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Thermodynamics and Thermal Stability of BTZ-ModMax Black Holes

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Abstract:
Motivated by a new interesting nonlinear electrodynamics (NLED) model known as Modification Maxwell (ModMax) theory, we obtain an exact analytic BTZ black hole solution in the presence of a new NLED model and the cosmological constant. Afterward, by considering the obtained solution, we Hawking temperature, entropy, electric charge, mass, and electric potential. We extract the first law of thermodynamics for the BTZ-ModMax black hole. We study thermal stability by evaluating the heat capacity (local stability) and Helmholtz free energy (global stability). By comparing the local and global stabilities, we find the common areas that simultaneously satisfy the local and global stabilities.

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1. Introduction

The nonlinear electrodynamics (NLED) theories are generalizations of Maxwell's theory and describe some of the phenomena that Maxwell's theory is unable to explain carefully. For example, NLED theories can explain the self-interaction of virtual ee^{+} (i.e., electron-positron) pairs [1–3]. Moreover, NLED theories can affect the gravitational redshift around super-strong magnetized compact objects [4, 5]. By applying NLED in cosmology, we can eliminate the Big Bang's singularity. For a black hole, the singularity of spacetime is removed using NLED (see Refs. [6–9], for more details). In this regard, a few NLED theories were introduced, such as Born-Infeld (BI) [10], Euler-Heisenberg (EH) [1], and Power-Law (PL) [11, 12]. BI-NLED preserves the electromagnetic duality invariance of Maxwell's theory and solves the self-energy of point particles' divergence [10]. However, BI-NLED theory is not conformally invariant. EH-NLED theory explains the effect of vacuum polarization, but it is not conformal invariant and dual invariant [1]. PL-NLED is conformal invariance [11, 12] and also can remove the self-energy of a point particle's divergence [13, 14]. Recently, Bandos et al. proposed an NLED theory with two fundamental symmetries, i.e., electromagnetic duality and conformal invariance [15]. This NLED theory had two types of solutions. One of them leads to Bialynicki-Birula electrodynamics, and the other one yields a generalization of Maxwell electrodynamics, which is known as a modification of Maxwell (ModMax) theory [15]. Notably, the ModMax NLED theory is characterized by a dimensionless parameter γ, and it turns to Maxwell's theory for $\gamma = 0$ [15, 16] (see Refs. [17–19] for recent extensions of the ModMax NLED theory). By coupling the ModMax electrodynamics and gravity, different black hole solutions have been studied in some literature [20–34].

The first black hole solution was found by Banados, Teitelboim, and Zanelli in three-dimensional spacetime [35], which is known as BTZ black hole. Then, the study of gravity in three-dimensional spacetime attracted a lot of attention due to different aspects of their physics; for example, there are relations between effective action in string theory and three-dimensional black holes [36–38]. Anti-Hawking phenomena [39, 40] provide a better understanding of gravitational systems in three-dimensional spacetime [41], AdS/CFT correspondence [42, 43], quantum aspect, entanglement, and quantum entropy [44–47], and holographic aspects [48–50]. In this regard, different three-dimensional black holes have been obtained in Einstein's gravity theory, and modified theories of gravity are coupled with linear and nonlinear matters [51–66].

In this paper, an analytic BTZ black hole solution is obtained by coupling Einstein's gravity with the ModMax

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NLED field. Then, the Hawking temperature, entropy, electric charge, electric potential, and mass of the BTZ-ModMax black hole are calculated. Finally, to evaluate the thermal stability of this black hole, the heat capacity and Helmholtz free energy are studied.

2. Field Equation and Black Hole Solution

The action of Einstein's theory of gravity coupled with the ModMax NLED and the cosmological constant in three-dimensional spacetime is:

$$\mathfrak{I} = \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-g}[R - 2\Lambda - 4\mathcal{L}], \tag{1}$$

where R is the Ricci scalar, and Λ is the cosmological constant. In the above action, $g = \det(g_{\mu\nu})$ is devoted to the determinant of the metric tensor $(g_{\mu\nu})$. In addition, $\mathcal{L}$ is the ModMax Lagrangian [15, 16]. Here, we assume that the ModMax Lagrangian in three-dimensional spacetime resembles the ModMax Lagrangian in four-dimensional spacetime. i.e.,

$$\mathcal{L} = \mathcal{S}cosh\gamma - \sqrt{\mathcal{S}^2 + \mathcal{P}^2}sinh\gamma, \tag{2}$$

where γ is a dimensionless parameter known as the ModMax parameter. In the ModMax Lagrangian, $\mathcal{S}$ and $\mathcal{P}$, respectively, are a true scalar, and a pseudoscalar. They are defined as:

$$\mathcal{S} = \frac{\mathcal{F}}{4}, \tag{3}$$

$$\mathcal{P} = \frac{\tilde{\mathcal{F}}}{4}, \tag{4}$$

where $\mathcal{F} = \mathcal{F}_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant. Also, $\mathcal{F}_{\mu\nu}$ is called the electromagnetic tensor field and is given as:

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

in which A_μ is the gauge potential. In addition, $\tilde{\mathcal{F}} = \mathcal{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\tilde{F}^{\mu\nu} = \frac{1}{2}\in_{\mu\nu}{}^{\rho\lambda} F_{\rho\lambda}$. Notably, the ModMax Lagrangian turns to linear Maxwell's theory, i.e., $\mathcal{L} = \frac{\mathcal{F}}{4}$, when $\gamma = 0$.

In this paper, we want to extract the electrically charged BTZ-ModMax black hole solutions in Einstein-Λ gravity, so $\mathcal{P} = 0$. The Einstein-Λ-ModMax equations are written in the following form:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{5}$$

$$\partial_\mu\left(\sqrt{-g}\tilde{E}^{\mu\nu}\right) = 0, \tag{6}$$

where

$$8\pi T^{\mu\nu} = 2(F^{\mu\sigma}F^{\nu}{}_{\sigma}e^{-\gamma}) - 2e^{-\gamma}\mathcal{S}g^{\mu\nu}, \tag{7}$$

$$\tilde{E}_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial F^{\mu\nu}} = 2\left(\mathcal{L}_{\mathcal{S}}F_{\mu\nu}\right), \tag{8}$$

and $\mathcal{L}_{\mathcal{S}} = \frac{\partial\mathcal{L}}{\partial\mathcal{S}}$. For the charged case, the ModMax field equation (Eq. (6)) turns to:

$$\partial_\mu\left(\sqrt{-g}e^{-\gamma}F^{\mu\nu}\right) = 0. \tag{9}$$

Notably, according to the physical restriction, we consider $\gamma \geq 0$ [15].

Here, we consider a three-dimensional static spacetime as

$$ds^2 = -\psi(r)dt^2 + \frac{dr^2}{\psi(r)} + r^2d\varphi^2, \tag{10}$$

and $\psi(r)$ is a function (known as the metric function) that we want to find.

To extract a radial electric field, we suppose A_μ in the following form:

$$A_\mu = h(r)\delta_\mu^t, \tag{11}$$

whereby considering the metric (10) and the MaxMax field Equation 9, we can get:

$$h'(r) + rh''(r) = 0, \tag{12}$$

and the prime is related to the first derivative concerning r, and the double prime is devoted to the second derivative with respect to r. Using Eq. (12), we extract

$$h(r) = q\ ln\left(\frac{r}{r_0}\right), \tag{13}$$

in which q is an integration constant. This constant (q) is related to the electric charge. Also, r_0 is another constant with length dimension and confirms that the logarithmic argument is dimensionless. The obtained electric field of ModMax NLED in three-dimensional spacetime is:

$$E(r) = \frac{q}{r}e^{-\gamma}, \tag{14}$$

To get the metric function, i.e., $\psi(r)$, we use Equation 10 with Eqs. (5), (7), and (13). We can extract the following differential equations:

$$eq_{tt} = eq_{rr} = \psi'(r) + 2\Lambda r + \frac{2q^2}{r}e^{-\gamma} = 0, \tag{15}$$

$$eq_{\varphi\varphi} = \psi''(r) + 2\Lambda - \frac{2q^2}{r^2}e^{-\gamma} = 0, \tag{16}$$

which eq_{tt}, eq_{rr}, and $eq_{\varphi\varphi}$, respectively, belong to tt, rr, $\varphi\varphi$ components of the field equation of motion (Eq. (5)). Using Eqs. (15) and (16), we obtain the metric function as:

$$\psi(r) = -m_0 - \Lambda r^2 - 2q^2e^{-\gamma}\ln\left(\frac{r}{r_0}\right), \tag{17}$$

where m_0 is an integration constant. This constant is related to the total mass of black holes. Moreover, the obtained metric function (17) turns to BTZ black holes in Einstein-Λ-Maxwell theory when the parameter of ModMax is zero, i.e. $\gamma = 0$.

After finding the solution (17), we want to evaluate the existence of essential singularity(ies). For this purpose, we evaluate two well-known scalars which are the Ricci and Kretschmann scalars. These scalars, respectively, are:

$$R = 6\Lambda + \frac{2q^2}{r^2}e^{-\gamma}, \tag{18}$$

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 12\Lambda^2 + \frac{8\Lambda q^2}{r^2}e^{-\gamma} + \frac{12q^4}{r^4}e^{-2\gamma}. \tag{19}$$

The results indicate different behavior for the Ricci and Kretschmann scalars. These are:

i. For the finite and small value of γ, these quantities reveal that there is a curvature singularity at $r = 0$.

ii. In the limit of $r \to \infty$, the Ricci and Kretschmann scalars turn to 6Λ and 12Λ, respectively, which indicate that the asymptotical behavior of the solution is $(A)dS$ for $\Lambda > 0$ $(\Lambda < 0)$.

iii. For very large value of γ or in the limit $\gamma \to \infty$, by considering $r \to 0$, we have

$$\lim_{r\to 0} R \to 6\Lambda, \tag{20}$$

$$\lim_{r\to 0} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \to 12\Lambda^2, \tag{21}$$

where indicates that they cannot reveal a curvature singularity at $r = 0$.

To find a singularity of the obtained solution (17), we calculate some of the components of Ricci and Riemannian tensors, which are:

$$R^{\varphi\varphi} = \frac{2\Lambda r^2 + 2q^2 e^{-\gamma}}{r^4}, \tag{22}$$

$$R^{r\varphi r\varphi} = \frac{\psi(r)(\Lambda r^2 + q^2 e^{-\gamma})}{r^4}, \tag{23}$$

and diverge in the limit $\gamma \to \infty$, and $r \to 0$. Therefore, an essential curvature singularity exists at $r = 0$, for different values of γ.

In order to get the real positive roots (or regular horizons) of the metric function, we have plotted Figure 1. We can find two horizons (event horizon and inner horizon), one extreme horizon, and also the naked singularity (without horizon) for the obtained solutions. In other words, the singularity is covered with at least one horizon (event horizon). So, we can interpret these solutions as black holes.

The effects of various parameters reveal that i) by fixing the parameters of q, r_0, γ, and Λ, the massive black holes have two roots, one of which is related to the event horizon (see the up-left panel in Figure 1). ii) dS BTZ black hole does not include the event horizon (see the up-right panel in Figure 1). So, the obtained solution (17) cannot be a black hole when $\Lambda > 0$. iii) small electrical charge BTZ black holes include two roots, an inner horizon, and an outer (event) horizon (see the down-left panel in Figure 1). iv) the effect of the ModMax parameter indicates that by increasing γ, the number of roots increases when other parameters are fixed (see the down-right panel in Figure 1).

As a result, small charged AdS BTZ black holes with large values of the ModMax parameter (γ) and mass (m_0) have two roots, which are the inner root and event horizon, respectively.

3. Thermodynamic Quantities

We obtain the thermodynamic quantities of BTZ-ModMax black holes in this section.

The Hawking temperature is defined as:

$$T = \frac{k}{2\pi} = \frac{\sqrt{-\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)}}{2\pi}, \tag{24}$$

where $\chi = \partial_t$ is the Killing vector, and also k is the surface gravity. Using the metric (10), we can get the surface gravity k, which is given by $k = \frac{\psi'(r)}{2}\Big|_{r=r_+}$, in which r_+ is the outer (event) horizon. So, the Hawking temperature of BTZ-ModMax black hole is:

$$T = \frac{\psi'(r)|_{r=r_+}}{4\pi} = -\frac{\Lambda r_+}{2\pi} - \frac{q^2}{2\pi r_+}e^{-\gamma}. \tag{25}$$

As one can see, the temperature depends on the cosmological constant (Λ), the electrical charge (q), and the parameter of ModMax (γ). The cosmological term has a positive effect on temperature due to the existence of black hole solutions for the AdS case (i.e., $\Lambda < 0$). Therefore, the Hawking temperature is an increasing function of the cosmological constant. In addition, by increasing q, the temperature decreases because the electrical charge has a negative effect on T. By increasing γ, the effect of the electrical charge disappears, and the temperature will be independent of q. On the other hand, the roots of temperature determine the bound points because, from a classical thermodynamics point of view, the negative of temperature is devoted to non-physical solutions. So, the roots of temperature can separate physical from non-physical solutions. We find the roots of the temperature, which are:

$$r_\pm\big|_{T=0} = \frac{\pm q}{\sqrt{-\Lambda e^{-\gamma}}}, \tag{26}$$

which indicates that there is only one positive root (r_+). Also, a real root for the temperature reveals that the cosmological constant has to be negative $\Lambda < 0$. This root (or the bound point) depends on three parameters of our system, i.e., Λ, q, and γ. This bound point decreases (increases) by increasing $|\Lambda|$ and $\gamma(q)$.

In the high energy limit of the temperature is given by

$$\lim_{\text{very small } r_+} T \propto -\frac{q^2}{2\pi r_+}e^{-\gamma}, \tag{27}$$

which indicates that the ModMax parameter plays a significant role in the high energy limit of the temperature. In other words, by increasing γ, this limit decreases, and finally, it will be zero (see Figure 2, for more details).

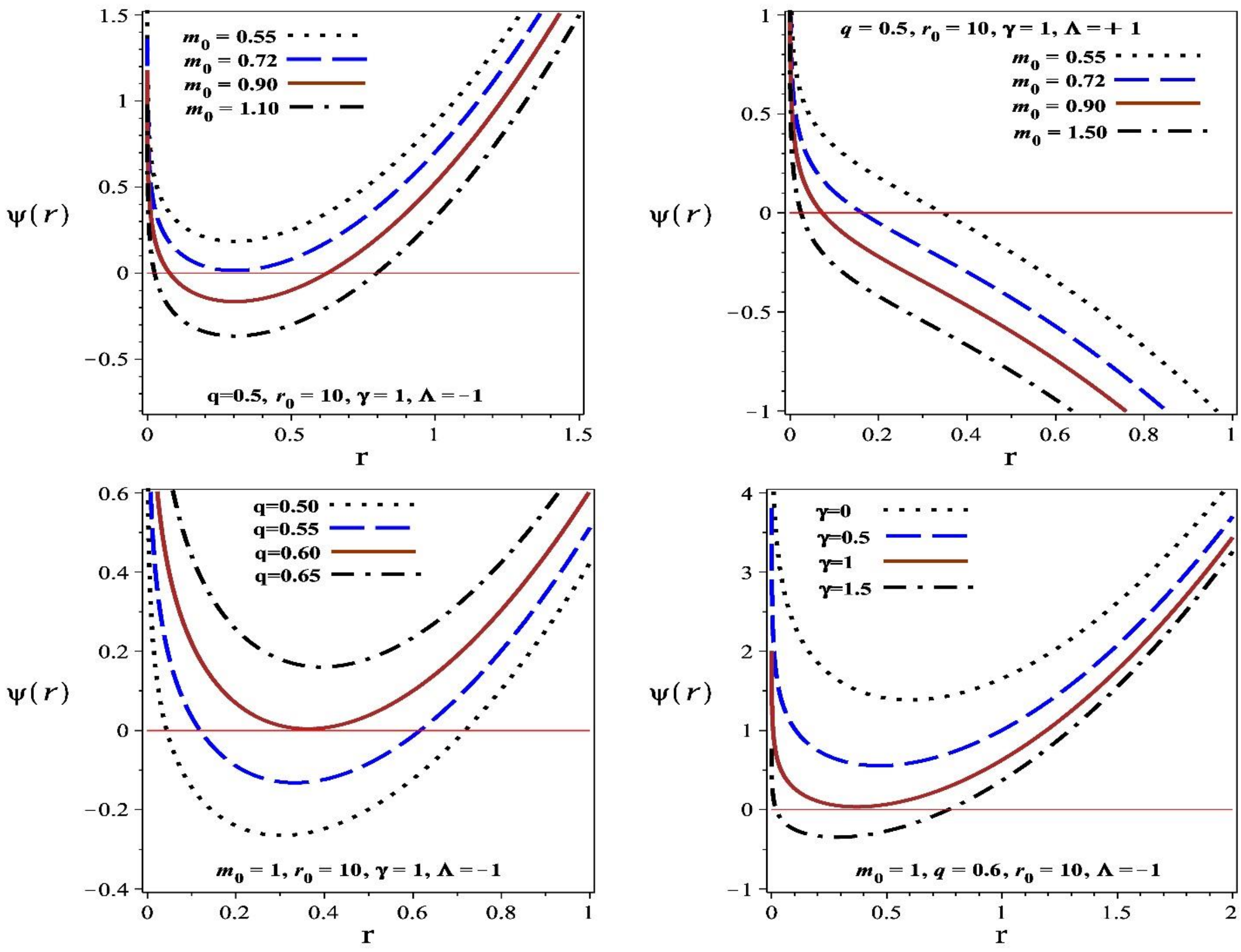

Figure 1. $\psi(r)$ **(metric function) versus** r **for different values of parameters**

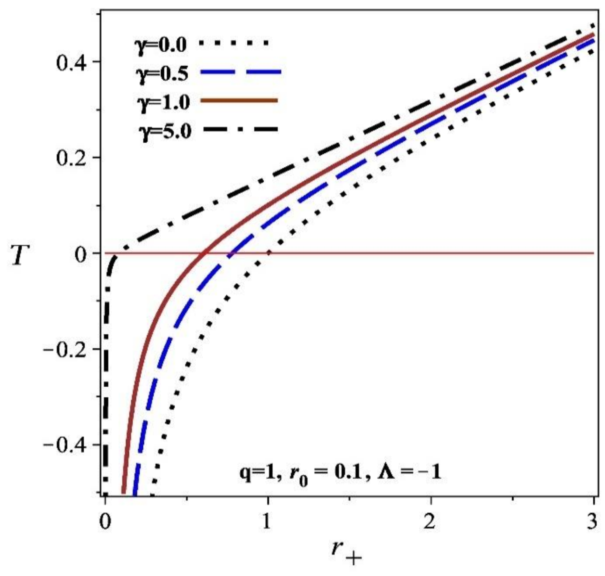

Figure 2. T **versus** r_+ **for different values of parameters**

For the asymptotic limit of the temperature, we have:

$$\lim_{very\ large\ r_+} T \propto -\frac{\Lambda r_+}{2\pi}, \tag{28}$$

which is dependent on the cosmological constant. According to this fact, the cosmological constant is negative, so the asymptotic limit of the temperature will be positive. Indeed, the temperature of very large BTZ-ModMax black holes is always positive when $\Lambda < 0$.

The electric charge, Q is obtained by using Gauss's law in the following form:

$$Q = \frac{q}{2} e^{-\gamma}, \tag{29}$$

where depends on the ModMax parameter.

In Einstein's theory of gravity, the entropy of black holes (S) can be extracted using the area law. In other words, we can obtain the entropy as a quarter of the event horizon area $S = \frac{A}{4}$. Considering the metric (10), we can get the event horizon area (A) which is given by:

$$A = \int_0^{2\pi} \sqrt{g_{\varphi\varphi}} d\varphi = \left(\int_0^{2\pi} d\varphi\right) r_+ = 2\pi r_+, \tag{30}$$

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so, the entropy of BTZ-ModMax black holes is:

$$S = \frac{\pi r_+}{2}. \tag{31}$$

By applying the Hamiltonian approach and also the counterterm method, we are able to obtain the total mass of solutions, which leads to:

$$M = \frac{m_0}{8}, \tag{32}$$

where m_0 is geometrical mass and gets from the metric function (17) on the horizon ($\psi(r = r_+) = 0$), which leads to:

$$m_0 = -\Lambda r_+^2 - 2q^2 e^{-\gamma} \ln\left(\frac{r_+}{r_0}\right),$$

so, the total mass turns to

$$M = -\frac{\Lambda r_+^2}{8} - \frac{q^2 e^{-\gamma}}{4} \ln\left(\frac{r_+}{r_0}\right). \tag{33}$$

Similar to the temperature (25), the total mass depends on the cosmological constant, the electrical charge, and the ModMax parameter. On the other hand, it was discussed that the total mass of black holes might be considered as the internal energy of a system. It is clear that the internal energy (on total mass) is always positive for large black holes. To study the effects of the MadMax parameter on the internal energy of our system, we plot the total mass (33) versus the radius of black holes in Figure 3.

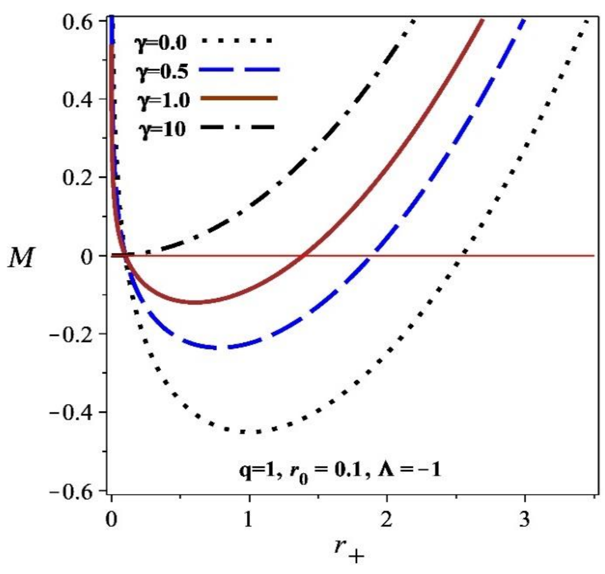

Figure 3. M versus r_+ for different values of parameters

Our findings indicate that by increasing γ, the negative values of the mass disappear, and we encounter the positive mass everywhere. Indeed, by increasing the ModMax parameter, the negative area decreases, and in the limit of very large values of γ, there is no negative area for the total mass.

We can get the electric potential, U, by the difference of gauge potential between the reference and the horizon, which yields [67, 68]:

$$U = A_\mu \mathcal{X}^\mu\big|_{r\to reference} - A_\mu \mathcal{X}^\mu\big|_{r\to r+} = -q \ln\left(\frac{r_+}{r_0}\right). \tag{34}$$

Now, we are able to check the relation of the first law of thermodynamics. Using the obtained thermodynamic quantities such as electric charge (29), entropy (31), and mass (32), and after some calculations, the first law of black hole thermodynamics is in the following relation:

$$dM = TdS + UdQ, \tag{35}$$

where S and Q, respectively, are the temperature and the electric potential in the following forms:

$$T = \left(\frac{\partial M}{\partial S}\right)_Q, \tag{36}$$

$$U = \left(\frac{\partial M}{\partial Q}\right)_S, \tag{37}$$

4. Thermal Stability

Here, we would like to study the thermal stability of BTZ-ModMax black holes in the context of the canonical ensemble. In this regard, the heat capacity and the Helmholtz free energy play a significant role in determining thermal stability. Using the heat capacity and the Helmholtz free energy, we can evaluate the local and global stability, respectively. So, we discuss the thermal stability of BTZ black holes using the heat capacity and the Helmholtz free energy.

4.1. Local Stability

By studying the heat capacity, we can extract two significant properties of the solutions, which are phase transition points, and the thermal stability of the solutions. Indeed, the phase transition points are related to the divergences of the heat capacity because these divergences belong to where the under-studying system goes under phase transitions. Also, the signature of heat capacity (i.e. bound point) determines the thermal instability/stability of the system in the canonical ensemble. In other words, the positivity (the negative) of heat capacity reveals that the black hole is in a thermally (un)stable state. So, we have to find bound (roots of the heat capacity is where the sign of temperature is changed) and phase transition (divergences point of the heat capacity) points.

The heat capacity with fixed charge is defined by:

$$C_Q = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_Q} = \frac{\left(\frac{\partial M(S,Q)}{\partial S}\right)_Q}{\left(\frac{\partial^2 M(S,Q)}{\partial S^2}\right)_Q}. \tag{38}$$

to extract the heat capacity, we re-write the Hawking temperature (25) and the total mass (33) of BTZ-ModMax black hole in terms of the electrical charge (29), and the entropy (31), in the following forms:

$$T = \left(\frac{\partial M(S,Q)}{\partial S}\right)_Q = \frac{-\Lambda S}{\pi^2} - \frac{Q^2 e^\gamma}{S}, \tag{39}$$

$$M(S,Q) = \frac{\Lambda S^2}{2\pi^2} + Q^2 e^\gamma \ln\left(\frac{2S}{\pi r_0}\right). \tag{40}$$

Now, we can get the heat capacity by using Eqs. (38) to (40), which is:

$$C_Q = \frac{(\Lambda S^2+\pi^2 Q^2 e^{\gamma})S}{\Lambda S^2-\pi^2 Q^2 e^{\gamma}}, \tag{41}$$

which indicates the heat capacity depends on the cosmological constant, electrical charge, and the ModMax parameter. To find the bound and phase transition points, we have to solve the following relations:

$$\begin{cases} \left(\frac{\partial M(S,Q)}{\partial S}\right)_Q = 0, & bound\ points \\ \left(\frac{\partial^2 M(S,Q)}{\partial S^2}\right)_Q = 0, & phase\ transition\ points \end{cases} \tag{42}$$

We first get the bound point by solving $T = 0$. For this purpose, we consider Equation 39 and solve it in terms of the entropy, which leads to:

$$S_{root} = \pi Q\sqrt{\frac{e^{\gamma}}{-\Lambda}}, \tag{43}$$

which states that there is one real root (or one bound point) for the heat capacity. In other words, there are two different (thermal stable/unstable) areas for the system that are before and after this bound point. It is notable that the bound point depends on the electrical charge, the cosmological constant, and the ModMax parameter. On the other hand, the root of temperature reveals a limitation point. Indeed, the root of the temperature separates physical (i.e. positive temperature) from non-physical solutions (i.e. negative temperature). For determining the thermal stability/instability and physical black holes, we plot the heat capacity versus entropy in Figure 4.

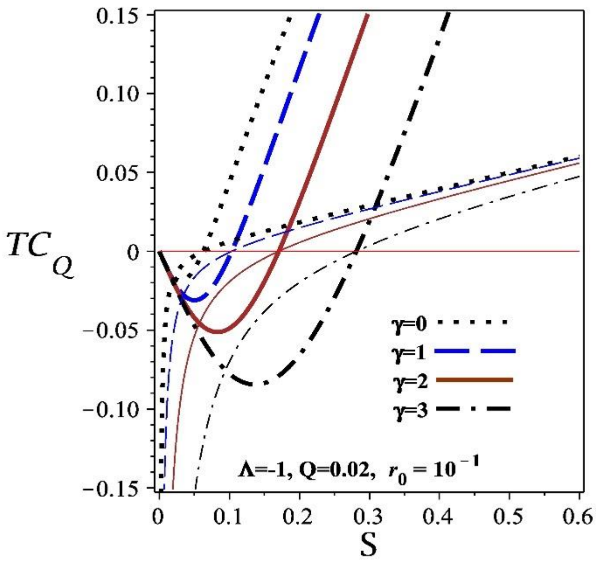

Figure 4. Heat capacity C_Q (thick lines), and the Hawking temperature T (thin lines) versus S for different values of the ModMax parameter

Our results in Figure 4 show that there are two different behaviors for the heat capacity, which are: i) In the range $S < S_{root}$, black holes are not physical objects because the temperature is negative. Since the heat capacity is negative in this area, our system is thermally unstable. ii) In the range $S < S_{root}$, the temperature and the heat capacity are positive. So, black holes in this area are physical and stable objects. The effect of the ModMax parameter reveals that the physical and thermal stable area decrease by increasing γ (see Figure 4 for more details). As a result, AdS BTZ-ModMax black holes with a large radius (or entropy) are physical and thermal stable objects.

4.2. Global Stability

The Helmholtz free energy in the usual case of thermodynamics, is defined as:

$$F = U - TS, \tag{44}$$

where in the context of the black holes, we can consider U=M. So, in a canonical ensemble with a fixed charge Q, the Helmholtz free energy is given by:

$$F(T,Q) = M(S,Q) - TS, \tag{45}$$

after some calculation, we get:

$$F = -Q^2 e^{\gamma}\left(\ln\left(\frac{2S}{\pi r_0}\right) - 1\right) + \frac{\Lambda S^2}{2\pi^2}, \tag{46}$$

It is notable that in the context of the canonical ensemble, $F < 0$ (i.e. the negative of the Helmholtz free energy) determines the global stability of a thermodynamic system. In this regard, we find one real root of the Helmholtz free energy in the following form:

$$S_{root_F} = \frac{\pi r_0}{2} e^{-\frac{LambertW\left(\frac{-\Lambda r_0^2}{4Q^2 e^{\gamma-2}}\right)}{2}+1}, \tag{47}$$

which indicates that this root depends on all of our system's parameters. To determine the global stability, we plot the Helmholtz free energy versus entropy in Figure 5.

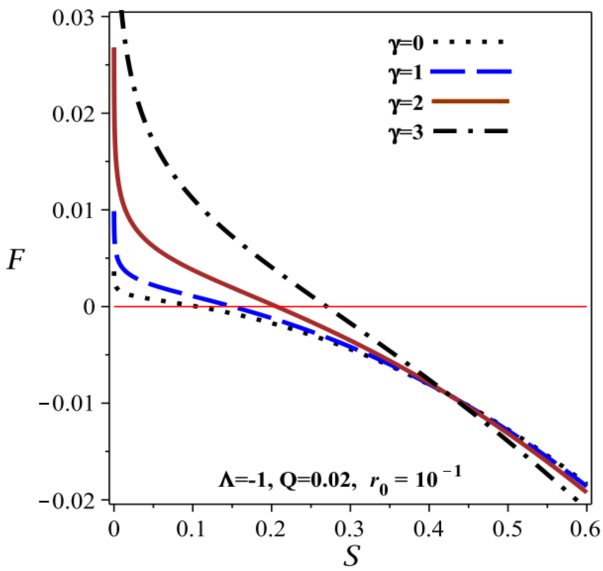

Figure 5. Helmholtz free energy F versus S for different values of the ModMax parameter

The results in Figure 5 indicate that there are two different areas: i) AdS BTZ-ModMax black holes do not satisfy the global stability in the range $S < S_{root_F}$ because $F > 0$ in this area. ii) In the range $S > S_{root_F}$, the Helmholtz free energy is negative, so the black holes satisfy the global stability.

The effect of the ModMax parameter indicates that the global stable area decreases by increasing γ (similar to the heat capacity). As a result, AdS BTZ-ModMax black holes with large radius (or entropy) are within the global stable area.

By comparing the local and global stabilities of AdS BTZ-ModMax black holes simultaneously, we find that the AdS BTZ-ModMax black holes with large radii (or entropy) satisfy the local and global stability conditions simultaneously.

5. Conclusions

In this paper, we obtained exact analytical three-dimensional solutions in the presence of a new NED model, which is known as ModMax NED. Our analysis indicated that this solution belongs to the black hole solution, which included a singularity at $r = 0$, which is covered by an event horizon for the AdS case. Also, we showed that although the ModMax parameter could remove the singularity of the electrical field at $r = 0$, however, it did not remove the curvature singularity at $r = 0$. Next, we evaluated the effects of various parameters on the root of the metric function. Our analysis in Figure 1 revealed that the small charged AdS BTZ black holes with big values of the ModMax parameter (γ) and mass (m_0) had two roots, which were related to the inner root and event horizon, respectively. Moreover, by increasing (decreasing) γ and m_0 (q), the radius of the event horizon increases, and we encounter large black holes.

We calculated the conserved and thermodynamic quantities of the solution, such as Hawking temperature, electric charge, electric potential, entropy, and mass. Our analysis of the Hawking temperature indicated that it depended on the cosmological constant (Λ), the electrical charge (q), and the parameter of ModMax (γ). Λ (q) had a positive (negative) effect on temperature. Indeed, the temperature was an increasing (decreasing) function of the cosmological constant (the electrical charge). In addition, by increasing γ, the Hawking temperature was independent of q. We studied the high energy and asymptotic limit of the Hawking temperature. Our analysis indicated that the high energy limit of the temperature was dependent on γ, whereas the asymptotic limit of the temperature was dependent on the cosmological constant. Another interesting thermodynamics quantity was related to the total mass of black holes because this quantity gives us information about internal energy. Our findings in Fig. 3 indicated that by increasing the ModMax parameter, the negative area of the total mass decreased, and in the limit of very large values of γ, there was no negative area for the total mass. Then, we checked the validity of the first law of thermodynamics for BTZ-ModMax black holes.

To study the thermal stability (i.e. the local and global stabilities) of BTZ-ModMax black holes in the canonical ensemble, we evaluated simultaneously the heat capacity and the Helmholtz free energy for these black holes.

Our analysis of the heat capacity indicated that there was one bound point (S_{root}), which was dependent on the electrical charge, the cosmological constant, and the ModMax parameter. This bound point separated two different behaviors for the heat capacity, which were before and after this point. In other words, before the bound point, i.e., in the range $S < S_{root}$, black holes were not physical and stable objects because the temperature and the heat capacity were negative. After the bound point, i.e., in the range $S > S_{root}$, the temperature and the heat capacity were positive. Therefore, large black holes were physical and stable objects. The effect of the ModMax parameter on the bound point in Fig. 4 revealed that the physical and thermal stable area decreased by increasing γ.

We studied the Helmholtz free energy to evaluate global stability. Our results in Fig. 5 indicated that the same behavior was observed for Helmholtz free energy, similar to the heat capacity. There was one real root for the Helmholtz free energy, in which before and after this root, Helmholtz free energy was positive and negative, respectively. Indeed, the large black holes satisfied the global stability condition. In addition, the effect of the ModMax parameter showed that the global stable area decreased by increasing γ.

Finally, we compared the local and global stability, together. We found that the AdS BTZ-ModMax black hole with a large radius (or entropy) could satisfy the local and global stability conditions, simultaneously.

6. Statements & Declarations

6.1. Acknowledgments

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6.2. Funding

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6.3. Author Contributions

Authors are encouraged to include a statement that specifies the contribution of every author to the research and preparation of the manuscript.

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