

The Stability of Some Systems of Harvested Lotka-Volterra Predator-Prey Equations

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ABSTRACT. Some scientists are interesting to study in area of harvested ecological modelling. The harvested population dynamics is more realistic than other ecological models. In the present paper, some of the Lotka-Volterra predator-prey models have been considered. In the said models, existing species are harvested by constant or variable growth rates. The behavior of their solutions has been analyzed in the stability sense. The employed methods are linearization and Lyapunov function.

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1. INTRODUCTION

The problem of predator-prey is an well-known problem which first was introduced by A.J. Lotka [2] and V. Volterra [8] around 1925 independently. The Lotka-Volterra predator-prey model is as follows:

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dx) \end{cases} \quad (1.1)$$

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where the variables x and y are considered as the densities of prey and predator species respectively. The coefficients a, b, c and d are positive constant. As the density of species can't be negative, the orbit of solutions of Lotka-Volterra model and also the harvest factor is located in the area $intR_+^2 = \{(x, y) | x, y > 0\}$. The standard Lotka-Volterra model has been widely investigated [3]. Considering situation that j^{th} -population may effect upon to the i^{th} -population by linearly way, one can find following general form of lotka- Volterra model

$$\frac{dx_i}{dt} = x_i \left(r_i + \sum_{j=1}^n a_{ij} x_j \right), \quad i = 1, 2, 3, \dots, n; \quad (1.2)$$

which the effect of j^{th} -population upon to the i^{th} -population is shown by a_{ij} . Above system is known as multi-species Lotka-Volterra model. The said system may distinguish into three cases: predator-prey, competition or coexistence system worked out in [4,5,6,7]. A globally dynamic for a harvested predator-prey system has been analyzed in [1]. Note that all presented parameters $a, b, c, d, h, i, v, e, p, q, \alpha, \beta, \gamma_1, \gamma_2$ are positive constant.

2. ANALYSIS OF HARVESTED LOTKA-VOLTERRA SYSTEMS; SIMPLE MODELS

2.1. Model Having Constant Harvesting Factors.

Consider the Lotka-Volterra model with two species prey and predator (1.1). Moreover, let other species such as human factor has effect on prey and predator species. In this section we show this effect by constant parameters h and i in which its parameters show efficiency of third species on prey and predator species in same time. Thus, the above assumption may be stated as follows:

$$\begin{cases} \frac{dx}{dt} = x(a - by) - h \\ \frac{dy}{dt} = y(-c + dx) - i \end{cases} \quad (2.1)$$

Let $y^* = \frac{(ac-ib) + \sqrt{(ac-ib)^2 + 4bc(ia-dh)}}{2bc}$ where $\frac{i}{h} > \frac{d}{a}$. Then the equilibrium point for above system is $(x^*, y^*) = (\frac{h}{a-by^*}, y^*)$ where $a - by^* > 0$.

Proposition 2.1. *The solution of system (2.1) is unstable.*

Proof: Define Lyapunove function as

$$V(x, y) = (x - x^*) - x^* Ln \frac{x}{x^*} + k[(y - y^*) - y^* Ln \frac{y}{y^*}]$$

where $k = \frac{b}{d}$. And so, $\frac{dV}{dt}(x, y) = h \frac{(x-x^*)^2}{xx^*} + ki \frac{(y-y^*)^2}{yy^*}$. Since $\frac{dV}{dt} > 0$, The system (2.1) is unstable. One special cases of the system (2.1) is as follows:

$$\begin{cases} \frac{dx}{dt} = x(a - by) - h \\ \frac{dy}{dt} = y(-c + dx) \end{cases} \quad (2.2)$$

which is known as the Lotka-Volterra model with having constant prey harvesting. Its equilibrium points are $(\frac{h}{a}, 0)$, $(\frac{c}{d}, \frac{a}{b} - \frac{dh}{cb})$ where $ah > cd$. Since the eigenvalues corresponding into the equilibrium point $(\frac{h}{a}, 0)$ are $\lambda_1 = a$ and $\lambda_2 = -c$. As $\lambda_1\lambda_2 < 0$ the said point is a saddle point for system (2.1). For the second equilibrium point $(\frac{c}{d}, \frac{a}{b} - \frac{dh}{cb})$, the eigenvalues are

$$\frac{dh}{c} \pm \frac{\sqrt{(\frac{dh}{c})^2 - 4(ac - dh)}}{2}$$

and so system (2.1) is unstable at the said point.

The second special case of model (2.1) is

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dx) - i \end{cases} \quad (2.3)$$

in which known as the Lotka-Volterra model having constant harvest factor for predator species. One may sake above system similarly to system (2.2).

2.2. Model Having Effort Rate Harvesting Factors.

First, consider the lotka-Volterra predator-prey model (1.1). Moreover, let the effort rate harvesting factors exist. The following model can be obtained

$$\begin{cases} \frac{dx}{dt} = x(a - by) - ex \\ \frac{dy}{dt} = y(-c + dx) - ey \end{cases} \quad (2.4)$$

The above model, with making variations $a - e$ and $c + e$ to a and c respectively, can be transferred to the model that if $a > e$ is the simplest ordinary Lotka-Volterra model, else the system (2.4) can't represent a predator-prey system. The equilibrium points of system (2.4) is given as $O = (0, 0)$, $F = (\frac{c}{d}, \frac{-e+a}{b})$. Using the linearization method, the Jacobian matrix implies that both of its eigenvalues are negative and so, origin is stability stable. Furthermore, the eigenvalues at the point F are $\pm\sqrt{c(a - e)}$. Hence, the point F is saddle point for system (2.4). Therefor, the following proposition is proved:

Proposition 2.2. *For system (2.4), origin is stable and the point $F = (\frac{c}{d}, \frac{-e+a}{b})$ is unstable point.*

Two special cases of the system(2.4) are called as

i) Predator-prey model having prey harvesting (corresponding to prey

population)

$$\begin{cases} \frac{dx}{dt} = x(a - by) - ex \\ \frac{dy}{dt} = y(-c + dx) \end{cases} \quad (2.5)$$

ii) Predator-prey model having predator harvesting (corresponding to predator population)

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dx) - ey \end{cases} \quad (2.6)$$

2.3. Model Having Variable harvesting Factors (Inverse Ratio).

In this section, consider the Lotka-Volterra model having inverse ratio harvest factor. Let moreover interaction between prey and predator species exist. Their density may be reduced by the terms $\frac{p}{x}$ and $\frac{q}{y}$ respectively. This model may be shown by following system:

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \frac{p}{x} \\ \frac{dy}{dt} = y(-c + dx) - \frac{q}{y} \end{cases} \quad (2.7)$$

Proposition 2.3. *The solution of system (2.7) is unstable if exists.*

Proof: Let (\bar{x}, \bar{y}) be it's equilibrium point
Defining Lyapunove function

$$v(x, y) = \int_{\bar{x}}^x \frac{s - \bar{x}}{s} ds + h \int_{\bar{y}}^y \frac{t - \bar{y}}{t} dt$$

implies that

$$\begin{aligned} \frac{dv}{dt} &= -p(x - \bar{x})\left(\frac{1}{x^2} - \frac{1}{\bar{x}^2}\right) - hq(y - \bar{y})\left(\frac{1}{y^2} - \frac{1}{\bar{y}^2}\right) \\ &= p \frac{(x - \bar{x})^2(x + \bar{x})}{(x\bar{x})^2} + hq \frac{(y - \bar{y})^2(y + \bar{y})}{(y\bar{y})^2} \end{aligned}$$

where $h = \frac{b}{d}$. It is clear that $\frac{dv}{dt} > 0$. Therefor, the proof is done.

A special case of system (2.7) is as follows:

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \frac{h}{x} \\ \frac{dy}{dt} = y(-c + dx) \end{cases} \quad (2.8)$$

The above system is known as the Lotka-Volterra model having variable prey harvesting(inverse ratio of prey population). If $\frac{a}{h} > \frac{d^2}{c^2}$, then the equilibrium points of system (2.8) in the first phase quadratic are

$(\bar{x}, \bar{y}) = (\sqrt{\frac{h}{a}}, 0)$, $(x^*, y^*) = (\frac{c}{d}, \frac{a}{b} - \frac{hd^2}{bc^2})$. The Jacobian matrix at the point (\bar{x}, \bar{y}) may be found as the follow

$$J|_{(\bar{x}, \bar{y})} = \begin{pmatrix} 2a & -b\sqrt{\frac{h}{a}} \\ 0 & -c + d\sqrt{\frac{h}{a}} \end{pmatrix}$$

Hence, its eigenvalues are $\lambda_1 = 2a$ and $\lambda_2 = -c + d\sqrt{\frac{h}{a}}$. Thus, the related point is saddle provided $\sqrt{\frac{h}{a}} < \frac{c}{d}$. And for the second point (x^*, y^*) , we have

$$J|_{(x^*, y^*)} = \begin{pmatrix} \frac{2hd^2}{c^2} & -\frac{bc}{d} \\ \frac{d}{b}(a - \frac{hd^2}{c^2}) & 0 \end{pmatrix}$$

Proposition 2.4. For system (2.8) the following statements are held:

- i) If $\frac{a}{h} > \frac{d^2}{c^2}$, then said system has two equilibrium point.
- ii) If $\frac{a}{h} > \frac{d^2}{c^2}$, then the equilibrium point (\bar{x}, \bar{y}) is a saddle point.
- iii) The equilibrium point (x^*, y^*) is a saddle point.

3. ANALYSIS OF HARVESTED PREDATOR-PREY SYSTEMS; COMPLEX MODELS

3.1. Complex Harvested Model (1).

Let diseases exists in a ecosystem or the lake in ecosystem is pollution and the density of prey and predator species reduce by terms βx^2 and αy^2 respectively. One may modularize the above assumption can be reduced as follows:

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \beta x^2 \\ \frac{dy}{dt} = y(-c + dx) - \alpha y^2 \end{cases} \quad (3.1)$$

The above system is known as competition Lotka-Volterra model too. One may sake stability of this system by linearization method or refer to references [8,10].

3.2. Complex Harvested Model(2).

Let it is cold and also food is rare, Furthermore predator species is reduced by the terms $\frac{y}{vx}$. Therefore the harvested model may show by the following system:

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dx) - \frac{y}{vx} \end{cases} \quad (3.2)$$

The above system has just one equilibrium point is as follows:

$$(x^*, y^*) = \left(\frac{cv + \sqrt{(cv^2) + 4dv}}{2dv}, \frac{a}{b} \right)$$

The Jacobian matrix of system (3.2) at the equilibrium point (x^*, y^*) as follows:

$$J|_{(x^*, y^*)} = \begin{pmatrix} 0 & -bx^* \\ \frac{d}{b}a + \frac{a}{bv x^{*2}} & 0 \end{pmatrix}$$

A simple calculating shows that the equilibrium point (x^*, y^*) is a spiral point for system (3.2).

3.3. Complex Harvested Model (3).

Now assume that in system (3.2), the preys species can't find food and place and some of them migrate to another ecosystem. In other word, there is competition factor between preys species for attain the food and place. The term $-\beta x^2$ in following system shows this reduction

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \beta x^2 \\ \frac{dy}{dt} = y(-c + dx) - \frac{y}{vx} \end{cases} \quad (3.3)$$

One of the equilibrium points of system (3.2) is $(\frac{a}{\beta}, 0)$ and if $a > \beta x^*$, then another equilibrium pint is (x^*, y^*) where

$$\begin{cases} x^* = \frac{cv + \sqrt{(cv^2) + 4dv}}{2dv} \\ y^* = \frac{a - \beta x^*}{b} \end{cases}$$

The Jacobian matrix of system (3.3) at equilibrium point $(\frac{a}{\beta}, 0)$ is as follows:

$$J|_{(\frac{a}{\beta}, 0)} = \begin{pmatrix} -a & -\frac{ab}{\beta} \\ 0 & -c + \frac{ad}{\beta} - \frac{\beta}{av} \end{pmatrix}$$

And so, its related eigenvalues are $\lambda_1 = -a$ and $\lambda_2 = -c + \frac{ad}{\beta} - \frac{\beta}{av}$.

Then the equilibrium point $(\frac{a}{\beta}, 0)$ is stable if $\frac{ad}{\beta} < c + \frac{\beta}{av}$.

The Jacobian matrix at the second equilibrium point (x^*, y^*) is as follows:

$$A = J|_{(x^*, y^*)} = \begin{pmatrix} -\beta x^* & -bx^* \\ \frac{d}{b}(a - \beta x^*) + \frac{a - \beta x^*}{bv x^{*2}} & 0 \end{pmatrix}$$

Thus, $tracA < 0$ and $detA > 0$, and so the related system has two eigenvalue having distinct negative real parts.

3.4. Complex Harvested Model(4).

Now let the harvested factors be $\frac{x}{y}$ and $\frac{y}{x+q}$ for prey and predator species respectively, which can be seen in the following harvested system

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \frac{x}{y} \\ \frac{dy}{dt} = y(-c + dx) - \frac{y}{x+q} \end{cases} \quad (3.4)$$

If $a^2 > 4b$, then the related equilibrium points are given by (\bar{x}, \bar{y}) and (\bar{x}, y^*) where

$$\bar{x} = \frac{(c - dq) + \pm \sqrt{(dq - c)^2 + 4d(cq + 1)}}{2d}$$

,

$$\bar{y} = y^* = \frac{a + \sqrt{a^2 - 4b}}{2b}$$

Linearization method shows that the point (\bar{x}, \bar{y}) is stable if the parameters A and B are negative, where

$$A = a - b\bar{y} - \frac{1}{\bar{y}} - c + d\bar{x} - \frac{1}{q + \bar{x}}$$

$$B = (a - b\bar{y} - \frac{1}{\bar{y}})(-c + d\bar{x} - \frac{1}{q + \bar{x}}) - (d\bar{y} + \frac{\bar{y}}{(\bar{x} + q)^2})(-b\bar{x} + \frac{\bar{x}}{\bar{y}^2})$$

Similarly, for the second point (\bar{x}, y^*) , we will find the eigenvalues as

$$\lambda_1, \lambda_2 = \frac{A_1 \pm \sqrt{A_1^2 - 4B_1}}{2}$$

where

$$A_1 = a - by^* - \frac{1}{y^*} - c + d\bar{x} - \frac{1}{q + \bar{x}}$$

$$B_1 = -(a - by^* - \frac{1}{y^*})(-c + d\bar{x} - \frac{1}{q + \bar{x}}) + (dy^* + \frac{y^*}{(\bar{x} + q)^2})(-b\bar{x} + \frac{\bar{x}}{y^{*2}})$$

3.5. Complex Harvested Model(5).

Now add the harvested factors βx^3 and $\frac{y^2}{vx}$ for prey and predator species respectively; for lotka-Volterra predator-prey model (1.1). This can be modelled in the following system

$$\begin{cases} \frac{dx}{dt} = x(a - by) - \beta x^3 \\ \frac{dy}{dt} = y(-c + dx) - \frac{y^2}{vx} \end{cases} \quad (3.5)$$

The equilibrium points for this model are given by $(\sqrt{\frac{a}{\beta}}, 0)$ and (\bar{x}, \bar{y}) , where

$$\bar{x} = \frac{bcv + \sqrt{(bcv)^2 + 4a(bdv + \beta)}}{2(bdv + \beta)}$$

$$\bar{y} = v\bar{x}(-c + d\bar{x})$$

The linearization method shows that $\lambda_1 = -2a$ and $\lambda_2 = -c + d\sqrt{\frac{a}{\beta}}$,

Then the solution of system (3.5) at the point $(\sqrt{\frac{a}{\beta}}, 0)$ is stable provided

$$\frac{c}{d} > \sqrt{\frac{a}{\beta}}.$$

For the second equilibrium point (\bar{x}, \bar{y}) , we use the Lyapunove function.

Proposition 3.1. *The system (3.5) is globally stable provided $x > \bar{x}$.*

Proof: Define the following function for system (3.5) as follow:

$$w(x, y) = \int_{\bar{x}}^x \frac{s - \bar{x}}{s} ds + h \int_{\bar{y}}^y \frac{t - \bar{y}}{t} dt$$

$$\Rightarrow \frac{dw}{dt} = -\beta(x + \bar{x})(x - \bar{x})^2 + (-b + hd)[(x - \bar{x})(y - \bar{y})] - \frac{h}{v}(y - \bar{y})\left(\frac{y}{x} - \frac{\bar{y}}{\bar{x}}\right)$$

The system (3.5) is globally stable provided the last derivative is negative. As regarding h is arbitrary, we can assume that $h = \frac{b}{d}$.

Now by a using simple calculating, we obtain

$$\begin{aligned} \frac{dw}{dt} &= -\beta(x + \bar{x})(x - \bar{x})^2 - \frac{b}{dvx}(y - \bar{y})^2 - \frac{b\bar{y}}{dv}\left(\frac{1}{x} - \frac{1}{\bar{x}}\right) \\ &= -\beta(x + \bar{x})(x - \bar{x})^2 - \frac{b(y - \bar{y})^2}{dvx} - \frac{b\bar{y}(x - \bar{x})}{dvx\bar{x}} \end{aligned}$$

which is negative provided $x > \bar{x}$.

Therefore proposition has been proved.

3.6. Complex Harvested Model (6).

In the section 2.2 the model having effort rate harvesting factors 2.4. is studied. By adding the density cubic of prey species and the density quadratic of predator species the, we find out the following system:

$$\begin{cases} \frac{dx}{dt} = x(a - by) - c_1ex - \gamma_1x^3 \\ \frac{dy}{dt} = y(-c + dx) - c_2ey - \gamma_2y^2 \end{cases} \quad (3.6)$$

Now, let the growth rate for both species be zero i.e. $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. Then the equilibrium points of system 3.6 are $(0, 0)$, $(\frac{c+c_2e}{d}, 0)$, (x^*, y^*) where

$$x^* = \frac{-bd}{\gamma_2} + \sqrt{\left(\frac{bd}{\gamma_2}\right)^2 + 4\gamma_1\left(a + \frac{bc}{\gamma_2} + \frac{c_2be}{\gamma_2} - c_1e\right)}$$

$$y^* = \frac{dx^* - c - c_2e}{\gamma_2}.$$

$$\Rightarrow J|_{(0,0)} = \begin{pmatrix} a - c_1e & 0 \\ 0 & -c - c_2e \end{pmatrix}$$

$$\Rightarrow \lambda_1 = a - c_1e, \quad \lambda_2 = -c - c_2e$$

Hence, the solutions of system (3.6) at origin is stable provided

$$a < c_1e + 3\gamma_1\left(\frac{c + c_2e}{d}\right)^2$$

$$\Rightarrow J|_{(\frac{c+c_2e}{d}, 0)} = \begin{pmatrix} a - c_1e - 3\gamma_1\left(\frac{c+c_2e}{d}\right)^2 & -b\left(\frac{c+c_2e}{d}\right) \\ 0 & -c - c_2e + d\left(\frac{c+c_2e}{d}\right) \end{pmatrix}$$

$$\Rightarrow \lambda_1 = a - c_1 e - 3\gamma_1 \left(\frac{c + c_2 e}{d}\right)^2, \lambda_2 = 0$$

Thus, the system (3.6) at the second equilibrium point $(\frac{c+c_2e}{d}, 0)$ is stable provided

$$a < c_1 e - 3\gamma_1 \left(\frac{c + c_2 e}{d}\right)^2$$

Finally, the last equilibrium point (x^*, y^*) is stable provided

$$\frac{a}{\beta} > \frac{cv + \sqrt{(cv)^2 + 4dv}}{2dv}.$$

Now by using the Lyapunov function we have

$$v(x, y) = [(x - x^*) - x^* \text{Ln}\left(\frac{x}{x^*}\right)] + h[(y - y^*) - y \text{ln}\left(\frac{y}{y^*}\right)]$$

Selecting $h = \frac{b}{d}$ implies that $\frac{dv}{dt} < 0$.

Therefore, the following proposition is proved:

Proposition 3.2. For system (3.6) the following statements are held:

- i) Origin is stable provided $a < c_1 e$.
- ii) The point $(\frac{c+c_2e}{d}, 0)$ is stable provided $a < c_1 e - 3\gamma_1 (\frac{c+c_2e}{d})^2$.
- iii) The point (x^*, y^*) is globally stable.

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