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New Approach of Non-Linear Fractional Differential Equations Analytical Solution by Akbari-Ganji's Method

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Abstract:
The present study examines fractional differential equations (FDEs) as these types of equations are widely used in modeling. FDEs are more difficult to solve than differential equations, which haveinteger order. There are a variety of mathematical methods for solving these equations. An analytical method is utilized in this paper to solve non-linear FDEs. Akbari-Ganji's method (AGM) has been used as a new method for solving FDEs. Comparisons are made between the AGM and previously published papers. Several non-linear FDE examples have been solved to illustrate the high performance and quality of the proposed method. Findings show that the AGM is a highly useful and powerful method that can be utilized for FDEs.

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1. Introduction

FDEs can be used for mathematical modeling of many natural phenomena. These equations are widely used in physics, chemistry, biology, and engineering. Hence, studies have been conducted on the methods of solving these equations. Since there is no exact solution to FDEs, the researchers used various approximate methods to solve this type of equation. To solve FDEs, Sinan et al. [1] employed the Homotopy Perturbation Method (HPM) and found that this method effectively solves higher-order non-linear partial differential equations (PDE). To solve fractional Volterra-Fredholmintegro-differential equations, Das et al. [2] employed the HPM, concluding that taking this method may lead to the model's exact solution. The HPM was investigated by Javeed et al. [3] for the solution of fractional partial differential equations (FPDEs). For non-linear FPDEs, Aguilar et al. [4] employed the Homotopy Perturbation Transform Method (HPTM). This method has also been used to solve equations in other studies [5 and 6]. Another method for solving FDEs is the Variational Iteration Method (VIM). DjelloulZiane et al. [7] employed a combination of the VIM and the Laplace transform method to solve linear and non-linear PDEs with time-fractional derivatives; they claimed that the suggested method is appropriate for such issues and that it is highly efficient. Sontakke et al. [8] employed the VIM to solve time FPDEs and concluded that the VIM is a useful method for solving non-linear FPDEs [9 and 10]. Demir et al. [11] used the differential transform method (DTM) to estimate solutions of fractional Bratu-type differential equations. Their findings show that the DTM can be used for non-linear FDEs in a simple and effective manner, concluding that the current method produces good results. Bansal et al. [12] employed the extended DTM to find analytical exact solutions to two key specific examples of the Bagley Torvik equation, both having Caputo fractional derivatives. Their research shows how the DTM can be used to solve FDEs [13 and 14]. Panda et al. [15] used an Adomian decomposition method (ADM) and HPM to solve a time-fractional partial integrodifferential equation. Abdollah Sadeghinia et al. [16] used the ADM to solve non-linear multi-term FDEs. Haq et al. [17] and Verma et al. [18] conducted more research on the Adomian decomposition method. Other methods, such as the Shehu method [19], the Adams Bashforth-Moulton method [20], the fractional residual power series method (FRPSM) [21], the B-SPLINE OPERATIONAL MATRIX method [22], the Bernoulli polynomials method [23], and Lubich's fractional linear multi-step methods [24], have been proposed in other studies. This form of FDEs [25] is investigated in this paper.

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$$D_*^{\alpha}y(t) + g(t)[y(t)]^m = f(t), \quad 0 \le t \le 1, \quad 0 < \alpha \le 2, \tag{1}$$

The initial boundary conditions of this equation are:

$$y(0) = \rho, \tag{2}$$

$$y(1) = \delta. \tag{3}$$

In boundary conditions 2 and 3, ρ and δ have constant values, y(t) is the solution for Equation 1 in (0, 1), g(t) and f(t) are known functions, and m is a positive integer. This problem is investigated using an alternate Legendre polynomials technique in [26] when g(t) =1.

This study utilized AGM to solve FDEs, and this method has been shown to be powerful. The following presents the basic definitions of fractional differential equations and explains AGM. The effectiveness of the AGM for solving non-linear FDEs is demonstrated by examples.

2. FDEs: Basic Definitions

The definitions of FDEs that will be utilized in this study are supplied in this section.

Definition 2.1

$\mathbf{f(t)}$ as a real function , $\mathbf{t > 0}$ is said to be in the space $\mathbf{C_\mu}$, $\boldsymbol{\mu} \in \mathbf{R}$ if there is a real number $\boldsymbol{p > \mu}$, such that $\mathbf{f(t) = t^p h(t)}$, where $\mathbf{h(t)} \in \mathbf{C}\,[\mathbf{0}, \infty)$, and it is said to be in the space $\mathbf{C_\mu^n}$if $\mathbf{f(n)} \in \mathbf{C_\mu}, \mathbf{n} \in \mathbf{N}.$

Definition 2.2

$\mathbf{I^\alpha}$of order $\boldsymbol{\alpha}$, is the operator of the Riemann-Liouville fractional integral for a function $\mathbf{f} \in \mathbf{C_\mu}, \ \boldsymbol{\mu} \ge -\mathbf{1}$.

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \alpha > 0, t > 0, \\ f(t) \qquad\qquad \alpha = 0. \end{cases}$$

Definition 2.3

The Caputo differential fractional derivative of $\mathbf{f(t)}$ when $\mathbf{f} \in \mathbf{C_{-1}^n}$, $\boldsymbol{n} \in \boldsymbol{N}$ is defined by:

$$D_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-1-\alpha} f^{(n)}(\tau)d\tau, n-1 < \alpha \le n, t > 0, \\ \frac{d^n f(t)}{dt^n} \qquad\qquad \alpha = n \in N. \end{cases}$$

Now, some of the features of the fractional operator are presented below [27].

Properties. For$\mathbf{f(t)} \in \mathbf{C_\mu}, \boldsymbol{\mu} \ge -\mathbf{1}, \boldsymbol{\alpha}, \boldsymbol{\beta} \ge \mathbf{0}, \boldsymbol{\gamma} > -\mathbf{1}$:

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t) = I^\beta I^\alpha f(t);$$

$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma};$$

$$D_*^\alpha[I^\alpha f(t)] = f(t);$$

$$I^\alpha[D_*^\alpha f(t)] = f(t) - \sum_{k=0}^{n-1} f^k(0^+)\frac{t^k}{k!}, 0 \le n-1 < \alpha \le n \in N;$$

3. Akbari-Ganji's Method Procedure

To solve differential equations by AGM, a total answer with constant coefficients is needed. Boundary conditions and initial conditions are needed to solve any linear and non-linear differential equation, and they are consistent with the physics of the issue. Equation 4 shows the general form of a differential equation [28].

$$p_k: f\left(y, y', y'' \ldots, y^{(m)}\right) = 0, y = y(t) \tag{4}$$

where $\boldsymbol{p_k}$ introduced the equation's name and $\boldsymbol{m}$ is the derivative's order.

Equation 5 shows the boundary conditions of the Equation 4.

$$\begin{cases} y(0) = y_0, y'(0) = y_1, \ldots, y^{(m-1)}(0) = y_{n-1} \\ y(L) = y_{L_0}, y'(L) = y_{L_1}, \ldots, y^{(m-1)}(L) = y_{L_{m-1}} \end{cases} \tag{5}$$

The series shown in Equation 6 is considered the solution of the first differential equation at t=L [28].

$$y(t) = \sum_{i=0}^n a_i t^i = a_0 + a_1 t^1 + a_2 t^2 + \cdots + a_n t^n \tag{6}$$

Using more sentences in Equation 6 leads to a more precise solution for equation 4. The series introduced in Equation 6 is of order n. To calculate (n + 1) unknown coefficients in this equation, we need (n + 1) equations. The boundary conditions of Equation 5 are used to solve these equations. These equations are solved using Equation 5.

The boundary conditions are applied as follows:

If $\boldsymbol{t = 0}$

$$\begin{cases} u(0) = a_0 = u_0 \\ u'(0) = a_1 = u_1 \\ u''(0) = a_2 = u_2 \\ \vdots \qquad \vdots \qquad \vdots \end{cases} \tag{7}$$

and when $\boldsymbol{t = L}$

$$\begin{cases} \boldsymbol{u(L) = a_0 + a_1 L + a_2 L^2 + \cdots + a_n L^n = u_{L_0}} \\ \boldsymbol{u'(L) = a_1 + 2a_2 L + 3a_3 L^2 + \cdots + n a_n L^{n-1} = u_{L_1}} \\ \boldsymbol{u''(L) = 2a_2 + 6a_3 L + 12 a_4 L^2 + \cdots + n(n-1) a_n L^{n-2} = u_{L_2}} \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \end{cases} \tag{8}$$

Equation 8 is substituted in Equation 4, and the boundary conditions are then applied to Equation 4 as follows:

$$\boldsymbol{p_0 = f\left(y(0), y'(0), y''(0), \ldots, y^{(m)}(0)\right)}$$

$$\boldsymbol{p_1 = f\left(y(L), y'(L), y''(L), \ldots, y^{(m)}(L)\right)} \tag{9}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

In order to create ($\boldsymbol{n}$+1) equations and ($\boldsymbol{n}$+1) unknown coefficients, the $\boldsymbol{n}$ ($\boldsymbol{n < m}$) terms have been chosen for Equation 6. There are also a number of additional unknowns that are undoubtedly the coefficients of Equation 6. We need m times derivation in Equation 4 to solve this problem. Now, the boundary conditions of Equation 5 have been applied.

$$p'_k = f\left(y', y'', \ldots, y^{(m+1)}\right)$$

$$p''_k = f\left(y'', y^{(3)}, \ldots, y^{(m+2)}\right) \quad (10)$$

$$\vdots \qquad \vdots \quad \vdots \qquad \vdots$$

Application of the boundary conditions on the derivatives of the differential equation p_k in Equation 10 is done in the form of

$$p'_k = \begin{cases} f\left(y'(0), y''(0), \ldots, y^{(m+1)}(0)\right) \\ f\left(y'(L), y''(L), \ldots, y^{(m+1)}(L)\right) \end{cases} \quad (11)$$

$$p''_k = \begin{cases} f\left(y''(0), y^{(3)}(0), \ldots, y^{(m+2)}(0)\right) \\ f\left(y''(L), y^{(3)}(L), \ldots, y^{(m+2)}(L)\right) \end{cases} \quad (12)$$

Equations 7 to 12 can be used to create (n + 1) equations. As a result, the unknown coefficients of Equation 6 will be determined. The coefficients of Equation 6 will subsequently be utilized to solve Equation 4.

4. Examples

In this section, AGM has solved three examples of non-linear FDEs. The obtained results using this method are compared with the VIM results by Nagdy et al. [25]. Also, the effect of various fractional order has been studied.

4.1. Example 1

Consider the following non-linear FDE [25].

$$D^{\alpha} y(t) + t y^2(t) = 2 + t^5, \quad 0 < \alpha \le 2 \quad (13)$$

According to the boundary conditions:

$$[y(t)]_{t=0} = 0, \quad [y(t)]_{t=1} = 1 \quad (14)$$

In AGM, the answer of the differential equations is considered as a finite series with constant coefficients as follows:

$$y(t) = \sum_{i=0}^{2} a_i t^i = a_0 + a_1 t + a_2 t^2 \quad (15)$$

$$y(\tau) = \sum_{i=0}^{2} b_i \tau^i = b_0 + b_1 \tau + b_2 \tau^2 \quad (16)$$

Substituting equations 15 and 16 in Equation 13 as follows:

$$y(t) = 2.256758334 * \sqrt{t} * b_2 + t * (t^2 * a_2 + t * a_1 + a_0)^2 - 2 - t^5 \quad (17)$$

Then, the boundary conditions will be applied as follows:

$$y(0) = 0 \rightarrow a_0 = 0 \quad (18)$$

$$\boldsymbol{y(1) = 1 \rightarrow a_2 + a_1 + a_0 = 1} \quad (19)$$

$$f\left(y(1)\right) = 2.256758334 * b_2 + (a_2 + a_1 + a_0)^2 - 3 = 0 \quad (20)$$

$$f'\left(y(1)\right) = 1.128379167 * b_2 + (a_2 + a_1 + a_0)^2 + \left(2 * (a_2 + a_1 + a_0)\right) * (2 * a_2 + a_1) - 5 = 0 \quad (21)$$

The unknown coefficients of Equation 17 will be gained by solving these four equations:

$$\{a_0 = 0, a_1 = 0.5000000000, \; a_2 = 0.5000000000, \; b_2 = 0.8862269255\} \quad (22)$$

By substituting the calculated coefficients in Equation 15, the differential equation will be obtained:

$$y(t) = 0.5000000000 * t^2 + 0.5000000000 * t \quad (23)$$

The comparison between AGM results and the results obtained by Nagdy et al. [25] when α=1.5 has been shown in Table 1 and Figure 1. As can be seen, the results obtained by AGM are in good agreement with the results obtained by Nagdy et al. [25].

Table 1. Comparison of the results obtained by AGM and VIM at α = 1.5

t	Present work	Nagdy et al.	Error (%)
0	0	0	0
0.1	0.055	0.07042	21.89736
0.2	0.12	0.144303	16.84152
0.3	0.195	0.223131	12.60758
0.4	0.28	0.307962	9.079757
0.5	0.375	0.399672	6.172965
0.6	0.48	0.499107	3.828178
0.7	0.595	0.607245	2.016417
0.8	0.72	0.725386	0.742542
0.9	0.855	0.855397	0.046424
1	1	1	0

The results of Example 4.1 when $\alpha = 0.25, 0.5, 0.75, 1.25$, and 1.75 are shown in Figure 2. As shown in Figure 2, by decreasing α, $y(t)$ has been increased.

4.2. Example 2

Consider the following non-linear FDE [25].

$$D^{\alpha} y(t) + e^t y^3(t) = e^t + e^{4t}, \quad 0 < \alpha \le 2 \quad (24)$$

According to the boundary conditions:

$$[y(t)]_{t=0} = 1, \quad [y(t)]_{t=1} = e \quad (25)$$

In AGM, the answer of the differential equations is considered as a finite series with constant coefficients as follows:

$$y(t) = \sum_{i=0}^{1} a_i t^i = a_0 + a_1 t \quad (26)$$

$$y(\tau) = \sum_{i=0}^{3} b_i \tau^i = b_0 + b_1 \tau + b_2 \tau^2 + b_3 \tau^3 \quad (27)$$

Substituting Equations 26 and 27 in Equation 24 as follows:

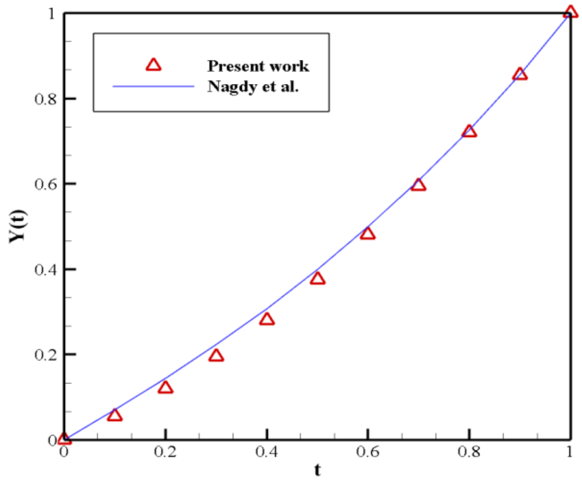

Figure 1. **The results of Example 4.1 when α=1.5, compared with the results obtained by VIM**

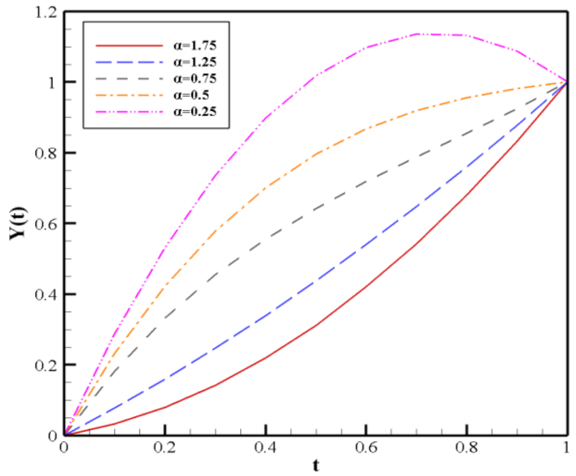

Figure 2. **The results of Example 4.1 when $\alpha = 0.25, 0.5, 0.75, 1.25$ and 1.75**

$$y(t) = 4.513516668 * t^{3/2} * b_3 + 2.256758334 * \sqrt{t} * b_2 + e^t * (t * a_1 + a_0)^3 - e^t - e^{4t} \quad (28)$$

Then, the boundary conditions will be applied as follows:

$$y(0) = 1 \rightarrow a_0 = 1 \quad (29)$$

$$y(1) = 1 \rightarrow a_1 + a_0 = e \quad (30)$$

$$f\big(y(1)\big) = 4.513516668 * b_3 + 2.256758334 * b_2 + e * (a_1 + a_0)^3 - e - e^4 = 0 \quad (31)$$

$$f'\big(y(1)\big) = 6.770275002 * b_3 + 1.128379167 * b_2 + e * (a_1 + a_0)^3 + 3 * e * (a_1 + a_0)^2 * a_1 - e - 4 * e^4 = 0 \quad (32)$$

The unknown coefficients of Equation 28 will be gained by solving these four equations:

$$\{a_0 = 1.000000000, \quad (33)$$

$$a_1 = 1.718281828,$$

$$b_2 = -26.09826182,$$

$$b_3 = 13.65138455\}$$

By substituting the calculated coefficients in Equation 26, the differential equation will be obtained.

$$y(t) = 1.718281828 * t + 1.000000000 \quad (34)$$

The comparison between AGM results and the results obtained by Nagdy et al. [25] when α=1.5 has been shown in Table 2 and Figure 3. As can be seen, the results obtained by AGM are in good agreement with the results obtained by Nagdy et al. [25].

Table 2. **Comparison of the results obtained by AGM and VIM at $\alpha = 1.5$**

t	Present work	Nagdy et al.	Error (%)
0	1	1	0
0.1	1.171828	1.157112	1.271768
0.2	1.343656	1.322714	1.583271
0.3	1.515485	1.491825	1.585917
0.4	1.687313	1.661503	1.553402
0.5	1.859141	1.829563	1.616645
0.6	2.030969	1.994985	1.803733
0.7	2.202797	2.158847	2.035824
0.8	2.374625	2.325785	2.099946
0.9	2.546454	2.50605	1.612233
1	2.718282	2.7183	0.000669

The results of Example 4.2 when $\alpha = 0.25, 0.5, 0.75, 1.25$, and 1.75 are shown in Figure 4. As shown in Figure 4, by decreasing α, $y(t)$has been increased.

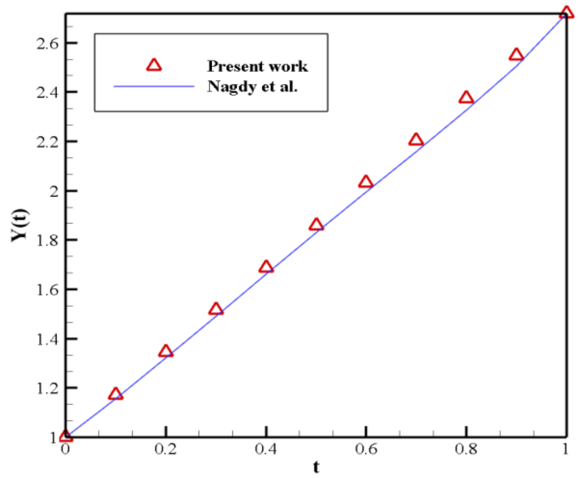

Figure 3. **The results of Example 4.2 when α=1.5, compared with the results obtained by VIM**

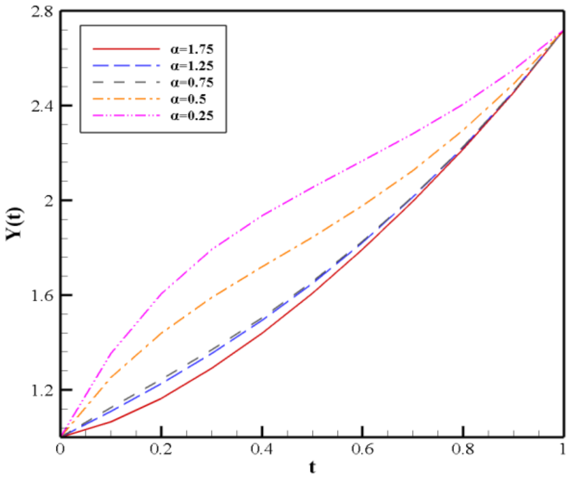

Figure 4. The results of Example 4.2 when $\alpha = 0.25, 0.5, 0.75, 1.25$, and 1.75

4.3. Example 3

Consider the following non-linear FDE [25].

$$D^{\alpha}y(t) + ty^2(t) = -sint + tsin^2(t),\ 0 < \alpha \leq 2, \quad (35)$$

According to the boundary conditions:

$$[y(t)]_{t=0} = 0,\ [y(t)]_{t=\frac{\pi}{2}} = 1 \quad (36)$$

In AGM, the answer of the differential equations is considered as a finite series with constant coefficients as follows:

$$y(t) = \sum_{i=0}^{3} a_i\, t^i = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \quad (37)$$

$$y(\tau) = \sum_{i=0}^{3} b_i\, \tau^i = b_0 + b_1 \tau + b_2 \tau^2 + b_3 \tau^3 \quad (38)$$

Substituting Equations 37 and 38 in Equation 35 as follows:

$$y(t) = 4.513516668 * t^{3/2} * b_3 + 2.256758334 * \sqrt{t} * b_2 + t * (t^3 * a_3 + t^2 * a_2 + t * a_1 + a_0)^2 + sin(t) - t * sin(t)^2 = 0 \quad (39)$$

Then, the boundary conditions will be applied as follows:

$$y(0) = 0 \rightarrow a_0 = 0 \quad (40)$$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_1 + a_0 = 1 \quad (41)$$

$$f(y(1)) = \left(1.128379167 * \sqrt{2} * \pi^{\frac{3}{2}}\right) * b_3 + 1.128379167 * \sqrt{2} * \sqrt{\pi} * b_2 + \frac{1}{2} * \pi * \left(\frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_1 + a_0\right)^2 + 1 - \frac{1}{2} * \pi = 0 \quad (42)$$

$$f'(y(1)) = 3.385137501 * \sqrt{2} * \sqrt{\pi} * b_3 + \frac{1.128379167*\sqrt{2}*b_2}{\sqrt{\pi}} + (\frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_1 + a_0)^2 + \pi * (\frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_1 + a_0) * (\frac{3}{4} * \pi^2 * a_3 + \pi * a_2 + a_1) - 1 = 0 \quad (43)$$

$$f''(y(1)) = \frac{3.385137501*\sqrt{2}*b_3}{\sqrt{\pi}} - \frac{1.128379167*\sqrt{2}*b_2}{\sqrt{\pi^{3/2}}} + 4 * (\frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_1 + a_0) * (\frac{3}{4} * \pi^2 * a_3 + \pi * a_2 + a_1) + \pi * (\frac{3}{4} * \pi^2 * a_3 + \pi * a_2 + a_1)^2 + \pi * (\frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_0 + a_0) * (3 * \pi * a_3 + 2 * a_2) - 1 + \pi = 0 \quad (44)$$

$$f'(y(1)) = \frac{3.385137501*\sqrt{2}*b_3}{\sqrt{\pi^{3/2}}} + \frac{3.385137501*\sqrt{2}*b_2}{\sqrt{\pi^{5/2}}} + 6 * \left(\frac{3}{4} * \pi^2 * a_3 + \pi * a_2 + a_1\right)^2 + 6 * \left(\frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_1 + a_0\right) * (3 * \pi * a_3 + 2 * a_2) + 3 * \pi * \left(\frac{3}{4} * \pi^2 * a_3 + \pi * a_2 + a_1\right) * (3 * \pi * a_3 + 2 * a_2) + 6 * \pi * \left(\frac{1}{8} * \pi^3 * a_3 + \frac{1}{4} * \pi^2 * a_2 + \frac{1}{2} * \pi * a_1 + a_0\right) * a_3 + 6 = 0 \quad (45)$$

The unknown coefficients of Equation 39 will be gained by solving these six equations.

$$a_0 = 0,\quad a_1 = 1.147465869,\quad a_2 = -0.2828944409,\quad a_3 = -0.02694193005,\quad b_2 = -0.4268734354,\quad b_3 = 0.02333849508 \quad (46)$$

By substituting the calculated coefficients in Equation 37, the differential equation will be obtained.

$$y(t) = -0.02694193005 * t^3 - 0.2828944409 * t^2 + 1.147465869 * t \quad (47)$$

The comparison between AGM results and the results obtained by Nagdy et al. [25] when α=1.5 has been shown in Table 3 and Figure 5. As can be seen, the results obtained by AGM are in good agreement with the results obtained by Nagdy et al. [25].

Table 3. Comparison of the results obtained by AGM and VIM at $\alpha = 1.5$

t	Present work	Nagdy et al.	Error (%)
0	0	0	0
0.2	0.214194	0.190696	12.32212
0.4	0.405351	0.372123	8.929335
0.6	0.572248	0.540419	5.889781
0.8	0.713661	0.692453	3.062817
1	0.828367	0.824106	0.517031

1.2	0.915141	0.928384	1.42648
1.4	0.97276	0.993341	2.071898
1.6	1	1.000094	0.009422

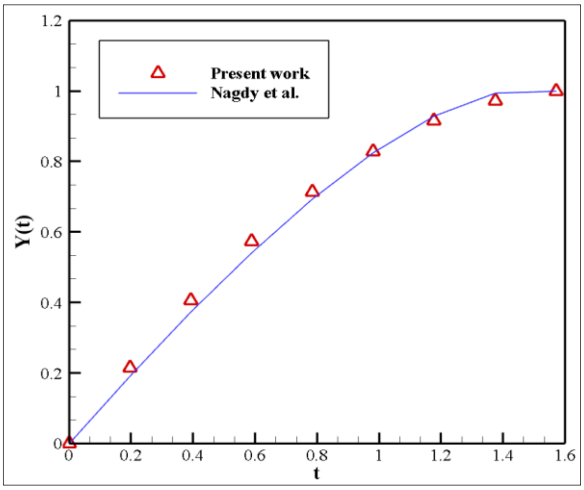

Figure 5. The results of Example 4.3 when α=1.5, compared with the results obtained by VIM

The results of Example 4.3 when $\alpha = 0.25, 0.5, 0.75, 1.25, 1.5$ and 1.75 are shown in Figure 6.

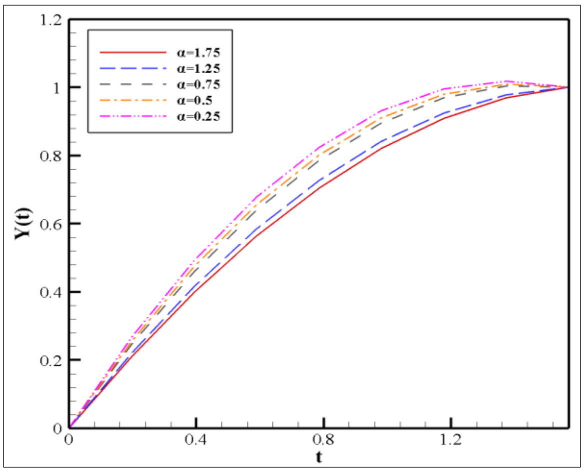

Figure 6. The results of Example 4.3 when $\alpha = 0.25, 0.5, 0.75, 1.25$ and 1.75

As shown in Figure 6, by decreasing α, $y(t)$ has been increased.

5. Conclusion

The AGM was successfully employed to find a solution to non-linear FDEs in this study. Three examples have been chosen to demonstrate the method's efficiency and accuracy. The VIM was compared with the approximate answers generated by the AGM, and the results of the equations are shown in Tables 1 to 3 to demonstrate the method's stability. Then, the equations were solved for various fractional orders, and the results show that as the α decreases, the value of y(t) increases. Results revealed that AGM is useful and powerful for solving FDEs.

6. References

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