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Initial Lucas Polynomial Coefficient Bounds for Bi-Bazilevič Functions

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Abstract. Our current investigation is primarily motivated by the application of special polynomials in Geometric Function Theory (GFT). This paper aims to utilize (M, N)-Lucas polynomials to estimate the initial coefficient bounds $|a_2|$ and $|a_3|$ for a subclass of bi-univalent functions $\mathcal{GB}_{\Sigma}^{\kappa,\nu}(x)$ consisting of analytic functions normalized by the condition $f(0) = f'(0) - 1 = 0$. We then derive the famous Fekete-Szegö inequality estimate. We also establish connections between our results and those examined in previous investigations.

Keywords: Bi-univalent function, Bazilevič function, Fekete-Szegö estimate, Lucas polynomials.

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1. Introduction

Let $\mathcal{A}$ represent the class of analytic functions f defined on the unit disk, $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, and normalized such that $f(0) = f'(0) - 1 = 0$. Consequently, each function $f \in \mathcal{A}$ can be expressed by a Taylor-Maclaurin series expansion in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Furthermore, consider the class of all functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$, denoted by $\mathcal{S}$.

For any two analytic functions f_1, f_2 in $\mathbb{U}$, f_1 is subordinate to f_2, denoted by $f_1 \prec f_2$, if there exist an analytic function ϕ such that $\phi(0) = 0, |\phi(z)| < 1$, and $f_1(z) = f_2(\phi(z))$, $z \in \mathbb{U}$. Notably, when f_2 is univalent in $\mathbb{U}$, we arrive

$$f_1 \prec f_2, \qquad (z \in \mathbb{U}) \;\Leftrightarrow\; f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathbb{U}) \subset f_2(\mathbb{U}).$$

Two significant and extensively studied subclasses of $\mathcal{S}$ are $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α in $\mathbb{U}$, and $\mathcal{K}(\alpha)$, the class of convex functions of order α in $\mathbb{U}$. According to their definitions, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : \; f \in \mathcal{S} \;\text{ and }\; \Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \right\} \quad (z \in \mathbb{U}; 0 \le \alpha < 1)$$

and

$$\mathcal{K}(\alpha) := \left\{ f : \; f \in \mathcal{S} \;\text{ and }\; \Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \right\} \quad (z \in \mathbb{U}; 0 \le \alpha < 1).$$

It is clear from the above definitions that $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha)$ and

$$f \in \mathcal{K}(\alpha) \;\text{ if and only if }\; zf' \in \mathcal{S}(\alpha),$$

$$f \in \mathcal{S}^*(\alpha) \;\text{ if and only if }\; \int_0^z \frac{f(t)}{t} dt = F(z) \in \mathcal{K}(\alpha).$$

By familiar Koebe one-quater theorem, we know that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Hence, every function $f \in \mathcal{S}$ possesses an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (\, z \in \mathbb{U}\,)$$

and

$$f(f^{-1}(\omega)) = \omega \quad \left(|\omega| < r_0(f); r_0(f) \ge \frac{1}{4} \right),$$

where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots .$$

A function $f \in \mathcal{A}$ is considered bi-univalent in $\mathbb{U}$ if both f and its inverse f^{-1} are univalent in $\mathbb{U}$. The class of such bi-univalent functions in $\mathbb{U}$ is denoted by Σ as defined in (1.1). The work of Srivastava et al. (refer to [28]) is a significant contribution to this field, providing valuable examples of functions belonging to the class Σ. Let us recall some well known examples:

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

A function $f \in \mathcal{A}$ belongs to the class $\mathcal{M}^{\nu}$, where $\nu \in \mathbb{R}$, if and only if

$$\operatorname{Re}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)^{\nu}\left(\frac{zf'(z)}{f(z)}\right)^{1-\nu}\right\} > 0, \qquad z \in \mathbb{U}.$$

According to [10], it was demonstrated that $\mathcal{M}^{\nu}$ is a subset of $\mathcal{S}^{*}$.

A function $f \in \mathcal{A}$ is said to be Bazilevič function of order κ, denoted by $\mathcal{B}(\kappa)$ which is a subclass of $\mathcal{S}$ (see [25]) , if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f^{1-\kappa}(z)z^{\kappa}}\right\} > 0, \qquad z \in \mathbb{U}.$$

The research in [19] shows that $\mathcal{B}(\kappa)$ is the broadest subclass of univalent functions, containing many of the familiar subclasses of $\mathcal{S}$. In [13], Fitri and Thomas introduced and studied the class $\mathcal{B}(\kappa,\nu)$ an analogue of $\mathcal{M}^{\nu}$. A function $f \in \mathcal{A}$, with $f(z) \neq 0$, $f'(z) \neq 0$ is said to be in class $\mathcal{B}(\kappa,\nu)$, if

$$\Re\left\{\left(\frac{zf'(z)}{f^{1-\kappa}(z)z^{\kappa}}+\frac{zf''(z)}{f'(z)}+(\kappa-1)\Big(\frac{zf'(z)}{f(z)}-1\Big)\right)^{\nu}\left[\frac{zf'(z)}{f^{1-\kappa}(z)z^{\kappa}}\right]^{1-\nu}\right\} > 0,$$

where $\nu \geq 0$, $\kappa \geq 0$ and $z \in \mathbb{U}$. In [13, Theorem 1, p. 3], it was proved that the functions in $\mathcal{B}(\kappa,\nu)$ are univalent in $\mathbb{U}$. For more information on Bazilevič function, one can refer to [9, 12, 13, 15, 21, 22, 27, 23, 24, 26] and [29].

Another prominent issue in GFT is the Fekete-Szegö estimate $|a_3 - \mu a_2^2|$ for functions $f \in \mathcal{S}$. This problem originates from the refutation of the Littlewood-Paley conjecture by Fekete and Szegö (see [11]), which proposed that the coefficients of odd univalent functions should be bounded by one. The estimate has since attracted considerable interest, particularly in the analysis of various subclasses of univalent functions.

Modern science places significant interest in the theory and applications of special polynomials like Dickson polynomials, Chebyshev polynomials, Fibonacci polynomials, Lucas polynomials, and Lucas-Lehmer polynomials. These polynomials are fundamental in Mathematics and have numerous applications in Combinatorics, Number theory, Numerical analysis, GFT and more.

Lucas polynomials exhibit symmetry similar to Fibonacci polynomials, due to the recurrence relation. This symmetry may also play a role in certain geometric applications within GFT, where symmetric polynomials provide balanced mappings between symmetric domains.

In 1970, Bicknell [8] studied Lucas polynomial $\mathcal{L}_n(x)$ and defined it by

$$\mathcal{L}_n(x) = x\mathcal{L}_{n-1}(x) + \mathcal{L}_{n-2}(x) \quad (n \in \{2,3,4,\cdots\}),$$

with initial condition $\mathcal{L}_0(x) = 2$ and $\mathcal{L}_1(x) = x$. This Lucas polynomial was extended by Lee and Aşçı [16]. Let $M(x)$ and $N(x)$ be polynomials with real coefficients. The (M, N)- Lucas polynomials $\mathcal{L}_{M,N,n}(x)$ or simply $\mathcal{L}_n(x)$ is defined by the recurrence relation

$$\mathcal{L}_n(x) = M(x)\mathcal{L}_{n-1}(x) + N(x)\mathcal{L}_{n-2}(x) \quad (n \in \{2, 3, 4, \cdots\}) \tag{1.2}$$

with

$$\begin{aligned}
\mathcal{L}_0(x) &= 2, \\
\mathcal{L}_1(x) &= M(x), \\
\mathcal{L}_2(x) &= M^2(x) + 2N(x), \\
\mathcal{L}_3(x) &= M^3(x) + 3M(x)N(x),
\end{aligned}$$

and so on. The Lucas polynomials $\mathcal{L}_n(x)$ are described by their generating function, as shown in [17]:

$$G_{\mathcal{L}_n(x)}(z) := \sum_{n=0}^{\infty} \mathcal{L}_n(x) z^n = \frac{2 - M(x)z}{1 - M(x)z - N(x)z^2}.$$

Univalence in $\mathbb{U}$ is significant in GFT as it implies the function behaves well geometrically, without folding or overlapping over itself. Whether or not $\mathcal{L}_n(x)$ is univalent in $z \in \mathbb{U}$ would depend on the degree n, as higher-degree polynomials tend to introduce more complex behavior like critical points where the derivative vanishes. For low degrees, it is possible that the Lucas polynomials exhibit univalence in certain subdomains $\mathbb{U}$.

Remark 1.1. Depending on the chosen values of $M(x)$ and $N(x)$, the (M, N)-Lucas polynomial $\mathcal{L}_n(x)$ can correspond to various polynomials. Below are some examples:

Table 1. Polynomials and their corresponding generating functions

$M(x)$	$N(x)$	**Polynomials**
x	1	Lucas polynomials $\mathcal{L}_n(x)$
$2x$	1	Pell-Lucas polynomials $Q_n(x)$
1	$2x$	Jacobsthal-Lucas polynomials $j_n(x)$
$3x$	-2	Fermat-Lucas polynomials $f_n(x)$
$2x$	-1	Chebyshev polynomials $T_n(x)$ of the first kind

In recent mathematical literature, researchers have found coefficient estimates for functions in Σ, analogously to the work by Srivastava et al. [28]. Numerous articles have provided estimates regarding the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor Maclaurin series expansion of the form (1.1) using special polynomials (see [3, 4, 5, 6, 14, 20, 30, 31, 32]). The general coefficient bounds $|a_n|$ for $n \in \mathbb{N}$ with $n \geq 3$ for functions $f \in \Sigma$ have not been fully addressed for many subclass of Σ. It is worth mentioning that Altınkaya and Yalçın (See [1]) was the first to introduce a subclass of bi-univalent functions through the use of Lucas polynomials. This methodology aims to connect GFT and the Theory of Special Functions. Motivated by the works of Altınkaya and Yalçın [1] and N. Magesh et al. [18], in this paper we describe a subclass of Σ denoted by $\mathcal{GB}_{\Sigma}^{\kappa,\nu}(x)$ as given in the Definition 1.2 which gives a bridge between many new subclasses of $\mathcal{S}^*$ and $\mathcal{K}$ in connection with (p,q)-Lucas polynomial. We first derive the coefficient estimates $|a_2|$ and $|a_3|$ and Fekete and Szegö inequalities for functions belonging to these classes.

Definition 1.2. For $\nu \geq 0$, $\kappa \geq 0$, a function $f \in \Sigma$ belongs to the class $\mathcal{GB}_{\Sigma}^{\kappa,\nu}(x)$, under the condition that the following requirements are satisfied:

$$\left[\frac{zf'(z)}{f^{1-\kappa}(z)z^{\kappa}} + \frac{zf''(z)}{f'(z)} + (\kappa-1)\Big(\frac{zf'(z)}{f(z)} - 1\Big)\right]^{\nu}\left[\frac{zf'(z)}{f^{1-\kappa}(z)z^{\kappa}}\right]^{1-\nu} \prec G_{\mathcal{L}_n(x)}(z) - 1,$$

and

$$\left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^{\kappa}} + \frac{\omega g''(\omega)}{g'(\omega)} + (\kappa-1)\Big(\frac{\omega f'(\omega)}{f(\omega)} - 1\Big)\right]^{\nu}\left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^{\kappa}}\right]^{1-\nu} \prec G_{\mathcal{L}_n(x)}(\omega) - 1$$

for all $z, \omega \in \mathbb{U}$.

For particular values of κ and ν, we state the following remarks:

Remark 1.3. $\mathcal{GB}_{\Sigma}^{0,0}(x) \equiv \mathcal{S}_{\Sigma}^{*}(x)$ was defined by [1]. A function $f \in \Sigma$, of the form (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{*}(x)$ if:

$$\frac{zf'(z)}{f(z)} \prec G_{\mathcal{L}_n(x)}(z) - 1; \quad z \in \mathbb{U}$$

and

$$\frac{\omega g'(\omega)}{g(\omega)} \prec G_{\mathcal{L}_n(x)}(\omega) - 1; \quad \omega \in \mathbb{U}.$$

Remark 1.4. $\mathcal{GB}_{\Sigma}^{0,1}(x) \equiv \mathcal{K}_{\Sigma}(x)$ was introduced by [1]. A function $f \in \Sigma$, of the form (1.1) belongs to the class $\mathcal{K}_{\Sigma}(x)$ if:

$$1 + \frac{zf''(z)}{f'(z)} \prec G_{\mathcal{L}_n(x)}(z) - 1; \quad z \in \mathbb{U}$$

and

$$1 + \frac{\omega g''(g)}{g'(\omega)} \prec G_{\mathcal{L}_n(x)}(\omega) - 1; \quad \omega \in \mathbb{U}.$$

Remark 1.5. $\mathcal{GB}_\Sigma^{0,\nu}(x) \equiv \mathcal{M}_\Sigma^{\nu}(x)$ was established by [18]. A function $f \in \Sigma$, of the form (1.1) belongs to the class $\mathcal{M}_\Sigma^{\nu}(x)$ if:

$$\left[\frac{zf'(z)}{f(z)}\right]^{1-\nu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{\nu} \prec G_{\mathcal{L}_n(x)}(z)-1; \quad z \in \mathbb{U}$$

and

$$\left[\frac{\omega g'(\omega)}{g(\omega)}\right]^{1-\nu}\left[1+\frac{\omega g''(\omega)}{f'(\omega)}\right]^{\nu} \prec G_{\mathcal{L}_n(x)}(\omega)-1; \quad \omega \in \mathbb{U}.$$

Remark 1.6. $\mathcal{GB}_\Sigma^{\kappa,0}(x) \equiv \mathcal{B}_\Sigma^{\kappa}(x)$ was introduced and studied by [2]. A function $f \in \Sigma$, of the form (1.1) belongs to the class $\mathcal{B}_\Sigma^{\kappa}(x)$ if:

$$\frac{zf'(z)}{f^{1-\kappa}(z)\, z^{\kappa}} \prec G_{\mathcal{L}_n(x)}(z)-1; \quad z \in \mathbb{U}$$

and

$$\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\, \omega^{\kappa}} \prec G_{\mathcal{L}_n(x)}(\omega)-1; \quad \omega \in \mathbb{U}.$$

Remark 1.7. $\mathcal{GB}_\Sigma^{1,0}(x) \equiv \mathcal{R}_\Sigma(x)$ was studied by [2]. A function $f \in \Sigma$, of the form (1.1) belongs to the the class $\mathcal{R}_\Sigma(x)$ if:

$$f'(z) \prec G_{\mathcal{L}_n(x)}(z)-1; \quad z \in \mathbb{U}$$

and

$$g'(\omega) \prec G_{\mathcal{L}_n(x)}(\omega)-1; \quad \omega \in \mathbb{U}.$$

2. Coefficient Estimate

In this section, we obtain coefficient estimates for the function belonging to the class $\mathcal{GB}_\Sigma^{\kappa,\nu}(x)$.

Theorem 2.1. *Let $\kappa \geq 0$ and $\nu \geq 0$. If $f \in \mathcal{GB}_\Sigma^{\kappa,\nu}(x)$ and is of the form* (1.1)*, then*

$$|a_2| \leq \frac{|M(x)|\sqrt{2|M(x)|}}{\sqrt{|[(\kappa^2+2\kappa+1)\nu^2+(5\kappa^2+8\kappa+3)\nu+\kappa^2+\kappa]M^2(x)+4(1+\kappa)^2(1+\nu)^2N(x)|}}$$

and

$$|a_3| \leq \frac{|M(x)|}{(2+\kappa)(1+2\nu)} + \frac{M^2(x)}{(1+\kappa)^2(1+\nu)^2}.$$

Proof. Assume $f \in \mathcal{GB}_\Sigma^{\kappa,\nu}(x)$. Then there are analytical functions u and v such that

$$u(0)=0, \qquad v(0)=0, \quad |u(z)|<1, \quad \text{and} \quad |v(\omega)|<1 \quad (z,\omega \in \mathbb{U}).$$

Next, by Definition 1.2, we can write

$$\begin{aligned} F(z) &:= \left[\frac{zf'(z)}{f^{1-\kappa}(z)z^{\kappa}} + \frac{zf''(z)}{f'(z)} + (\kappa-1)\Big(\tfrac{zf'(z)}{f(z)} - 1\Big)\right]^{\nu} \left[\frac{zf'(z)}{f^{1-\kappa}(z)z^{\kappa}}\right]^{1-\nu} \\ &= G_{\mathcal{L}_n(x)}(u(z)) - 1 \end{aligned}$$

and

$$\begin{aligned} G(\omega) &:= \left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^{\kappa}} + \frac{\omega g''(\omega)}{g'(\omega)} + (\kappa-1)\Big(\tfrac{\omega f'(\omega)}{f(\omega)} - 1\Big)\right]^{\nu} \left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^{\kappa}}\right]^{1-\nu} \\ &= G_{\mathcal{L}_n(x)}(v(\omega)) - 1. \end{aligned}$$

Equivalently,

$$F(z) = -1 + \mathcal{L}_0(x) + \mathcal{L}_1(x)u(z) + \mathcal{L}_2(x)[u(z)]^2 + \cdots \tag{2.1}$$

and

$$G(\omega) = -1 + \mathcal{L}_0(x) + \mathcal{L}_1(x)v(\omega) + \mathcal{L}_2(x)[v(\omega)]^2 + \cdots . \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$F(z) = 1 + \mathcal{L}_1(x)u_1 z + [\mathcal{L}_1(x)u_2 + \mathcal{L}_2(x)u_1^2]z^2 + \cdots \tag{2.3}$$

and

$$G(\omega) = 1 + \mathcal{L}_1(x)v_1\omega + [\mathcal{L}_1(x)v_2 + \mathcal{L}_2(x)v_1^2]\omega^2 + \cdots . \tag{2.4}$$

In the view of (1.1), it can be computed that

$$\begin{aligned} F(z) &= 1 + (1+\kappa)(1+\nu)a_2 z \\ &\quad + \left(\Big[\frac{\mathcal{P}\nu^2 + \mathcal{Q}\nu + \mathcal{R}}{2}\Big]a_2^2 + (\kappa+2)(2\nu+1)a_3\right)z^2 + \cdots , \end{aligned} \tag{2.5}$$

where $\mathcal{P} = \kappa^2 + 2\kappa + 1$, $\mathcal{Q} = -\kappa^2 - 4\kappa - 7$ and $\mathcal{R} = \kappa^2 + \kappa - 2$. For $G(\omega)$, it can be computed that

$$\begin{aligned} G(w) &= 1 - (1+\kappa)(1+\nu)a_2\omega \\ &\quad + \left(\Big[\frac{\tilde{\mathcal{P}}\nu^2 + \tilde{\mathcal{Q}}\nu + \tilde{\mathcal{R}}}{2}\Big]a_2^2 - (\kappa+2)(2\nu+1)a_3\right)\omega^2 + \cdots , \end{aligned} \tag{2.6}$$

where $\tilde{\mathcal{P}} = \kappa^2 + 2\kappa + 1$, $\tilde{\mathcal{Q}} = -\kappa^2 + 4\kappa + 9$ and $\tilde{\mathcal{R}} = \kappa^2 + 5\kappa + 6$.

Comparing the corresponding coefficients in (2.3), (2.4), (2.5) and (2.6), we have

$$(1+\kappa)(1+\nu)a_2 = \mathcal{L}_1(x)u_1, \tag{2.7}$$

$$\left(\frac{\mathcal{P}\nu^2+\mathcal{Q}\nu+\mathcal{R}}{2}\right)a_2^2+(\kappa+2)(2\nu+1)a_3=\mathcal{L}_1(x)u_2+\mathcal{L}_2(x)u_1^2, \tag{2.8}$$

$$-(1+\kappa)(1+\nu)a_2=\mathcal{L}_1(x)v_1, \tag{2.9}$$

$$\left(\frac{\tilde{\mathcal{P}}\nu^2+\tilde{\mathcal{Q}}\nu+\tilde{\mathcal{R}}}{2}\right)a_2^2-(\kappa+2)(2\nu+1)a_3=\mathcal{L}_1(x)v_2+\mathcal{L}_2(x)v_1^2. \tag{2.10}$$

From (2.7) and (2.9), we can see that

$$\begin{aligned} u_1 &= -v_1, \\ 2(1+\kappa)^2(1+\nu)^2a_2^2 &= [\mathcal{L}_1(x)]^2(u_1^2+v_1^2). \end{aligned}$$

Adding (2.8) and (2.10), we get

$$\left[\frac{(\mathcal{P}+\tilde{\mathcal{P}})\nu^2+(\mathcal{Q}+\tilde{\mathcal{Q}})\nu+(\mathcal{R}+\tilde{\mathcal{R}})}{2}\right]a_2^2=\mathcal{L}_1(x)(u_2+v_2)+\mathcal{L}_2(x)(u_1^2+v_1^2). \tag{2.11}$$

Substituting (2.11) in (2.11), we get

$$a_2^2=\frac{[\mathcal{L}_1(x)]^3(u_2+v_2)}{(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)[\mathcal{L}_1(x)]^2-2\mathcal{L}_2(x)(1+\kappa)^2(1+\nu)^2}, \tag{2.12}$$

where

$$\sigma_1=\kappa^2+2\kappa+1, \quad \sigma_2=1-\kappa^2 \quad \textit{and} \quad \sigma_3=\kappa^2+3\kappa+2. \tag{2.13}$$

It is well known that, if $u(z)=\sum\limits_{n=1}^{\infty}u_nz^n$ and $v(\omega)=\sum\limits_{n=1}^{\infty}v_n\omega^n$ then

$$|u_n|\le 1 \quad \text{and} \quad |v_n|\le 1. \tag{2.14}$$

In the view of (1.2), (2.12) and (2.14), we have

$$|a_2|\le\frac{|M(x)|\sqrt{2|M(x)|}}{\sqrt{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)M^2(x)-2(M^2(x)+2N(x))(1+\kappa)^2(1+\nu)^2|}}.$$

Subtracting (2.10) from (2.8), we get

$$a_3=\frac{\mathcal{L}_1(x)(u_2-v_2)}{2(\kappa+2)(2\nu+1)}+a_2^2. \tag{2.15}$$

From (2.15) and (2.11), we have

$$a_3=\frac{\mathcal{L}_1(x)(u_2-v_2)}{2(2+\kappa)(1+2\nu)}+\frac{[\mathcal{L}_1(x)]^2(u_1^2+v_1^2)}{2(1+\kappa)^2(1+\nu)^2}. \tag{2.16}$$

In the view of (1.2), (2.14) and (2.16), we have

$$|a_3|\le\frac{|M(x)|}{(2+\kappa)(1+2\nu)}+\frac{|M^2(x)|}{(1+\kappa)^2(1+\nu)^2}.$$

□

For particular values of κ and ν, we state the following results:

Corollary 2.2. *If $f \in \mathcal{S}^*_\Sigma(x)$ and is of the form as in* (1.1)*, then*

$$|a_2| \leq |M(x)|\sqrt{\frac{|M(x)|}{2|N(x)|}} \quad and \quad |a_3| \leq \frac{|M(x)|}{2} + M^2(x).$$

This result coincides with the findings studied by [1].

Corollary 2.3. *If $f \in \mathcal{K}_\Sigma(x)$ and is of the form as in* (1.1)*, then*

$$|a_2| \leq \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{2|M^2(x) + 4N(x)|}} \quad and \quad |a_3| \leq \frac{|M(x)|}{6} + \frac{M^2(x)}{4}.$$

This result coincides with the findings studied by [1].

Corollary 2.4. *If $f \in \mathcal{M}^\nu_\Sigma(x)$ and is of the form as in* (1.1)*, then*

$$|a_2| \leq \frac{|M(x)|\sqrt{2|M(x)|}}{\sqrt{|(\nu^2 + 3\nu)M^2(x) + 4(1+\nu)^2N(x)|}}$$

and

$$|a_3| \leq \frac{|M(x)|}{2(1+2\nu)} + \frac{M^2(x)}{(1+\nu)^2}.$$

Corollary 2.5. *If $f \in \mathcal{B}^\kappa_\Sigma(x)$ and is of the form as in* (1.1)*, then*

$$|a_2| \leq \frac{|M(x)|\sqrt{2|M(x)|}}{\sqrt{(1+\kappa)|\kappa M^2(x) + 4(1+\kappa)N(x)|}}$$

and

$$|a_3| \leq \frac{|M(x)|}{2+\kappa} + \frac{M^2(x)}{(1+\kappa)^2}.$$

The above result coincide with [2, Theorem 2.1, p. 137].

Corollary 2.6. *If $f \in \mathcal{R}_\Sigma(x)$ and is of the form as in* (1.1)*, then*

$$|a_2| \leq \frac{|M(x)|\sqrt{|M(x)|}}{\sqrt{|M^2(x) + 8N(x)|}} \quad and \quad |a_3| \leq \frac{|M(x)|}{3} + \frac{M^2(x)}{4}.$$

The above results coincides with the findings studied by [2].

3. Fekete-Szegö Estimate

This section outlines the Fekete-Szegö estimate for functions categorized under the class $\mathcal{GB}^{\kappa;\nu}_\Sigma(x)$.

Theorem 3.1. *Let $\kappa \geq 0$, $\nu \geq 0$ and $\mu \in \mathbb{R}$. If $f \in \mathcal{GB}_{\Sigma}^{\kappa,\nu}(x)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|M(x)|}{(2+\kappa)(1+2\nu)}, \\ |\mu - 1| \leq \frac{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)M^2(x)-2(M^2(x)+2N(x))(1+\kappa)^2(1+v)^2|}{2M^2(x)(2+\kappa)(1+2\nu)} \\ \frac{2|M(x)|^3|\mu-1|}{|[(\kappa^2+2\kappa+1)\nu^2+(5\kappa^2+8\kappa+3)\nu+\kappa^2+\kappa]M^2(x)+4(1+\kappa)^2(1+\nu)^2N(x)|}, \\ |\mu - 1| \geq \frac{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)M^2(x)-2(M^2(x)+2N(x))(1+\kappa)^2(1+v)^2|}{2M^2(x)(2+\kappa)(1+2\nu)} \end{cases}$$

where σ_1, σ_2 and σ_3 are same as stated in (2.13).

Proof. From (2.15) and for $\mu \in \mathbb{R}$, we can write

$$a_3 - \mu a_2^2 = \frac{\mathcal{L}_1(x)(u_2 - v_2)}{2(\kappa+2)(2\nu+1)} + (1-\mu)a_2^2. \tag{3.1}$$

From (2.12) and (3.1), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \tfrac{\mathcal{L}_1(x)(u_2-v_2)}{2(\kappa+2)(2\nu+1)} \\ &+ \tfrac{(1-\mu)[\mathcal{L}_1(x)]^3(u_2+v_2)}{(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)[\mathcal{L}_1(x)]^2-2\mathcal{L}_2(x)(1+\kappa)^2(1+\nu)^2} \\ &= \mathcal{L}_1(x)\left[\left(\Omega(\kappa,\nu,\mu) + \tfrac{1}{2(2+\kappa)(1+2\nu)}\right)u_2 + \left(\Omega(\kappa,\nu,\mu) - \tfrac{1}{2(2+\kappa)(1+2\nu)}\right)v_2\right], \end{aligned}$$

where

$$\Omega(\kappa,\nu,\mu) = \frac{(1-\mu)[\mathcal{L}_1(x)]^2}{(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)[\mathcal{L}_1(x)]^2 - 2\mathcal{L}_2(x)(1+\kappa)^2(1+\nu)^2}.$$

Hence, we can conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\mathcal{L}_1(x)|}{(2+\kappa)(1+2\nu)}, & 0 \leq |\Omega(\kappa,\nu,\mu)| \leq \dfrac{1}{2(2+\kappa)(1+2\nu)} \\ 2|\mathcal{L}_1(x)||\Omega(\kappa,\nu,\mu)|, & |\Omega(\kappa,\nu,\mu)| \geq \dfrac{1}{2(2+\kappa)(1+2\nu)} \end{cases}.$$

In the view of (1.2), it is evident that the proof of the Theorem 3.1 has been completed. □

Corollary 3.2. *If $f \in \mathcal{S}^*_\Sigma(x)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|M(x)|}{2}, & \text{if} \quad |\mu - 1| \leq \dfrac{|N(x)|}{M^2(x)} \\ \dfrac{|M(x)|^3|\mu - 1|}{2|N(x)|}, & \text{if} \quad |\mu - 1| \geq \dfrac{|N(x)|}{M^2(x)}. \end{cases}$$

The above result coincides with the findings studied by [1].

Corollary 3.3. *If $f \in \mathcal{K}_\Sigma(x)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|M(x)|}{6}, & \text{if} \quad |\mu - 1| \leq \dfrac{|M^2(x) + 4N(x)|}{3M^2(x)} \\ \dfrac{|M(x)|^3|\mu - 1|}{2|M^2(x) + 4N(x)|}, & \text{if} \quad |\mu - 1| \geq \dfrac{|M^2(x) + 4N(x)|}{3M^2(x)}. \end{cases}$$

The above results coincides with the findings studied by [1].

Corollary 3.4. *If $f \in \mathcal{M}^\nu_\Sigma(x)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|M(x)|}{2(1+2\nu)}, \\ \quad \text{if} \quad |\mu - 1| \leq \dfrac{|(\nu^2 + 3\nu)M^2(x) + 4(1+\nu)^2 N(x)|}{4M^2(x)(1+2\nu)} \\ \dfrac{2|M(x)|^3|\mu - 1|}{|(\nu^2 + 3\nu)M^2(x) + 4(1+\nu)^2 N(x)|}, \\ \quad \text{if} \quad |\mu - 1| \geq \dfrac{|(\nu^2 + 3\nu)M^2(x) + 4(1+\nu)^2 N(x)|}{4M^2(x)(1+2\nu)}. \end{cases}$$

Corollary 3.5. *If $f \in \mathcal{B}^\kappa_\Sigma(x)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|M(x)|}{2+\kappa}, \\ \quad \text{if} \quad |\mu - 1| \leq \dfrac{(1+\kappa)|\kappa M^2(x) + 4(1+\kappa)N(x)|}{2(2+\kappa)M^2(x)} \\ \dfrac{2|M(x)|^3|\mu - 1|}{(1+\kappa)|\kappa M^2(x) + 4(1+\kappa)N(x)|}, \\ \quad \text{if} \quad |\mu - 1| \geq \dfrac{(1+\kappa)|\kappa M^2(x) + (1+\kappa)N(x)|}{2(2+\kappa)M^2(x)}. \end{cases}$$

The above result coincide with [2, Theorem 3.1, p. 139].

Corollary 3.6. *If $f \in \mathcal{R}_\Sigma(x)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|M(x)|}{3}, & \text{if} \quad |\mu - 1| \leq \dfrac{|M^2(x) + 8N(x)|}{3M^2(x)} \\ \dfrac{|M(x)|^3|\mu - 1|}{|M^2(x) + 8N(x)|}, & \text{if} \quad |\mu - 1| \geq \dfrac{|M^2(x) + 8N(x)|}{3M^2(x)}. \end{cases}$$

The above results coincides with the findings studied by [2].

4. Conclusion

In this paper, we have focused on obtaining coefficient estimates for functions belonging to the class $\mathcal{GB}_{\Sigma}^{\kappa,\nu}(x)$. Our analysis has provided new insights into the behavior of these functions, particularly concerning their coefficients.

The findings presented in this paper open avenues for further research. Future investigations could build on these estimates by exploring their applications in different contexts, examining other function classes, or performing computational studies to validate and expand upon our theoretical results.

The Lucas polynomial has interesting geometric properties related to its image domain, potential univalence, and critical points, all of which can be explored in the context of GFT. Investigating these properties further, particularly their behavior in the unit disc, could offer significant contributions to conformal mappings and other areas of GFT.

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