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# Radical formula for L-submodules

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Abstract. Let $R$ be a commutative ring with non-zero identity and $M$ be a unitary $R$-module. In this paper, we introduce the concepts of radical and $E$-radical (envelope) of $L$-submodules and investigate their properties. Additionally, we define $L$-radical formula and demonstrate that, under specific conditions, the $L$-radical formula can be derived using the concept of the envelope of an $L$-submodule.

Keywords: Prime $L$-submodules, Radical of $L$-submodules, Envelope of $L$-submodules.

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## 1. Introduction

Throughout this paper, $R$ denotes a commutative ring with identity. $M$ is assumed to be a unitary $R$-module. For each ideal $I$ of $R$, the radical of $I$ defined as the intersection of all prime ideals containing $I$, has the characterization $\sqrt{I} = \{r \in R | r^n \in I, \text{ for some } n \in \mathbb{Z}^+\}$. A proper

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submodule $N$ of an $R$-module $M$ is said to be prime if $ra \in N, a \notin N$ implies that $rM \subseteq N$. The concept of radical of a submodule $N$ of an $R$-module $M$ was defined in [3] as the intersection of all prime submodules of $M$ containing $N$ and is denoted by $rad(N)$. If there does not exist any prime submodule of $M$ containing $N$, then it is defined that $rad(N) = M$. $N$ is called radical submodule if $rad(N) = N$.
Zadeh in [15], introduced the notion of a fuzzy subset $\mu$ of a nonempty set $X$ as a function from $X$ to unit real interval $I = [0, 1]$. Goguen in [4] replaced $I$ with a complete lattice $L$ in the definition of fuzzy sets and introduced the notion of L-fuzzy sets. Rosenfeld introduced the notion of fuzzy groups [12] and fuzzy submodules of $M$ over $R$ were first introduced by Negoita and Ralescu [10]. Pan in [11] studied fuzzy finitely generated modules and fuzzy quotient modules (also see [13]). In the last few years, a considerable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals. $\sqrt{\mu}$ and $\Re(\mu)$ were defined for any $L$-ideals of a ring $R$ by Malik and Mordeson in [7] and [8] respectively, these were the generalization of the concept of the radical of an ideal $\sqrt{I}$. The notion of prime $L$-submodules of a module over a commutative ring with identity say $R$, where $L$ is a complete lattice, was introduced by Ameri and Mahjoob in [1]. They also introduced the notion of the radical of an $L$-submodule $\mu$ as the intersection of all prime $L$-submodules of $M$ containing $\mu$ and it is denoted by $Rad(\mu)$. In this paper we investigate some basic properties of $Rad(\mu)$, and also we introduce the notion of the envelope of an $L$-submodule $\mu$ of an $R$-module $M$ denoted by $E(\mu)$ in Sec. 4. Our definition is a generalization of the notion of the ordinary envelope of an $R$-module which appears in the current literature on algebra (for example refer to [3] and [5]). In the end, we will establish relationships between $Rad(\mu)$ and $E(\mu)$ in Section 5.

## 2. Preliminaries

In this section, to make it easier to follow, we recall some definitions and theorems from the book [9], which we need them for development of our paper.
$L$ denotes a complete lattice. By an $L$-subset $\mu$ of a non-empty set $X$, we mean a function $\mu$ from $X$ to $L$. $L^X$ denotes the set of all $L$-subsets of $X$. In particular, if $L = [0, 1]$, the L-subsets of X are called fuzzy subsets and are denoted by $F^X$. Let $A$ be a subset of $X$ and $y \in L$. Define $y_A \in L^X$ as follows:

$$y_A(x) = \begin{cases} y & if\ x \in A \\ 0 & otherwise \end{cases}$$

In special cases if $A = \{a\}$, we denote $y_{\{a\}}$ by $y_a$, and it is called an $L$-point of $X$.
For $\mu$, $\nu \in L^X$, we say that $\mu$ is contained in $\nu$ and we write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$, in particular $x_a \in \mu$ means $a \leqslant \mu(x)$ for all $x \in X$ and $a \in L$. For $\mu, \nu \in L^M$, the intersection and union, $\mu \cup \nu$, $\mu \cap \nu \in L^X$, are defined by

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) \text{ and } (\mu \cap \nu)(x) = \mu(x) \wedge \nu(x) \text{ for all } x \in X.$$

Also for $\mu \in L^X$, $a \in L$, $\mu_a$ is defined by,

$$\mu_a = \{x \in M | \mu(x) \geq a\},$$

where $\mu_a$ is called $a$-cut or $a$-level subset of $\mu$.
Let $f$ be a mapping from $X$ into $Y$ and let $\mu \in L^X$, $\nu \in L^Y$. Then $f(\mu) \in L^Y$ and $f^{-1}(\nu) \in L^X$ are defined as follows:

$$f(\mu)(y) = \begin{cases} \bigvee\{\mu(x) | x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(\nu)(x) = \nu(f(x)), \ \forall x \in X.$$

This is the extension principle.
Let $M$ and $N$ be $R$-modules and $f : M \to N$ be an $R$-module homomorphism. $\mu \in L^M$ is called $f$-invariant if $f(x) = f(y)$, implies that $\mu(x) = \mu(y)$ for all $x, y \in M$.

**Definition 2.1.** *Let $\mu$ be an $L$-subset of $R$. The radical of $\mu$ is denoted by* $\Re(\mu)(x) = \bigvee\limits_{n \in N} \mu(x^n), \ \forall x \in R.$

**Definition 2.2.** *Let $c \in L \setminus \{1\}$. $c$ is called a prime element of $L$ if $a \wedge b \leq c$, implies that $a \leq c$ or $b \leq c$, for all $a, b \in L$.*

**Definition 2.3.** *Let $\mu \in L^R$. Then $\mu$ is called the $L$-ideal of $R$ if for every $x, y \in R$ the following conditions are satisfied:*
(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$*;*
(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.
*The set of all $L$-ideals of $R$ is denoted by $LI(R)$.*

**Definition 2.4.** *Let $\mu, \nu \in LI(R)$. We define $\mu\nu \in LI(R)$ as follows:*

$$\mu\nu(x) = \bigvee\{\mu(y) \wedge \nu(z) \mid y, z \in R, x = yz\}, \ \forall x \in R$$

**Definition 2.5.** *Let $R$ be a ring and $\zeta \in LI(R)$. Then $\zeta$ is called prime $L$-ideal of $R$ if $\zeta$ is non-constant and for every $\mu, \nu \in LI(R), \mu\nu \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.*

**Theorem 2.6.** *Let $\zeta \in L^R$. Then $\zeta$ is prime $L$-ideal of $R$ if only if $\zeta(0) = 1$ and $\zeta = 1_{\zeta_*} \cup c_R$ such that $\zeta_*$ is a prime ideal of $R$ and $c$ is a prime element of $L$.*$\square$

**Definition 2.7.** *An $L$- submodule of $M$ is an $L$-subset $\mu \in L^M$ such that:*
(1)$\mu(0) = 1$;
(2)$\mu(rx) \geq \mu(x)$ *for all* $r \in R$ *and* $x \in M$ *and*
(3)$\mu(x + y) \geq \mu(x) \wedge \mu(y)$ *for all* $x, y \in M$.
*The set of all $L$-submodules of $M$ is denoted by $L(M)$.*

**Definition 2.8.** *Let $\mu \in L^M$. Then $\mu \in L(M)$ if and only if each non-empty level subset of $\mu$ is a submodule of $M$. Moreover if $\mu \in L(M)$ then*

$$\mu_* = \{x \in M \mid \mu(x) = 1\}$$

*is a submodule of $M$.*

Let $\zeta \in L^R$ and $\mu \in L^M$. Define $\zeta \cdot \mu \in L^M$ as follows:

$$(\zeta \cdot \mu)(x) = \bigvee \{\zeta(r) \wedge \mu(y) \mid r \in R, y \in M, ry = x\} \text{ for all } x \in M.$$

For $\mu, \nu \in L^M$ and $\zeta \in L^R$, define $(\mu : \nu) \in L^R$ and $(\mu : \zeta) \in L^M$ as follows:

$$(\mu : \nu) = \bigcup \{\eta \in L^R \mid \eta \cdot \nu \subseteq \mu\},$$

$$(\mu : \zeta) = \bigcup \{\nu \in L^M \mid \zeta \cdot \nu \subseteq \mu\}.$$

In [9] it was proved that if $\nu \in L^M$, $\mu \in L(M)$ and $\zeta \in LI(R)$ then

$$(\mu : \nu) = \bigcup \{\eta \in LI(R) \mid \eta \cdot \nu \subseteq \mu\},$$

$$(\mu : \zeta) = \bigcup \{\nu \in L(M) \mid \zeta \cdot \nu \subseteq \mu\}.$$

**Theorem 2.9.** *Let $c \in L$ and $N$ be a submodule of $M$. Then*

$$(1_N \cup c_M) : 1_M = 1_{(N:M)} \cup c_R.$$

We recall that an $L$-submodule $\mu$ of $M$ is called prime if for $\zeta \in LI(R)$ and $\nu \in L(M)$ such that $\zeta \cdot \nu \subseteq \mu$, then either $\nu \subseteq \mu$ or $\zeta \subseteq (\mu : 1_M)$([1]). By $L$-Spec(M), we denote the set of all prime $L$-submodules of $M$.

**Theorem 2.10.** [1] *Let $\mu$ be an $L$-submodule of $M$. Then $\mu$ is prime if and only if $\mu = 1_{\mu_*} \cup c_M$ such that $\mu_*$ is a prime submodule of $M$ and $c$ is a prime element of $L$.*

As we have mentioned, the radical of an $L$-submodule $\mu \in L(M)$ is the intersection of all prime $L$-submodules $\nu$ of $M$ containing $\mu$ and denoted by $Rad(\mu)$. In [3], the notion of the envelope of a submodule $N$ of $M$, $E(N)$, is defined to be the set

$$\{x \in M \mid x = ry, r^n y \in N, \text{ for some } r \in R,\ y \in M \textit{ and } n \in \mathbb{N}\}.$$

We are going to generalize this notion to $L$-submodules of $M$ and, we will investigate some basic properties of $Rad(\mu)$ and $E(\mu)$ and establish relationship between $Rad(\mu)$ and $E(\mu)$ by using of notions of radical $L$-submodule and $L$-radical formula.

## 3. On radical of $L$-submodules

The notion of radical of an $L$-submodule $\mu$ of $M$ is introduced in [1], now we recall it here and study some new results.

**Definition 3.1.** *Let $\mu \in L(M)$, and $\mathcal{P}_\mu$ be the family of all prime $L$-submodules $\nu$ of $M$ which contain $\mu$. The $L$-radical of $\mu$ denoted by $Rad(\mu)$ is defined by*

$$Rad(\mu) = \begin{cases} \cap\{\nu \mid \nu \in \mathcal{P}_\mu\} & \textit{if } \mathcal{P}_\mu \neq \varnothing \\ 1_M & \textit{if } \mathcal{P}_\mu = \varnothing \end{cases}$$

The concept of radical of an $L$-ideal was introduced in [9], and several results were investigated. Here we generalize their results to $L$-submodules. The proofs are almost the same as $L$-ideals.

**Theorem 3.2.** *Let $\mu$ be a constant $L$-submodule of $M$, then*

$$Rad(\mu) = 1_M$$

**Theorem 3.3.** *Let $\mu$ be a non-constant $L$-submodule of $M$. Then*
(1) $Rad(\mu)(0) = \mu(0)$;
(2) $Rad(\mu_*) \subseteq Rad(\mu)_*$;
(3) $Rad(\mu)(x) = 1, \forall x \in Rad(\mu)_*$;
(4) $\mu \subseteq Rad(\mu)$;
(5) $Rad(\mu \cap \nu) \subseteq Rad(\mu) \cap Rad(\nu)$;
(6) $Rad(Rad(\mu)) = Rad(\mu)$.

**Proposition 3.4.** *Let $M$ be an $R$-module and $\mu_i$ be an $L$-submodule of $M$, for each $i \in I$. Then*
$(i) Rad(\bigcap_{i\in I} \mu_i) \subseteq \bigcap_{i\in I} Rad(\mu_i) = Rad(\bigcap_{i\in I} Rad(\mu_i))$;
$(ii) \sum_{i\in I} Rad(\mu_i) \subseteq Rad(\sum_{i\in I} \mu_i) = Rad(\sum_{i\in I} Rad(\mu_i))$.

**Proof.** $(i)$ $\bigcap_{i\in I} \mu_i \subseteq \mu_i$, for all $i \in I$, so $Rad(\bigcap_{i\in I} \mu_i) \subseteq \bigcap_{i\in I} Rad(\mu_i)$

$$Rad(\bigcap_{i\in I} \mu_i) \subseteq \bigcap_{i\in I} Rad(\mu_i) \subseteq Rad(\bigcap_{i\in I} Rad(\mu_i)).$$

Hence $Rad(\bigcap_{i\in I}\mu_i) \subseteq \bigcap_{i\in I} Rad(\mu_i) = Rad(\bigcap_{i\in I} Rad(\mu_i))$.

$(ii)\mu_i \subseteq \sum_{i\in I}\mu_i$, so $\mu_i \subseteq Rad(\mu_i) \subseteq Rad(\sum_{i\in I}\mu_i)$ for all $i \in I$ and $\sum_{i\in I}\mu_i \subseteq \sum_{i\in I} Rad(\mu_i) \subseteq Rad(\sum_{i\in I}\mu_i)$. Hence

$$Rad(\sum_{i\in I}\mu_i) \subseteq Rad(\sum_{i\in I} Rad(\mu_i)) \subseteq Rad(Rad(\sum_{i\in I}\mu_i)) = Rad(\sum_{i\in I}\mu_i).$$

**Lemma 3.5.** *Let $L$ be a dense chain. Then $Rad(\mu)_a \subseteq rad(\mu_a)$ for each $a \in L$.*

**Proof.** Let $P$ be a prime submodule of $M$ containing $\mu_a$, and $\nu \in L(M)$ is defined by $\nu(x) = 1$ if $x \in P$ and $\nu(x) = b$ if $x \notin P$ that $a < b \in L$. Then $\nu \in \mathcal{P}_\mu$ and $\nu_a = P$, so $Rad(\mu)_a \subseteq rad(\mu_a)$.□
From the previous lemma, we can see for $\mu \in L(M)$; $Rad(\mu)_* \subseteq rad(\mu_*)$, and the following example shows that in general, $Rad(\mu_*) \neq Rad(\mu)_*$.

**Example 3.6.** *([9]) Consider $R = M = \mathbb{Z}$ and let $P_n =< 2^n >$ and $P = \cap P_n$. Let*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in P, \\ 1 - \dfrac{1}{i_x} & \text{if } x \notin P \text{ and } i_x \text{ is the smallest } i \text{ such that } x \notin P_i. \end{cases}$$

*Then $Rad(\mu_*) = \{0\}$ and $(Rad\mu)_* =< 2 >$, whis shows that $Rad(\mu_*) \neq (Rad\mu)_*$.*□

In the last example, we note that $\vee\{\mu(x) \mid x \notin Rad(\mu_*)\} = \mu(0)$.

**Theorem 3.7.** *Let $\mu$ be a non-constant $L$-submodule of $M$ and let $\vee\{\mu(x) \mid x \notin Rad(\mu_*)\} = a < \mu(0)$ and $a$ be prime. Then $Rad(\mu_*) = Rad(\mu)_*$.*

As mentioned in Theorem 3.3, it is not always true for $L$-submodules $\mu$ and $\nu$ that $Rad(\mu) \cap Rad(\nu) = Rad(\mu \cap \nu)$. Now we investigate sufficient conditions on $L$-submodules $\mu$ and $\nu$ which will guarantee that

$$Rad(\mu) \cap Rad(\nu) = Rad(\mu \cap \nu). \ (*)$$

It is quite simple to show that $(*)$ holds if $\mu$ and $\nu$ are prime or $\mu \subseteq \nu$.

**Definition 3.8.** *We say that an $L$-submodule $\mu$ of $M$ is a radical $L$-submodule if $Rad(\mu) = \mu$.*

**Theorem 3.9.** *Let $\mu$ and $\nu$ be $L$-submodules of an $R$-module $M$. Then $\mu \cap \nu = Rad(\mu) \cap Rad(\nu)$ if and only if the following conditions hold:*
*$(i)$ $\mu \cap \nu$ is radical;*
*$(ii)$ $Rad(\mu) \cap Rad(\nu) = Rad(\mu \cap \nu)$.*

**Proof.** Suppose that $\mu \cap \nu = Rad(\mu) \cap Rad(\nu)$. Then

$$Rad(\mu) \cap Rad(\nu) \subseteq \mu \cap \nu \subseteq Rad(\mu \cap \nu).$$

So we can conclude that,

$$Rad(\mu) \cap Rad(\nu) = Rad(\mu \cap \nu).$$

By $(i)$, $\mu \cap \nu = Rad(\mu \cap \nu) = Rad(\mu) \cap Rad(\nu)$. The converse is immediate.□

**Example 3.10.** *Let $N$ and $K$ be prime submodules of an $R$-module $M$ and $c$ be a prime element of the complete lattice $L$. Put $\mu = 1_N \cup c_M$ and $\nu = 1_K \cup c_M$. Then by Theorem 2.10, $\mu$ and $\nu$ are prime L-submodules and so by last Theorem $Rad(\mu \cap \nu) = Rad(\mu) \cap Rad(\nu)$.*□

**Example 3.11.** *Let $R = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$, $N =< (2,3) >$ and $K =< (4,0),(0,1) >$. It is shown in* [6] *that $N$ is prime submodule and $rad(K) =< (2,0),(0,1) >$. Also $N \cap K =< (4,6) >$ which is a radical submodule, and $rad(N) \cap rad(K) =< (2,3) >$. Suppose that $c$ is a prime element of a complete lattice $L$. Consider $\mu = 1_N \cup c_M$ and $\nu = 1_K \cup c_M$. Then*

$$Rad(\mu)(2,3) = \bigwedge\{\xi(2,3) | \xi \in \mathcal{P}_\mu\} = 1,$$

*and*

$$Rad(\nu)(2,3) = \bigwedge\{\zeta(2,3) | \zeta \in \mathcal{P}_\nu\} = 1$$

*since $(2,3) \in rad(k) \subseteq \zeta_*$ for each $\zeta \in \mathcal{P}_\nu$.*
*Also we have $\bigcap\limits_{\eta \in \mathcal{P}_{\mu\cap\nu}} (\eta)_* = rad(N \cap K)$, since if there is a prime submodule $P$ of $M$ containing $N \cap K$, then $(1_P \cup c_M) \in \mathcal{P}_{\mu\cap\nu}$, and so*

$$Rad(\mu \cap \nu)(2,3) = \bigwedge\{\eta(2,3) | \eta \in \mathcal{P}_{(\mu\cap\nu)}\} < 1$$

*because if $(2,3) \in \eta_*$ for every $\eta \in \mathcal{P}_{\mu\cap\nu}$, then $(2,3) \in \bigcap(\eta)_* = rad(N \cap K) = N \cap K$, that is a contradiction.*□

If $M$ is a finitely generated $R$-module, then every proper submodule of $M$ contained in a maximal submodule of $M$. Further, since every maximal submodule of $M$ is prime, thus every proper submodule of $M$ is contained in a prime submodule.
If we adapt the proof of [9, Lemma 3.7.7], then we get the following lemma.

**Lemma 3.12.** *Let $M$ be a finitely generated $R$-module, $L$ be a dense chain and $\xi$ be a proper L-submodule of $M$. Then there exists at least a prime L-submodule of $M$ containing $\xi$.*

**Theorem 3.13.** *Let $M$ be a finitely generated $R$-module, $L$ be a dense chain, and $\mu$, $\nu$ be $L$-submodules of $M$. Then $Rad(\mu)+Rad(\nu)=1_M$ if and only if $\mu+\nu=1_M$.*

**Proof.** Since $\mu\subseteq Rad(\mu)$ and $\nu\subseteq Rad(\nu)$, $\mu+\nu=1_M$ implies that $Rad(\mu)+Rad(\nu)=1_M$. Conversely, assume that $Rad(\mu)+Rad(\nu)=1_M$ and suppose that $\mu+\nu\neq 1_M$. Since $M$ is finitely generated, there exists a prime $L$-submodule $\eta$ such that $\mu+\nu\subseteq\eta$. Hence $Rad(\mu)\subseteq\eta$ and $Rad(\nu)\subseteq\eta$, so $Rad(\mu)+Rad(\nu)\subseteq\eta$, contradicting the assumption.□

**Definition 3.14.** [9]. *Let $\{\mu_\alpha|\alpha\in\Omega\}$ be a collection of finite-valued $L$-ideal of $R$ such that $\mu_\alpha(0)=\mu_\beta(0)$, for all $\alpha,\beta\in\Omega$. Then the $\mu_\alpha$ are said to be pairwise comaximal if $\mu_\alpha\neq a_R, a=\mu(0),\forall\alpha\in\Omega$ and $\mu_\alpha+\mu_\beta=a_R$ $\forall\alpha,\beta\in\Omega,\alpha\neq\beta$.*

**Lemma 3.15.** *Let $\mu_1,...,\mu_n$ be $L$-submodules of $M$ such that for $i=1,...,n,(\mu_i:1_M)$ are pairwise comaximal. Consider a prime $L$-submodule $\nu$ of $M$ containing $\mu_1\cap\mu_2\cap...\cap\mu_n$ then there is exactly one $i\in\{1,2,...,n\}$ such that $\mu_i\subseteq\nu$*

Before proof of the lemma, we mention a result about $L$-ideals.

**Lemma 3.16.** *Let $\mu_1,\mu_2,...,\mu_n$ be $L$-ideals and suppose that $\mu$ be a prime $L$-ideal of $R$ containing $\bigcap_{i=1}^{n}\mu_i$. Then $\mu_i\subseteq\mu$ for some $i=1,2,...,n$.*

**Proof.** Let $\mu_i\nsubseteq\mu$ for all $i$. Then there exists $x_i$ in $M$ such that $\mu_i(x_i)\nleqslant\mu(x_i)$for $i=1,2,...,n$. Thus $x_i\notin\mu_*$ and $\mu_i(x_i)\nleqslant c,$for $i=1,...,n$. So $\prod_{i=1}^{n}x_i\notin\mu_*$, since $\mu_*$ is a prime ideal of $R$ by Theorem 2.6. But

$$\mu_1(x_1)\wedge\mu_2(x_2)\wedge...\wedge\mu_n(x_n)\leqslant(\bigcap_{i=1}^{n}\mu_i)(\prod_{i=1}^{n}x_i)\leqslant\mu(\prod_{i=1}^{n}x_i)=c,$$

this is a contradiction since $c$ is a prime element of $L$.□

**Proof of Lemma 3.15** If $\nu$ is a prime $L$-submodule such that

$$\mu_1\cap,...,\cap\mu_n\subseteq\nu,$$

then

$$(\mu_1:1_M)\cap,...,\cap(\mu_n:1_M)\subseteq(\nu:1_M)$$

Since $(\nu:1_M)$ is a prime $L$-ideal, wlog we have $(\mu_1:1_M)\subseteq(\nu:1_M)$. If $(\mu_i:1_M)\subseteq(\nu:1_M)$ for some $i>1$, then both $(\mu_1:1_M)\subseteq(\nu:1_M)$ and $(\mu_i:1_M)\subseteq(\nu:1_M)$, contradicting comaximality. We thus have $(\mu_1:1_M)\subseteq(\nu:1_M)$ and $(\mu_i:1_M)\nsubseteq(\nu:1_M)$ for $i>1$. We will show that $\mu_1\subseteq\nu$. Suppose $x_a\in\mu_1$ but $x_a\notin\nu$ and choose $(r_i)_{a_i}\in$

$(\mu_i : 1_M)$ and $(r_i)_{a_i} \notin (\nu : 1_M)$ for $i > 1$. Put $r_b = (r_2)_{a_2}...(r_n)_{a_n}$. Then $r_b x_a \in \mu_1$ since $x_a \in \mu_1$ and also $r_b x_a \in \mu_i$ for $i > 1$ by the construction of $r_b$. Hence $r_b x_a \in \mu_1 \cap ... \cap \mu_n \subseteq \nu$ and $x_a \notin \nu$ implies that $r_b = (r_2)_{a_2}...(r_n)_{a_n} \in (\nu : 1_M)$, a contradiction. Thus $\mu_1 \subseteq \nu$.□

**Theorem 3.17.** *Let $\mu_1, ..., \mu_n$ be L-submodules of $M$ such that for $i = 1, ..., n, (\mu_i : 1_M)$ are pairwise comaximal. Then $Rad(\mu_1 \cap, ..., \cap \mu_n) = Rad(\mu_1) \cap, ..., \cap Rad(\mu_n)$.*

**Proof.** If $Rad(\mu_1 \cap, ..., \cap \mu_n) = 1_M$, then the result is immediate. If not, then clearly we have $Rad(\mu_1 \cap, ..., \cap \mu_n) \subseteq Rad(\mu_1) \cap, ..., \cap Rad(\mu_n)$. On the other hand, if $\nu$ is a prime $L$-submodule such that $\mu_1 \cap, ..., \cap \mu_n \subseteq \nu$, then by last lemma there is an $i \in \{1, 2, ..., n\}$ such that $\mu_i \subseteq \nu$. Thus $Rad(\mu_i) \subseteq Rad(\mu_1 \cap, ..., \cap \mu_n)$, and so $Rad(\mu_1) \cap, ..., \cap Rad(\mu_n) \subseteq Rad(\mu_1 \cap, ..., \cap \mu_n)$. □

## 4. $E$-radical of $L$-submodules

In this section, we introduce the notion of $E$-radical of $L$-submodules and give some results of it.

**Definition 4.1.** *Let $M$ be an R-module and $\mu$ be an L-subset of $M$. Then the L-subset $E(\mu) \in L^M$ is defined by*

$$E(\mu)(x) = \vee\{\mu(r^n y) | x = ry, n \in \mathbb{N}\}, \ \forall x \in M,$$

*and is called the E-radical (envelope) of $\mu$.*

Note that this definition is different from the definition of radicals of $L$-submodules given in [13] even $L = [0, 1]$. In fact this definition is a generalization of the notion of the envelope of submodules in module theory, as well as it agrees with the definition of $\Re$-radicals by the Definition 2.1.

**Theorem 4.2.** *Let $\mu \in L(M)$. Then the following assertions hold:*
(1) $E(\mu)(0) = 1$*;*
(2)$\mu \subseteq E(\mu) \subseteq Rad(\mu)$*;*
(3) $< E(\mu_b) > \subseteq E(\mu)_b$*;*
(4) *If* $\vee\{\mu(x) | x \notin \mu_*\} = b < 1$*, then* $< E(\mu_*) > = E(\mu)_*$*;*
(5)$\mu \subseteq \nu \Rightarrow E(\mu) \subseteq E(\nu)$*.;*
(6)$\mu + \nu \subseteq E(\mu) + E(\nu)$*;*
(7)$E(\mu) \cap E(\nu) = E(\mu \cap \nu)$*.*

**Proof.** (1) $E(\mu)(0) = \vee\{\mu(r^n y) | 0 = ry, n \in \mathbb{N}\} = \mu(0) = 1$.
(2). Clearly $\mu \subseteq E(\mu)$. If $Rad(\mu) = 1_M$, then the result is clear, If not, consider $\xi$ as a prime $L$-submodule of $M$ and containing $\mu$. Then $\xi(M) = \{1, c\}$, that is $c \in L \setminus \{1\}$ is a prime element of $L$. Let $x \in M$ such that $x = ry, n \in \mathbb{N}$ and $\xi(r^n y) = 1$. Then $r^n y \in \xi_*$, thus $r \in$

$(\xi_* : M)$ or $y \in \xi_*$, since $\xi_*$ is prime. Therefore $x = ry \in \xi_*$ and so $\xi(r^n y) = \xi(x) = 1$. Now if $\xi(r^n y) = c$, then $c = \xi(r^n y) \geqslant \xi(ry) = \xi(x)$. Thus $\xi(x) = \xi(ry) = c = \xi(r^n y)$. Hence

$$\forall n \in \mathbb{N},\ \xi(x) = \xi(r^n y) \geqslant \mu(r^n y).$$

So $E(\mu)(x) \leqslant \xi(x)$, therefore $E(\mu)(x) \subseteq Rad(\mu)(x)$.
(3) Let $x \in E(\mu_b)$, then there are $m \in \mathbb{N}, r \in R$ and $y \in M$ such that $x = ry$ and $r^m y \in \mu_b$. Thus $\mu(r^m y) \geqslant b$, Hence $E(\mu)(x) = \vee\{\mu(r^n y) \mid x = ry, n \in \mathbb{N}\} \geqslant b$. Therefore $x \in E(\mu)_b$, and so $< E(\mu_b) > \subseteq E(\mu)_b$.
(4) From (3) we have $< E(\mu_*) > \subseteq E(\mu)_*$. Now let $x \in E(\mu)_*$, then $E(\mu)(x) = E(\mu)(0)$. Thus

$$1 = \mu(0) = E(\mu)(0) = E(\mu)(x) = \vee\{\mu(r^n y) \mid x = ry, n \in \mathbb{N}\}.$$

Hence there are $n \in \mathbb{N}, s \in R$, and $y \in M$ such that $x = sy$ and $\mu(s^n y) = \mu(0) = 1$. Then $x = sy \in \mu_* \subseteq E(\mu_*) \subseteq < E(\mu_*) >$
(5) and (6) are straightforward.

$$\begin{aligned}
(7)(E(\mu) \cap E(\nu))(x) &= (\bigvee\{\mu(r^n y) \mid x = ry, n \in \mathbb{N}\}) \wedge (\bigvee\{\nu(r^n y) \mid x = ry, n \in \mathbb{N}\}) \\
&= \bigvee\{\mu(r^n y) \wedge \nu(r^n y) \mid x = ry, n \in \mathbb{N}\} \\
&= \{(\mu \cap \nu)(r^n y) \mid x = ry, n \in \mathbb{N}\} \\
&= E(\mu \cap \nu)(x). \square
\end{aligned}$$

**Theorem 4.3.** *Let $M$ be an $R$-module. If $\mu \in L(M)$, then $E(\mu) \in L(M)$*

**Proof.** Let $x, y \in M$ and $r \in R$. By previous theorem, $E(\mu)(0) = 1$,

$$\begin{aligned}
E(x+y) &= \bigvee\{\mu(r^n z) \mid x + y = rz, n \in \mathbb{N}\} \\
&\geqslant \bigvee\{\mu(r^n u + r^n v) \mid x = ru, y = rv, n \in \mathbb{N}\} \\
&\geqslant \bigvee\{\mu(r^n u) \wedge \mu(r^n v) \mid x = ru, y = rv, n \in \mathbb{N}\} \\
&= (\bigvee\{\mu(r^n u) \mid x = ru, n \in \mathbb{N}\}) \wedge (\bigvee\{\mu(r^n(v)) \mid, y = rv, n \in \mathbb{N}\}) \\
&= E(\mu)(x) \wedge E(\mu)(y)
\end{aligned}$$

and we have

$$\begin{aligned}
E(\mu)(rx) &= \vee\{\mu(s^n y) \mid rx = sy, n \in \mathbb{N}\} \\
&\geqslant \vee\{\mu(r^n(t^n y)) \mid rx = rty, n \in \mathbb{N}\} \\
&\geqslant \vee\{\mu(t^n y) \mid rx = rty, n \in \mathbb{N}\} \\
&\geqslant \vee\{\mu(t^n y) \mid x = ty, n \in \mathbb{N}\} = E(\mu)(x). \square
\end{aligned}$$

We know that for an $L$-ideal $\mu$ of $R$, we have $\Re(\Re(\mu)) = \Re(\mu)$. So in studying the $\Re$-radical of an $L$-ideal, the number of $\Re$-radicals is not important. But for $E(\mu)$, it is important. For example, see the following:

**Example 4.4.** *Consider $R = \mathbb{Z}[x]$ and let the $R$-module to be $M = R \oplus R$ and let $N = R(x, 4) + R(0, x) + x^2 M$. Then according to Example 1 in [2] for submodule $N$ we have,*

$$< E(N) >= R(0, 4) + xM \neq R(0, 2) + xM.$$

*Hence $(0, 2) \notin < E(N) >$, however*

$$2^2(0, 1) = (0, 4) \in < E(N) > .$$

*Now let $\mu = 1_N \cup c_M$. Then we have,*

$$E(\mu)(0, 2) = \bigvee\{\mu(r^n y) | (0, 2) = ry\} = c.$$

*But*

$$E(E(\mu))(0, 2) = \bigvee\{E(\mu)(s^m z) | (0, 2) = sz\} = 1.$$

*Because $(0, 4) = 2^2(0, 1) \in N$ and $E(\mu)(0, 4) = 1$. Therefore $E(\mu) \neq E(E(\mu))$.*

**Theorem 4.5.** *Let $\mu \in L(M)$. Then*

$$E(\mu)(x) = \bigvee\{a \mid a \in \mu(M), x \in E(\mu_a)\}.$$

**Proof.** Let $x \in M$ and $E(\mu)(x) = b = \bigvee\{\mu(r^n y) \mid x = ry, n \in \mathbb{N}\}$ and suppose that $c = \bigvee\{a \mid a \in \mu(M), x \in E(\mu_a)\}$. Then

$$\begin{aligned} c &= \bigvee\{a \mid a \in \mu(M), x = ry, r^n y \in \mu_a, \text{ for some } n \in \mathbb{N}\} \\ &= \bigvee\{a \mid a \in \mu(M), x = ry, \mu(r^n y) \geqslant a, \text{ for some } n \in \mathbb{N}\}. \end{aligned}$$

Therefore $b \geqslant c$. Now let $x \in M$ such that $x = ry$. For $n \in \mathbb{N}$ put $a = \mu(r^n y)$. Then $a \in \mu(M)$ and $r^n y \in \mu_a$. Hence $x \in E(\mu_a)$, so $b \leqslant c$.□

**Corollary 4.6.** *Let $\mu \in L(M)$ and $\mu(M) = \{a_i \mid i \in \mathbb{N}\}$ which $a_1 = 1$ and $\forall i \in \mathbb{N} \setminus \{1\}, a_{i+1} \leqslant a_i$. Then*
*(i)* $< E(\mu_{a_i}) >= E(\mu)_{a_i}, \ \forall i \in \mathbb{N}$
*(ii)*

$$E(\mu)(x) = \begin{cases} \mu(0) & if\ x \in E(\mu_*) \\ a_i & if\ x \in < E(\mu_{a_i}) > \setminus < E(\mu_{a_{i-1}}) > \ \ i \in \mathbb{N} \setminus \{1\} \end{cases}$$

**Proof.** By Theorem 4.2, $< E(\mu_{a_i}) > \subseteq E(\mu)_{a_i}$. Now let $i \in \mathbb{N}$ and $x \in E(\mu)_{a_i}$. Then $E(\mu)(x) \geqslant a_i$. Thus

$$\bigvee\{\mu(r^n y) \mid x = ry, n \in \mathbb{N}\} \geqslant a_i.$$

So there is $n \in \mathbb{N}$ such that $\mu(r^n y) \geqslant a_i$, $x = ry$. Then $r^n y \in \mu_{a_i}$, $x = ry$, so $x \in E(\mu_{a_i}) \subseteq < E(\mu_{a_i}) >$.
*(ii)*. Let $x \in < E(\mu_{a_i}) > \setminus < E(\mu_{a_{i-1}}) >$. Then $x \in E(\mu)_{a_i} \setminus < E(\mu)_{a_{i-1}}$

by $(i)$. Thus $a_i \leqslant E(\mu)(x) < a_{i-1}$, and $E(\mu)(x) = a_i$. $\square$
By Corollary 4.6, if $\mu$ is a fuzzy submodule, then $< E(\mu_b) >= E(\mu)_b$.

**Theorem 4.7.** *Let $\nu \in L^M$. Then*

$$E(\nu) = \bigcup\{x_a \mid x = ry, a \in L, (r^n y)_a \subseteq \nu, \textit{ for some } n \in \mathbb{N}\}.$$

**Proof**. Let $a \in L$, and let $x = sz$. If there exists $n \in \mathbb{N}$ such that $(s^n z)_a \subseteq \nu$, then

$$\begin{aligned} E(\nu)(x) &= \bigvee\{\nu(r^m y) \mid x = ry, m \in \mathbb{N}\} \\ &\geqslant \nu(s^n z) \\ &\geqslant a. \end{aligned}$$

Thus $E(\nu)(x) \geqslant x_a(x)$, hence

$$E(\nu) \supseteq \bigcup\{x_a \mid x = ry, a \in L, (r^n y)_a \subseteq \nu, \text{ for some } n \in \mathbb{N}\}.$$

Now Let $x \in M$ such that $x = r_i y_i$, which $y_i \in M, r_i \in R$ and $\forall n \in \mathbb{N}$, $\nu(r_i^n y_i) = a_{in}$. Then $(r_i^n y_i)_{a_{in}} \subseteq \nu$, $\forall n \in \mathbb{N}$. Therefore

$$\begin{aligned} E(\nu)(x) &= \bigvee\{\nu(r^n y) \mid x = ry,\ n \in \mathbb{N}\} \\ &= \bigvee\{(r_i^n y_i)_{a_{in}}(r_i^n y_i) \mid x = r_i y_i,\ n \in \mathbb{N}\} \\ &\leqslant (\bigcup\{x_a \mid x = ry,\ a \in L,\ (r^n y)_a \subseteq \nu,\ \text{for some } n \in \mathbb{N}\})(x). \end{aligned}$$

Hence $E(\nu) \subseteq \bigcup\{x_a \mid x = ry,\ a \in L,\ (r^n y)_a \subseteq \nu,\ \text{for some } n \in \mathbb{N}\}$. Therefore,

$$E(\nu)(x) = \bigcup\{x_a \mid x = ry,\ a \in L,\ (r^n y)_a \subseteq \nu,\ \text{for some } n \in \mathbb{N}\}.\square$$

In the next two Theorems, we investigate the behavior of $E$-radical under homomorphisms.

**Theorem 4.8.** *Let $N$ be an $R-$module and $f$ be an epimorphism from $M$ into $N$. Consider $\mu \in L^M$ and $\nu \in L^N$.*
*(i) If $\mu$ is $f-$invariant, then $f(E(\mu)) = E(f(\mu))$.*
*(ii) $f^{-1}(E(\mu)) \subseteq E(f(\mu))$.*

**Proof.** (i) Let $y \in N$. Then

$$\begin{aligned}
f(E(\mu))(y) &= \bigvee\{E(\mu)(x) \mid y = f(x)\}\\
&= \bigvee\{\bigvee\{\mu(r^n z) \mid x = rz,\ n \in \mathbb{N}\} \mid y = f(x)\}\\
&= \bigvee\{\bigvee\{\mu(r^n z) \mid f(r^n z) = r^{n-1}y\} \mid y = rf(z),\ n \in \mathbb{N}\}\\
&= \bigvee\{\bigvee\{\mu(u) \mid f(u) = r^{n-1}y\} \mid y = rf(z),\ n \in \mathbb{N}\}\\
&= \bigvee\{f(\mu)(r^{n-1}y) \mid y = rf(z),\ n \in \mathbb{N}\}\\
&= \bigvee\{f(\mu)(r^n f(z)) \mid y = rf(z),\ n \in \mathbb{N}\} = E(f(\mu))(y).
\end{aligned}$$

$(ii)$

$$\begin{aligned}
E(f^{-1}(\nu))(x) &= \bigvee\{f^{-1}(\nu)(r^n y) \mid x = ry, n \in \mathbb{N}\}\\
&= \bigvee\{\nu(f(r^n y)) \mid x = ry, n \in \mathbb{N}\}\\
&\leqslant \bigvee\{\nu((r^n f(y))) \mid f(x) = rf(y), n \in \mathbb{N}\}\\
&= E(\nu)(f(x)) = f^{-1}(E(\nu)(x)).\square
\end{aligned}$$

Now if $f$ is an isomorphism, then the equality holds in the second part.

**Theorem 4.9.** *Assume the hypothesis given of Theorem 4.8. Then*
*(i) If $\mu$ is $f-$invariant, then $E(f(\mu)) \in L(N)$.*
*(ii)If $f$ be an isomorphism, then $E(f^{-1}(\nu)) \in L(M)$.*

**Proof.** This is a direct consequence of the last theorem.$\square$

**Proposition 4.10.** *Let $\mu \in L(M)$. Then:*
*i)*$\Re(\mu : 1_M) \subseteq (E(\mu) : 1_M)$;
*ii)*$\sqrt{\mu : 1_M} \subseteq (Rad(\mu) : 1_M)$.

**Proof**.$(i)$

$$\begin{aligned}
(\Re(\mu:1_M)\cdot 1_M)(x) &= \bigvee_{x=ry}\{\Re(\mu:1_M)(r)\wedge 1_M(y)\}\\
&= \bigvee_{x=ry}\{\bigvee_{n\in\mathbb{N},\eta\cdot 1_M\subseteq\mu}(\eta(r^n)\wedge 1_M(y))\}\\
&= \bigvee_{x=ry,n\in\mathbb{N}}\{\bigvee_{\eta\cdot 1_M\subseteq\mu}(\eta(r^n)\wedge 1_M(y))\}\\
&\leq \bigvee_{x=ry,n\in\mathbb{N}}\{\bigvee_{\eta\cdot 1_M\subseteq\mu}\eta\cdot 1_M(r^n y)\}\\
&\leq \bigvee_{x=ry,n\in\mathbb{N}}\mu(r^n y)\\
&= E(\mu)(x), \forall x\in M.
\end{aligned}$$

$(ii)$ if $Rad(\mu)=1_M$, the result is immediate. Otherwise, if $\nu$ is any prime $L$-submodule of $M$ which contains $\mu$, we have $(\mu:1_M)\subseteq(\nu:1_M)$, and $(\nu:1_M)$ is prime $L$-ideal. Hence $\sqrt{\mu:1_M}\subseteq(\nu:1_M)$, and thus $\sqrt{\mu:1_M}\cdot 1_M\subseteq(\nu:1_M)\cdot 1_M\subseteq\nu$. Since $\nu$ is an arbitrary prime $L$-submodule containing $\mu$, we have $\sqrt{\mu:1_M}\cdot 1_M\subseteq Rad(\mu)$. $\square$

## 5. L-Radical formula

We say that $M$ satisfies the $L$-radical formula if for every $\mu\in L(M)$, $E(\mu)=Rad(\mu)$. In the folowing example, it can be seen that the $L$-radical formula is not generally valid in $L$-submodules:

**Example 5.1.** *Let $R$ denote the polynomial ring $\mathbb{Z}[x]$. Consider $M=R\oplus R$ and let $N=R(x,4)+R(0,x)+x^2M$. Then according to* [14]*, for submodule $N$, we have,*

$$<E(N)>=R(0,4)+xM\neq R(0,2)+xM=Rad(N).$$

*Now let $\mu=1_N\cup c_M$. If $\nu\in\mathcal{P}_\mu$, then $N=\mu_*\subseteq\nu_*$. So $Rad(N)\subseteq\nu_*$, therefore $(0,2)\in\nu_*$ for all $\nu\in\mathcal{P}_\mu$. Thus*

$$Rad(\mu)(0,2)=\bigwedge_{\nu\in\mathcal{P}_\mu}\nu(0,2)=1.$$

*But*

$$E(\mu)(0,2)=\bigvee\{\mu(r^n y)\mid(0,2)=ry,n\in\mathbb{N}\}=c.$$

*Since if there is $n\in\mathbb{N}$ such that $r^n y\in N$ and $(0,2)=ry$, then we will have $(0,2)\in E(N)\subseteq<E(N)>$, that is a contradiction.* $\square$

In the sequel, we will investigate some conditions for $E(\mu)=Rad(\mu)$ for any $L$-submodule $\mu$ of $M$.

**Theorem 5.2.** *Let $\mu$ be a radical L-submodule of $M$. Then*

$$E(\mu) = Rad(\mu).$$

**Proof**. $\mu \subseteq E(\mu) \subseteq Rad(\mu) = \mu$.□

**Corollary 5.3.** *Let $\mu$ be a radical L-submodule of $M$. Then*

$$\sqrt{\mu : 1_M} = (Rad(\mu) : 1_M) = (E(\mu) : 1_M) = \Re(\mu : 1_M)$$

**Theorem 5.4.** *Let $\mu$ be an L-submodule and $rad(\mu_a) =< E(\mu_a) >$, for each $a \in L$. Then $E(\mu) = Rad(\mu)$.*

**Proof**. Consider $x \in M$. Put $a = Rad(\mu)(x)$, then

$$x \in Rad(\mu)_a \subseteq rad(\mu_a) =< E(\mu_a) >\subseteq E(\mu)_a.$$

So $E(\mu)(x) \geq a = Rad(\mu)(x)$. Thus $Rad(\mu) \subseteq E(\mu)$.□

**Corollary 5.5.** *Let $M$ be an R-module that satisfies the radical formula, and $L$ be a dense chain. Then $E(\mu) = Rad(\mu)$.*

**Corollary 5.6.** *Let $rad(\mu_a) =< E(\mu_a) >$, for each $a \in L$. Then $rad(\mu_a) \subseteq Rad(\mu)_a$.*

**Proof**. $rad(\mu_a) =< E(\mu_a) >\subseteq E(\mu)_a \subseteq Rad(\mu)_a$. □

## References


[1] R. Ameri, R. Mahjoob, *Spectrum of prime L-submodules*, Fuzzy Sets and Systems 159 (2008)1107-1115.

[2] A. Azizi, *Radical formula and prime submodules*, Journal of Algebra 307 (2007) 454- 460.

[3] R. McCasland, *Some Commutative Ring Results Generalized to Unitary Modules*, Ph.D.Thesis, University of Texas at Arlington, 1983.

[4] J.A. Goguen, *L-fuzzy sets*, J. Math. Appl. 18 (1967) 145- 174.

[5] R. L. McCasland, M. E. Moore, *On radicals of submodules*, Communications in Algebra 19(5) (1991)1327-1341.

[6] M. E. Moore, S. J. Smith, *Prime and radical submodules of modules over commutative ring*, Communications in Algebra, 30(10)(2002)5037-5064.

[7] D.S. Malik, J.N. Mordeson, *Fuzzy maximal,radical,and primary ideals of a ring*, Inform.Sci.53(1991)237-250.

[8] D.S. Malik, J.N. Mordeson, *Radicals of fuzzy ideals*, Inform.Sci. 65(1992)239-252.

[9] J.N. Mordeson, D.S. Malik, *Fuzzy Commutative Algebra*, World Scientific Publishing, Singapore, 1998.

[10] C.V. Negoita, D.A. Ralescu, *Application of Fuzzy Systems Analysis*, Birkhauser, Basel, 1975.

[11] F.Z. Pan, *Fuzzy finitely generated modules*, Fuzzy Sets and Systems 21 (1987) 105- 113.

[12] R. Rosenfeld, *Fuzzy groups*, J. Math. Anal Appl. 35(1971)512- 517.

[13] F.I. Sidky, *On radical of fuzzy submodules and primary fuzzy submodules*, Fuzzy Sets and Systems 119 (2001) 419- 425.

[14] P.F. Smith, *Primary modules over commutative rings*, Glasgow Math. J.43(2001)103-111.

[15] L.A. Zadeh, *Fuzzy Sets*, Inform and Control 8 (1965) 338- 353.