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Bell polynomials for computing closed formulae of numbers and polynomials

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Abstract. Exponential partial Bell polynomials are a main tool for computing explicit formula for a large family of numbers and polynomials. Using these polynomials, we review most of the polynomials already studied in the book of Djordjevic - Milovanovic (2014). This study allows to unify the writing of these polynomials and obtain other extensions. Consequently the obtained results find their application in Number Theory; notably for the arithmetical properties of power products, Fermat numbers and Mersenne numbers.

Keywords: Exponential partial Bell polynomials; Bell polynomials, Cauchy product, Fermat and Mersenne numbers.

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1. Introduction

The Bell polynomials $B_{n,k}$ are defined as the coefficients of the following formal double series

$$\exp\left(u\sum_{k=1}^{+\infty}\frac{x_k}{k!}t^k\right)=\sum_{n,k\geq 0}B_{n,k}\left(x_1,x_2,\cdots\right)u^k\frac{t^n}{n!}. \tag{1.1}$$

The Bell polynomials play an important role in combinatorics and number theory and even in geometry. In fact, several special numbers such as Stirling numbers of first or of second kind, and the Bell numbers can be expressed as special values of Bell polynomials. It is interesting to ask whether there exists a unified approach to the study of special numbers. The goal of this work is to investigate this direction of research. We discovered some new formulas for some classical special numbers, and recovered some well-known identities in the literature. The method relies on the use of Bell polynomials. More precisely, we succeed to incorporate the techniques of the theory of generating functions mainly the Bell generating function into this investigation of some new identities for some special numbers. Some arithmetical applications are given. The paper is divided into three parts. The first part consists of the use the theory of generating functions in order to derive some new formulas. In the second part, we apply this method to recover some identities that appear in Djordjevic-Milovanovic s book. The last part focuses on some applications in number theory. The problem of finding a unified approach for the theory of generating functions is not new and goes back to Appell (1880). An Appell sequence $(P_n(x))_{n\in\mathbb{N}}$ is defined as follows

$$A(t)e^{xt}=\sum_{n=0}^{+\infty}P_n(x)\frac{t^n}{n}. \tag{1.2}$$

where $A(t)$ is a function satisfying certain conditions. For instance the Bernoulli polynomials form a special case of Appell polynomials. This path of research was continued by several mathematicians. By considering

$$\frac{te^{xt+yt^m}}{\lambda e^t\pm 1} \tag{1.3}$$

which is essentially the generating function for Bell polynomials with $x_1=x$, $x_m=m!y$ and $x_k=0$ for $k\neq 1,m$, we are led to the $2D$-dimensional Apostol-Bernoulli and the $2D$-dimensional Apostol-Euler, see reference [2]. Later, this approach was generalized in reference [3]. We considered a large class of generating functions:

$$\theta(t)\exp\left(-\sum_{k=1}^{m}x_kt^k\right), \tag{1.4}$$

where $\theta(t)$ is a function having an explicit Laurent expansion near $t = 0$. A special case of (1.4) is investigated in paragraph 2.1.

2. Bell polynomials and generating functions composition

Let $x_1, x_2, x_3, \cdots$ be a countable set of variables. Another generating function of exponential partial Bell polynomials $B_{n,k} = B_{n,k}(x_1, x_2, \cdots)$ [7] is

$$\frac{1}{k!}\left(\sum_{n\geq 1} x_n \frac{t^n}{n!}\right)^k = \sum_{n\geq k} B_{n,k}\frac{t^n}{n!}. \tag{2.1}$$

The exponential complete Bell polynomials $Y_n = Y_n(x_1, x_2, \cdots)$ are defined by:

$$\exp\left(\sum_{n\geq 1} x_n \frac{t^n}{n!}\right) = 1 + \sum_{n\geq 1} Y_n \frac{t^n}{n!}. \tag{2.2}$$

Consequently

$$Y_n = \sum_{k=1}^{n} B_{n,k},\ Y_0 = 1. \tag{2.3}$$

We assume that $B_{0,0} = 1$ and $B_{n,0} = 0$ if $n > 0$. Polynomials $B_{n,k}$ are homogeneous and have integral coefficients; their explicit formula is:

$$B_{n,k}(x_1, x_2, \cdots) = \frac{n!}{k!}\sum_{\substack{k_1+\cdots+k_n=k\\ k_1+2k_2+\cdots+nk_n=n}} \binom{k}{k_1, \cdots, k_j}\prod_{r=1}^{n}\left(\frac{x_r}{r!}\right)^{k_r}. \tag{2.4}$$

Some arithmetical properties of these polynomials are developed in [5]. This kind of polynomials and Cauchy product of generating functions [11] are useful for reviewing most of the polynomials already studied in [8], in order to unify their expressions. In what follows we recall the link of these polynomials to generating functions theory. Let $f(t) = \sum_{n\geq 0} a_n t^n$ and $g(t) = \sum_{n\geq 0} b_n t^n$ be two generating functions, with a_n and b_n two sequences of numbers. The Cauchy product of $f(t)$ and $g(t)$ is

$$f(t)\, g(t) = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} a_k b_{n-k} t^n. \tag{2.5}$$

Let A_n and B_n be the numbers generated respectively by the functions fog (for $b_0 \neq 0$) [14] and $\exp g$ (for $b_0 = 0$), which means that

$$f \circ g(t) = \sum_{n\geq 0} A_n \frac{t^n}{n!} \text{ and } \exp g(t) = \sum_{n\geq 1} B_n \frac{t^n}{n!}. \tag{2.6}$$

The expressions of sequences A_n and B_n by means of $B_{n,k}$ are respectively,

$$A_n = \sum_{k=0}^{n} B_{n,k}\left(b_1, 2!b_2, \cdots\right) f^{(k)}(b_0) \tag{2.7}$$

and

$$B_n = Y_n\left(b_1, 2!b_2, \cdots\right), \tag{2.8}$$

where $f^{(k)}$ is the k-th derivative of f. For the proof and more information we refer to [12, 13, 15, 16, 17, 19] and [20]. The q-analog case is developed in [18]. For α a complex number and $b_0 \neq 0$, let $b_n^{(\alpha)}$ be the sequence of numbers generated by the function g^{α}; namely $g^{\alpha}(t) = \sum_{n\geq 0} b_n^{(\alpha)} t^n$. Thereafter the explicit formula of $b_n^{(\alpha)}$ is

$$b_n^{(\alpha)} = \frac{1}{n!} \sum_{k=0}^{n} (\alpha)_k \, b_0^{\alpha-k} B_{n,k}\left(b_1, 2!b_2, \cdots\right), \tag{2.9}$$

where $(\alpha)_k = \alpha\,(\alpha-1)\cdots(\alpha-k+1)$ is a falling number. For the proof, we refer to [12] and [21, Identity 12 p.49]. In the case $\alpha = m$ a positive integer, we just have

$$b_n^{(-m)} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, k! \binom{m+k-1}{k} b_0^{-m-k} B_{n,k}\left(b_1, 2!b_2, \cdots\right), \tag{2.10}$$

Thereafter

$$b_n^{(-1)} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, k! b_0^{-1-k} B_{n,k}\left(b_1, 2!b_2, \cdots\right). \tag{2.11}$$

Letting the polynomials $A(t) = \sum_{j=0}^{m} a_j t^j$ and $B(t) = \sum_{j=0}^{r} b_j t^j$ of degree m and r respectively with coefficients in $\mathbb{C}[x]$. We consider the generating functions $A(t)B^{\alpha}(t) = \sum_{n\geq 0} b_n^{(A,\alpha)} t^n$ for $b_0 \neq 0$ and $A(t)\exp(B(t)) = \sum_{n\geq 0} b_n^{(A)} t^n$ for $b_0 = 0$. Here $a_j = 0$ and $b_k = 0$ for $j > m$ and $k > r$. By means of the previous properties and Cauchy product of generating functions we can easily show that

$$b_n^{(A,\alpha)} = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{a_{n-k}}{k!} (\alpha)_j b_0^{\alpha-j} B_{k,j}\left(b_1, 2!b_2, \cdots\right) \tag{2.12}$$

and

$$b_n^{(A)} = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{a_{n-k}}{k!} B_{k,j}\left(b_1, 2!b_2, \cdots\right). \tag{2.13}$$

If $b_0 = 1$ and $\alpha \in \mathbb{C}\backslash\mathbb{N}$, the identity (2.12) is reduced to

$$b_n^{(A,\alpha)} = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{a_{n-k}}{k!}(\alpha)_j B_{k,j}\left(b_1, 2!b_2, \cdots\right). \tag{2.14}$$

This approach of studying generating functions is different of that adopted in [3]; based on the use of the theory of zeta functions, which gives a new description of special polynomials and special numbers as special values of certain zeta functions such as the Riemann zeta function. One can consult [23] for a description of the theory of zeta functions, and [4] for some applications.

2.1. **Main results.** Let m be a positive integer. For polynomials of the form $g(t) := g_m(t) = 1 + b_1 t + b_m t^m$ with $m \geq 2$, we obtain the following theorem

Theorem 2.1. *Polynomials $b_n^{(\alpha)}$ and $b_n^{(A,\alpha)}$ are given respectively by the relations*

$$b_n^{(\alpha)} = \sum_{j=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \binom{\alpha}{n-(m-1)j}\binom{n-(m-1)j}{j} b_1^{n-mj} b_m^j \tag{2.15}$$

and

$$b_n^{(A,\alpha)} = \sum_{k=0}^{n}\sum_{j=0}^{\left\lfloor \frac{k}{m} \right\rfloor} \binom{\alpha}{n-(m-1)j}\binom{n-(m-1)j}{j} a_{n-k} b_1^{k-mj} b_m^j. \tag{2.16}$$

Proof. Since we have

$$B_{n,j}\left(b_1, 0, \cdots, 0, m!b_m\right) = \sum_{\substack{j_1+j_m=j \\ j_1+mj_m=n}} \frac{n!}{j_1!j_m!} b_1^{j_1} b_m^{j_m}.$$

Thus

$$B_{n,j}\left(b_1, 0, \cdots, 0, m!b_m\right) = \frac{n!}{j!}\binom{j}{i} b_1^{j-i} b_m^i,$$

with $n = j - i + mi$ and the expression of $b_n^{(\alpha)}$ is immediate. The second result follows from Cauchy product of $A(t)$ and $g^{\alpha}(t)$. $\square$

Taking $A(t) := A_r(t) = a_0 + a_r t^r, r \geq 1$ to deduce the following corollary

Corollary 2.2. *We have for $n < r$;*

$$b_n^{(A,\alpha)} = a_0 \sum_{j=0}^{\left\lfloor \frac{n}{m} \right\rfloor} \binom{\alpha}{n-(m-1)j}\binom{n-(m-1)j}{j} b_1^{n-mj} b_m^j, \ n < r$$

and for $n \geq r$ the formulation

$$\begin{aligned}
b_n^{(A,\alpha)} &= a_0 \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} \binom{\alpha}{n-(m-1)j} \binom{n-(m-1)j}{j} b_1^{n-mj} b_m^j \\
&+ a_r \sum_{j=0}^{\lfloor \frac{n-r}{m} \rfloor} \binom{\alpha}{n-r-(m-1)j} \binom{n-r-(m-1)j}{j} b_1^{n-r-mj} b_m^j.
\end{aligned}$$

If $m = 2$ and $(r, \alpha) = (1, -1)$ we have

$$\begin{aligned}
b_n^{(A,-1)} &= a_0 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n-j} \binom{n-j}{j} b_1^{n-2j} b_2^j \\
&- a_1 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-j} \binom{n-r-j}{j} b_1^{n-r-2j} b_2^j.
\end{aligned}$$

Regarding the identity

$$\binom{n-j}{j} = \frac{n-j}{n-2j} \binom{n-j-1}{j},$$

we get the following corollary

Corollary 2.3. *For $g(t) = g_2(t)$ we have*

$$b_{2n}^{(A,-1)} = a_0(-b_2)^n + \sum_{j=0}^{n} (-1)^j \binom{2n-1-j}{j} \left(\frac{2n-j}{2n-2j} a_0 b_1 - a_1 \right) b_1^{2n-1-j} b_2^j$$

and

$$b_{2n+1}^{(A,-1)} = \sum_{j=0}^{n-1} (-1)^{j+1} \binom{2n-j}{j} \left(\frac{2n+1-j}{2n+1-2j} a_0 b_1 - a_1 \right) b_1^{2n-j} b_2^j.$$

In order to compute polynomials $b_n^{(A)}$ generated by the function $A(t) \exp B^\alpha(t)$ we use the identities (2.8) and (2.9) as follows

$$\exp g^\alpha(t) = \sum_{n \geq 1} \sum_{k=1}^{n} B_{n,k} \left(b_1^{(\alpha)}, 2! b_2^{(\alpha)} \cdots \right) \frac{t^n}{n!}. \tag{2.17}$$

The Cauchy product of $A(t)$ and $\exp g^\alpha(t)$ conducts to

$$b_n^{(A)} = \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{a_{n-j}}{j!} B_{j,k} \left(b_1^\alpha, 2! b_2^{(\alpha)} \cdots \right). \tag{2.18}$$

Table 1. Examples of polynomials in the book [8]

Polynomial	Name	Polynomial	Name
$G_n^{(-\alpha)}(x)$	Gegenbauer [31, 33]	$P_{n,k}(x)$	Convolved Pell [8]
$A_n(x)$	Horadam [8, 25]	$J_n(x)$	Jacobsthal [27]
$f_n(x)$	Fermat (first kind) [8]	$B_n(x)$	Horadam [8]
$T_n(x)$	Chebyshev (second kind) [25]	$Q_n(x)$	Pell-Lucas [26]
$P_n(x)$	Pell [26]	$Q_{n,k}(x)$	Convolved Pell-Lucas [9]
$F_n(x)$	Fibonacci [32]	$\pi_n^{(a,b)}(x)$	Mixed Pell [24]

Table 2. Explicit formulae of polynomials in Table 1

Polynomial	Explicit formula
$G_n^{(-\alpha)}(x)\left[\left(1-2xt+t^2\right)^{-\alpha}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{-\alpha}{n-j}\binom{n-j}{j}(-2x)^{n-2j}$
$A_n(x)\left[\left(1-pxt-qt^2\right)^{-1}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}q^j(px)^{n-2j}$
$f_n(x)\left[\left(1-xt+2t^2\right)^{-1}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}(-2)^jx^{n-2j}$
$T_n(x)\left[\left(1-2xt+t^2\right)^{-1}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}(-1)^jx^{n-2j}$
$P_n(x)\left[\left(1-2xt-t^2\right)^{-1}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}(2x)^{n-2j}$
$F_n(x)\left[\left(1-xt-t^2\right)^{-1}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}x^{n-2j}$
$P_n^{(k)}(x)\left[\left(1-2xt-t^2\right)^{-k-1}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{-k-1}{n-j}\binom{n-j}{j}(-1)^{n-j}(2x)^{n-2j}$
$J_n(x)\left[\left(1-t+2xt^2\right)^{-1}\right]$	$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}(-2x)^j$

For $A(t)=1$, letting $\alpha=-1$ in the identity (2.18) to obtain

$$b_n^{(A)}=\frac{1}{n!}\sum_{k=1}^{n}b_{n,k}\left(b_1^{(-1)},2!b_2^{(-1)},\cdots\right). \tag{2.19}$$

With the generating functions AB^{α} and $A\exp B^{\alpha}$, we construct finitely many families of polynomials including Fibonacci, Gegenbauer, Jacobsthal, Fermat-Lucas polynomials (see Table 1 and Table 2) and other interesting polynomials of the book [8]. The notation $a_n(x)[f(x,t)]$ in Table 2 means that $f(x,t)=\sum_{n\geq 0}a_n(x)t^n$. For the polynomials $B_n(x),Q_n(x),Q_{n,k}(x)$ and $\pi_n^{(a,b)}(x)$ in Table 1, we recall that

$$\frac{1+qt^2}{1-pxt-qt^2}=\sum_{n\geq 0}B_n(x)t^n,$$

$$\frac{1+t^2}{1-2xt-t^2} = \sum_{n\geq 0} Q_n(x)t^n,$$

$$\left(\frac{2+2xt}{1-2xt-t^2}\right)^{k+1} = \sum_{n\geq 0} Q_{n,k}(x)t^n,$$

and

$$\frac{(2x+2t)^b}{(1-2xt-t^2)^{a+b}} = \sum_{n\geq 0} \pi_n^{(a,b)}(x)t^n,$$

where a, b are two integers such that $b \geq 1$ and $a+b \geq 1$. The closed formulae of $B_n(x)$ and $Q_n(x)$ are respectively

$$B_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} q^j (px)^{n-2j} + q \sum_{j=0}^{\lfloor n-2/2 \rfloor} \binom{n-j-2}{j} q^j (px)^{n-2j-2}$$

and

$$Q_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{j} (2x)^{n-2j} + \sum_{j=0}^{\lfloor n-2/2 \rfloor} \binom{n-j-2}{j} (2x)^{n-2j-2}.$$

To compute explicit formula of $Q_{n,k}(x)$, we can choose between two methods. The first consists to write

$$\left(\frac{2+2xt}{1-2xt-t^2}\right)^{k+1} = 2^{k+1}(x+t)^{k+1}\left(1-2xt-tp^2\right)^{-k-1}.$$

Since

$$(x+t)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k-j} t^j,$$

then

$$Q_{n,k}(x) = 2^{k+1} \sum_{j=0}^{n} \binom{k+1}{n-j} x^{k-n+j} P_{j,k}(x).$$

For the second we have

$$2\frac{x+t}{1-2xt-t^2} = 2(x+t)\sum_{n\geq n} P_n(x)t^n + 2x + 2\sum_{n\geq 1}\left(xP_n(x) - P_{n-1}(x)\right)t^n.$$

It is obvious to remark that

$$Q_n(x) = \frac{1}{n!}\sum_{j=0}^{n} j!\binom{k+1}{j}(2x)^{k-j+1} B_{n,j}\left(1!s_1(x), 2!s_2(x), \cdots\right),$$

where $s_r(x) = xP_n(x) - P_{n-1}(x)$. The recurrence relation [8] of $\pi_n^{(a,b)}$is

$$\pi_n^{(a,b)}(x) = 2^b \sum_{j=0}^{n} \binom{b}{j} \pi_{n-j}^{(a+b,0)}(x) x^{b-j}. \tag{2.20}$$

Since we have
$$\pi_{n+1}^{(a,b)}(x) = b_n^{(A,-a-b)},$$
where
$$A(t) = 2^b(x+t)^b = 2^b \sum_{j=0}^{b} \binom{b}{j} x^{b-j} x^j \text{ and } g(t) = 1 - 2xt - t^2.$$
Systematically we have
$$\begin{aligned} \pi_{n+1}^{(a,b)}(x) &= 2^b \sum_{j=0}^{n} \sum_{i=0}^{\lfloor (n-j)/2 \rfloor} \binom{b}{j} \binom{-b-a}{n-j-i} \binom{n-j-i}{i} \\ &\times (-1)^{n-j+i} 2^{n-j-2i} x^{n+b-2j-2j}. \end{aligned}$$
We end this section by examples of generating function of the form $\exp tg^{\alpha}$. Let the generating function
$$\exp(b_1 t + b_m t^m) = \sum_{n \geq 0} b_n^{(\star)} t^n. \tag{2.21}$$
Since
$$b_n^{(\star)} = \frac{1}{n!} \sum_{k=0}^{n} B_{n,k}\left(b_1, 0, \cdots, 0, m! b_m\right).$$
and
$$B_{n,k}\left(b_1, 0, \cdots, 0, m! b_m\right) = \frac{n!}{k!} \binom{k}{j} b_1^{k-j} b_m^j,$$
for $n = k - j + mj$ and zero otherwise. We conclude that
$$b_n^{(\star)} = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{1}{(n-(m-1)k)!} \binom{n-(m-1)k}{k} b_1^{n-mk} b_m^k. \tag{2.22}$$
The two variables Hermite Kamp de F riet polynomials $H_n(x,y)$ [1] are a special case of $b_n^{(\star)}$. We have
$$\exp\left(xt + yt^2\right) = \sum_{n \geq 0} H_n(x,y) \frac{t^n}{n!}. \tag{2.23}$$
Taking $m = 2, b_1 = x$ and $b_2 = y$ in the identity (2.22) to deduce that
$$H_n(x,y) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{x^{n-2k} y^k}{k!(n-2k)!}. \tag{2.24}$$
We have just another proof of the result already found in [1] and generalized in [19]. Regarding the identity
$$(1 + b_m t^m)^{-\alpha} = \sum_{n \geq 0} \binom{-\alpha}{n} b_m^n t^{mn},$$

we write

$$(1+b_m t^m)^{-\alpha} = \sum_{n\geq 0} I_{n,m} \binom{-\alpha}{\frac{n}{m}} b_m^{\frac{n}{m}} t^n.$$

The sequence $I_{n,m}$ is one if n is a multiple of m and zero otherwise. In the same way we have

$$\frac{a_1 t}{1+b_m t^m} = a_1 t \sum_{n\geq 1} \binom{-1}{\frac{n}{m}} I_{n,m} b_m^n t^n.$$

The homogenization of the terms of the above series gives

$$\frac{a_1 t}{1+b_m t^m} = \sum_{n\geq 1} a_1 \binom{-1}{\frac{n-1}{m}} I_{n-1,m} b_m^{n-1} t^n.$$

Letting

$$\exp\left(\frac{a_1 t}{1+b_m t^m}\right) = \sum_{n\geq 0} b_n^{(c)}(x) t^n.$$

Thus

$$b_n^{(c)} = \frac{1}{n!} \sum_{j=0}^{n} b_{n,j}\left(\cdots, a_1(-1)^{r-1} I_{r-1,m} b_m^{k-1}, \cdots\right).$$

The computation of this family of Bell polynomials states that

$$b_{n,j}\left(\cdots, a_1(-1)^{r-1} I_{r-1,m} b_m^{k-1}, \cdots\right) = a_1^k \left(-b_m\right)^{n-k} B_{n,k}\left(I_{0,m}, I_{1,m}, \cdots\right),$$

and

$$b_n^{(c)}(x) = \frac{1}{n!} \sum_{j=0}^{n} a_1^k (-b_m)^{n-k} B_{n,k}\left(I_{0,m}, I_{1,m}, \cdots\right). \tag{2.25}$$

Now we consider

$$(1+b_m t^m)^{-\alpha} \exp\left(\frac{a_1 t}{1+b_m t^m}\right) = \sum_{n\geq 0} b_n^{(A_1)} t^n.$$

The Cauchy product of $(1+b_m t^m)^{-\alpha}$ and $\exp\left(\frac{a_1 t}{1+b_m t^m}\right)$ conducts to

$$b_n^{(A_1)}(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{1}{k!} a_1^j (-1)^{k-j} I_{n-k,m} \binom{-\alpha}{n-k} b_m^{n-j} B_{k,j}\left(I_{0,m}, I_{1,m}, \cdots\right).$$

The explicit form of $B_{n,k}(I_{0,m}, I_{1,m}, \cdots)$ depends on the divisibility on m and remains an open question. Many authors are interested in this kind of problems, we can cite [30] about some explicit formulas of certain Bell polynomials.

3. Consequences

In this section, we use the results of the previous sections to establish some new identities in number theory. Recently we introduced the Fermat arithmetical function $f_n^+(a,b) = a^n + b^n$ for a and b real numbers and proved that

$$f_{2n}^+(a,b) = 2(-ab)^n + \sum_{j=0}^{n-1} \frac{n}{n-j} \binom{2n-j-1}{j} (-ab)^j (a+b)^{2(n-j)}$$

and

$$f_{2n+1}^+(a,b) = \sum_{j=0}^{n} \frac{2n+1}{2(n-j)-1} \binom{2n-j}{j} (-ab)^j (a+b)^{2(n-j)+1}.$$

It is reported that there is a misprint in the formula (2.7) of Theorem 2.2 [12]. The generating function is

$$\frac{2-(a+b)t}{1-(a+b)t+abt^2} = \sum_{n\geq 0} f_n^+(a,b)t^n. \tag{3.1}$$

This generating function corresponds to the function Ag_2^α, where $\alpha = -1$, $A(t) = 2-(a+b)t$ and $g_2(t) = 1-(a+b)t+abt^2$. Thus we can write

$$\frac{(a^n-(-b)^n)^2}{(a+b)^2} = \sum_{j=0}^{n-1} \frac{n}{n-j} \binom{2n-j-1}{j} (-ab)^j (a+b)^{2(n-j-1)} \tag{3.2}$$

and

$$\frac{a^{2n+1}+b^{2n+1}}{a+b} = \sum_{j=0}^{n} \frac{2n+1}{2(n-j)-1} \binom{2n-j}{j} (-ab)^j (a+b)^{2(n-j)}. \tag{3.3}$$

The quantities

$$\sum_{j=0}^{2n} \frac{2n+1}{2n-j+1} \binom{4n-j+1}{j} (-ab)^j (a+b)^{4n-2j}$$

and

$$\sum_{j=0}^{n} \frac{2n+1}{2(n-j)-1} \binom{2n-j}{j} (-ab)^j (a+b)^{2(n-j)}$$

are integers. For the pair (a, a) we find

$$\frac{(1-(-1)^n)^2}{n} = \sum_{j=0}^{n-1} \frac{(-1)^j}{n-j}\binom{2n-j-1}{j}4^{n-j}, \tag{3.4}$$

$$\frac{1}{2n+1} = \sum_{j=0}^{n} \frac{(-1)^j}{2n-2j+1}\binom{2n-j}{j}4^{n-j}. \tag{3.5}$$

Substituting successively $2n$, $2n+1$ in identity (3.4) to get

$$\sum_{j=0}^{2n-1} \frac{(-1)^j}{2n-j}\binom{4n-j-1}{j}4^{2n-j} = 0,$$

$$\frac{1}{2n+1} = \sum_{j=0}^{2n} \frac{(-1)^j}{2n-j+1}\binom{4n-j+1}{j}4^{2n-j}.$$

Thus

$$\sum_{j=0}^{2n} \frac{(-1)^j}{2n-j+1}\binom{4n-j+1}{j}4^{2n-j} = \sum_{j=0}^{n} \frac{(-1)^j}{2n-2j+1}\binom{2n-j}{j}4^{n-j}.$$

3.1. **Application to Jacobsthal-Lucas and Mersenne numbers.** The Jacobsthal-Lucas numbers [6] are defined by the recursive formula

$$j_{n+2} = j_{n+1} + 2j_n,\ j_0 = 2,\ j_1 = 1. \tag{3.6}$$

The binet formula of this sequence is given by

$$j_n = 2^n - (-1)^n. \tag{3.7}$$

$j_{2n} = M_{2n}$ is a Mersenne number. We recall that M_n is given by relation $M_n = 2^n - 1$. Eakin [10] gave the following combinatorial partition of M_n

$$M_n = \binom{2n}{n}^{-1} \sum_{j=0}^{n} \left(\frac{n+1}{j}\binom{2n-2j}{n-j} \sum_{k=0}^{\lfloor (j-1)/2 \rfloor} (-1)^j \binom{2j}{k}\binom{n+3j-2k}{j-2k-1} \right).$$

From (3.2) we obtain combinatorial formulations for the squares:

$$j_n^2 = \sum_{j=0}^{n-1} \frac{n}{n-j}\binom{2n-j-1}{j}(-2)^j 9^{n-j} \tag{3.8}$$

and

$$M_n^2 = \sum_{j=0}^{n-1} \frac{n}{n-j}\binom{2n-j-1}{j}2^j. \tag{3.9}$$

The Fermat numbers $F_n = 2^{2^n} + 1$ [29] lie to generalized sequence of numbers $F^\star_n = 2^n + 1$. From (3.3) we obtain the identity

$$F^\star_{2n+1} = 3\sum_{j=0}^{n} \frac{2n+1}{2n-2j+1}\binom{2n-j}{j}(-2)^j 9^{n-j} \tag{3.10}$$

and

$$M_{2n+1} = \sum_{j=0}^{n} \frac{2n+1}{2n-2j+1}\binom{2n-j}{j}2^j. \tag{3.11}$$

To compute M_{2n} we use the relation $M_{2n+1} - 1 = 2M_{2n}$:

$$M_{2n} = \sum_{j=1}^{n} \frac{2n+1}{2n-2j+1}\binom{2n-j}{j}2^{j-1}. \tag{3.12}$$

Since $j_n^2 = F^\star_{2n} - 2(-1)^n 2^n$, then from identity (3.2) we deduce that

$$F^\star_{2n} = (-1)^n 2^{n+1} + \sum_{j=0}^{n-1} \frac{2n+1}{2n-2j+1}\binom{2n-j}{j}(-2)^j 9^{n-j}. \tag{3.13}$$

Catarino et al. [6] proved that $M_n^2 = 4^n - M_{n+1}$, the expression of M_n without regarding the parity of n is

$$M_n = 4^{n-1} - \sum_{j=0}^{n-2} \frac{n-1}{n-j-1}\binom{2n-j-3}{j}2^j. \tag{3.14}$$

It is easy to verify that $F^\star_{2n+1}$ is a multiple of 3 and a multiple of $2n+1$ if and only if

$$\sum_{j=0}^{n} \frac{1}{2n-2j+1}\binom{2n-j}{j}(-2)^j 9^{n-j}$$

is an integer. The first three cases are $3|F^\star_3, 9|F^\star_9$ and $27|F^\star_{27}$. Similarly M_{2n} is a multiple of $2n+1$ if and only if

$$\sum_{j=1}^{n} \frac{1}{2n-2j+1}\binom{2n-j}{j}2^{j-1} \in \mathbb{N}.$$

The first cases are $3|M_2, 5|M_4, 7|M_6, 11|M_{10}, 13|M_{12}$ and $17|M_{16}$. So are the sequences $2n+1|F^\star_{2n+1}$ and $2n+1|M_{2n}$ infinite? Is the sequence $2n+1|M_{2n+1}$ without values or does it admit an infinity? For which value of n, the quantities

$$\sum_{j=0}^{n} \frac{1}{2n-2j-1}\binom{2n-j}{j}(-2)^j 9^{n-j} \text{ and } \sum_{j=1}^{n} \frac{1}{2n-2j-1}2^{j-1}$$

are integers?

3.2. **Application to power products.** Let m be a positive integer, in what follows we are interested by writing m^n as a linear combination of m^k,$0 \le k \le n$: $m^n = \sum_{k=0}^{n-1} a_k m^k$, $a_k \in \mathbb{Q}$. This writing is inspired from the representation of integers as linear combinations of power products [22]. First consider $c = a^2$ and $d = b^2$ to write identities (3.2) and (3.3) in the following forms

$$c^n + d^n - 2\left(-\sqrt{cd}\right)^n = \sum_{j=0}^{n-1} \frac{n}{n-j}\binom{2n-j-1}{j}\left(-\sqrt{cd}\right)^j\left(\sqrt{c}+\sqrt{d}\right)^{2n-2j}$$

and

$$\sqrt{c}c^n + \sqrt{d}d^n = \sum_{j=0}^{n} \frac{2n+1}{2n-2j+1}\binom{2n-j}{j}\left(-\sqrt{cd}\right)^j\left(\sqrt{c}+\sqrt{d}\right)^{2n-2j+1}.$$

Equalizing c and d to conclude that

$$4^n = \frac{1+(-1)^n}{2} - (-1)^n(n+1)^2 + \sum_{j=1}^{n-1} \frac{n+1}{n-j+1}\binom{2n-j+1}{j}(-1)^{j+1}4^{n-j}$$

and

$$4^n = 1 - (-1)^n(2n+1) + \sum_{j=1}^{n-1} \frac{2n+1}{2n-2j+1}\binom{2n-j}{j}(-1)^{j+1}4^{n-j}.$$

Letting $v_n(4) = (a_{n-1}, \cdots a_0) \in \mathbb{Q}^n$ for which we have $4^n = a_{n-1}4^{n-1} + a_{n-2}4^{n-1} + \cdots + a_0$, then 4^n is a linear combination of $1, 4, , 4^{n-1}$ in two different ways. Some values in the case $n = 2$ are given in Table 3. In the general case $v_n(m)$ depend on the parity of n. Taking $d = m^2c$ to write

$$m^{2n} = 2(-m)^n - 1 + \sum_{j=0}^{n-1} \frac{n}{n-j}\binom{2n-j-1}{j}(m+1)^{2n-2j}(-m)^j$$

and

$$m^{2n+1} = -1 + \sum_{j=0}^{n} \frac{2n+1}{2n-2j+1}\binom{2n-j}{j}(m+1)^{2n-2j+1}.$$

Consequently the power product m^n can be written in two different ways as a linear combination of the lower powers. The other two scripts are:

$$m^{2n} = \frac{m-2(-1)^n m^{n+1}}{2n} - \frac{1}{2}\sum_{1} \frac{1}{n-j}\binom{2n-j-1}{j}\binom{2n-2j}{k-j-1}(-1)^j m^k$$

and

$$m^{2n+1} = \frac{m}{2n+1} - \sum_{2} \frac{1}{2n-2j+1}\binom{2n-j}{j}\binom{2n-2j+1}{k-j-1}(-1)^j m^k,$$

Table 3. Few vectors $v_n(4)$

Form 1 of $v_n(4)$	Form 2 of $v_n(4)$
$4^2 = (6, -8)$	$4^2 = (5, -4)$
$4^3 = (8 - 20, 16)$	$4^3 = (7 - 14, 8)$
$4^4 = (10, -35, 50, -24)$	$4^4 = (9, -27, 30, -8)$
$4^5 = (12, -54, 112, -105, 36)$	$4^5 = (11, -44, 77, -55, 12)$
$4^6 = (14, -77, 210, -294, 196, -48)$	$4^6 = (13, -65, 156, -182, 91, -12)$
$4^7 = (16, 10, 352, -660, 672, -336, 64)$	$4^7 = (15, -90, 275, -450, 378, -140, 16)$

where

$$\sum_1 = \sum_{k=1}^{2n-1} \sum_{j=1}^{\lfloor (2n-k+1)/2 \rfloor} \text{ and } \sum_2 = \sum_{k=1}^{2n} \sum_{j=0}^{\lfloor (2n-k+2)/2 \rfloor} .$$

4. Discussion

This work unified the expression of a large family of polynomials known in the literature. This is the case with Gegenbauer, Horadam, Fermat, Pell, Jacobsthal, Humbert [28] and Morgan-Voyce polynomials. They are not the only ones, the list is long, the space available does not allow to quote them all. As we have seen, they are all linear combinations of exponential partial Bell polynomials. These polynomials are relevant tools of generating functions theory and number sequences. We can clearly see their importance in the number theory; as is the case Fermat and Mersenne numbers and power numbers. Using Bell polynomials; we succeeded in giving formal decomposition in product of two numbers for some number sequences and to ask some open questions concerning number fields and prime numbers.

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