directed path from u to v. Thus D must have a directed cycle C of length n. Such a cycle is a directed Hamiltonian cycle. We can see easily removing the edges of C from D reduce the indegree and the outdegree of each vertex in D by 1. That is every vertex in $D - E(C)$ has indegree and outdegree is $r - 1$. Thus $id(u) = od(u) = r - 1$ for each vertex u in $D - E(C)$. Hence $D - E(C)$ is $(r - 1)-$regular digraph, and C is removable directed cycle in D. □

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