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# Solving Inverse Optimization Problems in Linear Programming: A Geometric and Algorithmic Approach

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Abstract. This paper addresses the inverse optimization problem for linear programming, focusing on determining a cost vector that ensures a pre-specified solution is optimal. Two approaches are presented: (i) using the Karush-Kuhn-Tucker (KKT) conditions, and (ii) a geometric perspective leveraging first-order necessary conditions. The latter method results in a convex quadratic programming problem, solved efficiently using the gradient projection method. Numerical experiments, including a complex resource allocation problem, validate the proposed approach. This study extends the theory and application of inverse optimization across logistics, resource management, and supply chain optimization.

Keywords: Inverse optimization, Linear programming, Gradient projection method, Karush-Kuhn-Tucker (KKT) conditions, Convex quadratic programming, Resource allocation.

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## 1. Introduction

Consider the following optimization problem:

$$\min \{f(c, x) | x \in D(x)\}, \tag{1.1}$$

where $c \in \mathbb{R}^n$ is a parameter vector, $D(x)$ is the feasible region for $x$, and $f(c, x)$ is the objective function. If $D(x)$ is a bounded and closed set, and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then by Weierstrass Theorem, the problem (1.1) has an optimal solution [1]. Suppose $x^*$ is a feasible solution to (1.1). The goal is to determine $\bar{c} \in \mathbb{R}^n$ such that $x^*$ becomes the optimal solution of (1.1) when $\bar{c}$ replaces $c$. Define

$$F(x^*) = \{\bar{c} \in \mathbb{R}^n | \min\{f(\bar{c}, x) \mid x \in D\} = f(\bar{c}, x^*)\}. \tag{1.2}$$

It is evident that $0 \in F(x^*)$, and therefore $F(x^*) \neq \phi$. With this, we define the inverse problem associated with (1.1) as:

$$\min\{\|c - \bar{c}\|_p | \bar{c} \in F(x^*)\}, \tag{1.3}$$

where $\| \cdot \|_p$ denotes the $p-$norm. In this paper, we focus on the inverse problem (1.3) using the $l_2-$norm. For simplicity, $\|\cdot\|$ will represent $\|\cdot\|_2$.

The concept of inverse optimization was first introduced by Burton and Toint in 1992 in the context of the shortest path problem [2]. Since then, inverse optimization has been applied to various problems, including: linear optimization problems [3, 4], combinatorial optimization problems [2, 5], combinatorial optimization problems with bounded variables [6, 7], semidefinite quadratic programming and conic programming problems [8], multi-objective optimization [9].

Inverse optimization has numerous applications across science and engineering, including traffic modeling and seismic tomography [2, 5, 6], minimum spanning tree problems [11], shortest arborescence problems [10], perfect $k$-matching problems in bipartite graph [7], portfolio optimization, utility function identification [8], cancer therapy [9], and radiotherapy planning [12].

This paper is organized as follows: In Section 2, we provide the formal definition of the inverse problem in linear programming. Section 3 offers a geometric perspective for constructing the inverse problem in linear optimization. In Section 4, we present a solution methodology for the inverse problem. Finally, Section 5 provides numerical results to illustrate the proposed approach.

## 2. Inverse problem of linear programming

This section introduces the inverse problem of linear programming. We provide a general perspective on the problem and its formulation. In the next section, we focus on a geometric interpretation of the inverse problem, leveraging the fact that the optimal solution of a linear programming problem occurs at an extreme point. Let $x^*$ be a feasible solution. The objective is to approximate a cost vector such that $x^*$ becomes the optimal solution of problem (1.1) with this new cost vector. Specifically, we aim to determine a cost vector with minimal adjustments from the original, ensuring $x^*$ is optimal. Consider the general form of a linear programming problem:

$$\min\{c^T x \mid Ax \geq b\}, \tag{2.1}$$

where $b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $A$ is an $m \times n$ matrix, $m \geq n$. Suppose $x^*$ is a feasible solution to problem (2.1). The goal is to find a new coefficient vector with minimal norm changes such that $x^*$ becomes the optimal solution of the updated problem. According to (1.3), the inverse problem for (2.1) can be formulated as:

$$\min\left\{\|\bar{c} - c\|^2 \middle| \; x^* = \arg\min\{\bar{c}^T x | Ax \geq b\}\right\}. \tag{2.2}$$

To facilitate this, we define the active set $A(x^*)$ at a feasible point $x^*$ as:

$$A(x^*) = \{i | a^i x^* = b_i\},$$

where $a^i$ is $i-$th row of the matrix $A$ [14]. At the feasible point $x^*$, the $i-$ inequality constraint $i$ is said to be active if $a^i x^* = b_i$ and inactive if $a^i x^* > b_i$.

The following theorem reformulates the inverse problem (2.2) as a quadratic programming problem.

**Theorem 2.1.** *Let $c \in \mathbb{R}^n, b \in \mathbb{R}^m$, and $Ax^* \geq b$. Define $\theta = \bar{c} - c$. Then the inverse problem (2.2) is equivalent to the following problem:*

$$\begin{array}{llll} (ILP) & \min & \|\theta\|^2 & \\ & s.t. & a_j^T u - \theta_j = c_j, & \\ & & u_j \geq 0, & j \in A(x^*), \\ & & u_j = 0, & j \notin A(x^*). \end{array} \tag{2.3}$$

*Proof.* According to complementary slackness theorem [1], the feasible solution $x^*$ is optimal for the problem $\min\{\bar{c}^T x \mid Ax \geq b\}$ if and only if there exists a vector $u \in \mathbb{R}^m$ such that:

$$\begin{aligned} u^T a_j &= \bar{c}_j, && j = 1, 2, \cdots, n, \\ u_i(a^i x^* - b) &= 0, && i = 1, 2, \cdots, m, \\ u &\geq 0. \end{aligned}$$

Since $x^*$ is known (clear) and by definition of $A(x^*)$, we can write equation $u_i(a^i x^* - b) = 0$ as follows: $u_j \geq 0$ when $j \in A(x^*)$ and $u_j = 0$ when $j \notin A(x^*)$. By setting $\theta = \bar{c} - c$, the proof is completed. $\square$

## 3. Inverse Linear Optimization Problem: A Geometric Perspective

In this section, we examine the inverse peoblem of (2.1) from a geometric perspective. We know that $x^*$ is an unique optimal solution of a linear programming problem when $x^*$ is an extreme point of the feasible set, and the vector $-c$ aligns with one of the directions indicated in Figure 1. Furthermore, $x^*$ is an optimal solution if and only if the cost vector $c$ is a linear combination of the normal vectors of the active constraints at $x^*$ [1, 15]. This is illustrated in Figure 1. Since the vector

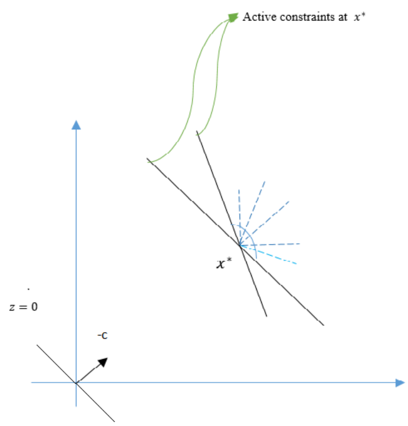


Figure 1. Graphical solution of the linear programming problem

$c$ must be a linear combination of the active constraints at $x^*$, then the inverse problem can be stated as follows:

$$\begin{aligned} \bar{c} = \quad & \arg\min \quad \|y - c\|_2 \\ & s.t. \quad \sum_{j \in A(x^*)} \lambda_j a_j - y = 0, \\ & \qquad\quad \lambda_j \geq 0, \quad j \in A(x^*). \end{aligned} \tag{3.1}$$

Let $\bar{c}$ be an arbitary solution to probelm (3.1). The following theorem demonstrates that $x^*$ is the optimal solution to (2.1) with $c = \bar{c}$:

**Theorem 3.1.** *The optimal solution of*

$$\min\{\bar{c}^T x | \ Ax \geq b\}, \tag{3.2}$$

*is $x^*$, where $\bar{c}$ is a feasible solution of problem* (3.1)*.*

*Proof.* Consider the Lagrangian function for the problem (3.2):

$$L(x, \lambda) = \bar{c}^T x - \lambda^T (Ax - b),$$

where $\lambda \in \mathbb{R}_+^{m+n}$. If $x^*$ is the optimal solution to the problem (3.2), then there exists a Lagrange multiplier vector $\lambda^*$ such that the following first-order necessary conditions hold at $(x^*, \lambda^*)$[14]:

$$\begin{array}{rcll} \nabla_x L(x^*, \lambda^*) & = & 0, & \\ Ax^* & \geq & b, & \\ \lambda_i^*(a^i x^* - b_i) & = & 0, & i = 1, 2, \cdots, m+n \\ \lambda & \geq & 0. & \end{array} \tag{3.3}$$

From (3.3), the Lagrange multipliers corresponding to inactive constraints are zero. Therefore, we can exclude the terms for indices $i \notin \bar{A}(x^*)$and rewrite the condition as:

$$0 = \nabla_x L(x^*, \lambda^*) = \bar{c} - \sum_{j \in A(x^*)} \lambda_j a^j.$$

Thus $x^*$ is the optimal solution for problem (3.2) if and only if $\bar{c} = \sum_{j \in a(x^*)} \lambda_j a^j$. The proof is complete. □

Let the matrix $B$ contains all active constraints, such that $B^T \lambda = \sum_{j \in \bar{a}(x^*)} \lambda_j \bar{A}^j$. Here, the size of $B$ is at most $(m+n) \times n$. Using this, we can reformulate the inverse optimization problem (3.1) as:

$$\begin{array}{rcl} \bar{c} = & \arg\min & \|y - c\|_2^2 \\ & s.t. & B^T \lambda = y, \\ & & \lambda \geq 0, \end{array} \tag{3.4}$$

where $\lambda \in \mathbb{R}^{|\bar{A}(x^*)|}$ and $B \in \mathbb{R}^{|\bar{A}(x^*)| \times n}$. Removing the auxiliary variable $y$, we obtain:

$$\begin{array}{ll} \min & \|B^T \lambda - c\|_2^2 \\ s.t. & \lambda \geq 0. \end{array} \tag{3.5}$$

To simplify the objective function, we expand $\|B^T \lambda - c\|^2$ as:

$$\begin{array}{rcl} \|B^T \lambda - c\|^2 & = & (B^T \lambda - c)^T (B^T \lambda - c) = (\lambda^T B - c^T)(B^T \lambda - c) \\ & = & \lambda^T B B^T \lambda - 2c^T B^T \lambda + c^T c \end{array}$$

Thus, problem (3.5) is reformulated as:

$$\min_{\lambda \geq 0} \lambda^T B B^T \lambda - 2c^T B^T \lambda + \|c\|^2.$$

The matrix $BB^T$ is symmetric, but it may not necessarily be positive definite, implying that problem (3.5) can be either convex or nonconvex. The feasible region is the nonnegative orthant. Various methods, such as the gradient projection method [14], can be used to solve this problem.

Letting $D = B^T$, the problem (3.5) can be equivalently written as:

$$\begin{array}{ll} \min & \|D\lambda - c\|_2 \\ s.t. & \lambda \geq 0. \end{array} \tag{3.6}$$

This is the nonnegative least squares problemfor which several algorithms are available [16, 17, 18, 19]. In the next section, we solve problem (3.5) using the gradient projection method.

## 4. The Gradient Projection Method

In this section, we solve problem (3.5) using the gradient projection method, as outlined in [14].

**Algorithm 4.1** (Gradient Projection Method for Solving Problem (3.5))**.**

**Step 1** *(Initialization:) Choose an initial feasible point* $\lambda^0 \geq 0$*. If no such point is available, initialize with* $\lambda^0 = 0$*.* $\varepsilon > 0$ *is a pre-specified tolerance level.*

**Step 2** *(Gradient Descent:) Compute the gradient of the objective function:*

$$\nabla f(\lambda) = 2BB^T\lambda - 2Bc.$$

*Update* $\lambda$ *using a step size* $\alpha$ *in the negative gradient direction:*

$$\lambda^{k+1} = \lambda^k - \alpha \nabla f(\lambda^k)$$

**Step 3** *(Projection Step:) After each gradient descent step, project* $\lambda^{k+1}$ *onto the feasible region* $\lambda > 0$*. This is done by setting:*

$$\lambda^{k+1} = \max(0, \lambda^{k+1}),$$

*where the max function is applied element-wise.*

**Step 3** *(Convergence Check:) Repeat the process until convergence criteria are met, such as:*

$$\|\nabla f(\lambda^k)\|\varepsilon.$$

In the next section, we demonstrate the application of the gradient projection method to solve problem (3.5) with numerical examples.

## 5. The Gradient Projection Method

In this section, we solve problem (3.5) using the gradient projection method, as outlined in [14]. This iterative method ensures convergence to a solution within the feasible region of the inverse problem.

**Algorithm 5.1** (Gradient Projection Method for Solving Problem (3.5))**.** **Step 1** ***Initialization:*** *Choose an initial feasible point* $\lambda^0 \geq 0$*. If no such point is available, initialize with* $\lambda^0 = 0$*. Let* $\varepsilon > 0$ *be a pre-specified tolerance level.*

**Step 2** ***Gradient Descent:*** *Compute the gradient of the objective function:*

$$\nabla f(\lambda) = 2BB^T\lambda - 2Bc.$$

*Update $\lambda$ using a step size $\alpha$ in the negative gradient direction:*

$$\lambda^{k+1} = \lambda^k - \alpha \nabla f(\lambda^k).$$

**Step 3** ***Projection Step:*** *After each gradient descent step, project $\lambda^{k+1}$ onto the feasible region $\lambda \geq 0$. This is done by setting:*

$$\lambda^{k+1} = \max(0, \lambda^{k+1}),$$

*where the maximum function is applied element-wise.*

**Step 4** ***Convergence Check:*** *Repeat the process until the convergence criteria are met, such as:*

$$\|\nabla f(\lambda^k)\| \leq \varepsilon.$$

## 6. Numerical Results

In this section, we apply the proposed gradient projection method to solve a resource allocation problem. This example demonstrates the effectiveness of the algorithm in deriving an adjusted cost vector that ensures the optimality of a given solution.

**Example: Resource Allocation Problem.** Consider a scenario where a company manages $m = 5$ resources and allocates them to $n = 4$ competing projects. The goal is to minimize the total cost of resource allocation while satisfying project-specific constraints. The problem is formulated as:

$$\min\{c^T x \mid Ax \geq b,\ x \geq 0\},$$

where:

- $c \in \mathbb{R}^n$ is the cost vector representing resource consumption per unit allocation,
- $x \in \mathbb{R}^n$ is the decision vector for the allocation,
- $A \in \mathbb{R}^{m \times n}$ is the constraint matrix defining resource limits and project requirements,
- $b \in \mathbb{R}^m$ is the resource and project-specific demand vector.

The problem data is given as:

$$A = \begin{bmatrix} 2 & 3 & 1 & 5 \\ 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 4 \\ 2 & 4 & 3 & 1 \\ 5 & 2 & 1 & 3 \end{bmatrix}, \quad b = [20, 15, 25, 10, 30]^T, \quad c = [8, 7, 6, 9]^T.$$

The company specifies that the optimal allocation should be $x^* = [4, 2, 5, 3]^T$.

The active constraints correspond to the first three rows of $A$, where equality holds. These constraints form the matrix $A(x^*)$:

$$A(x^*) = \begin{bmatrix} 2 & 3 & 1 & 5 \\ 1 & 2 & 4 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}.$$

Using $B^T = A(x^*)^T$, the inverse problem becomes:

$$\min_{\lambda \geq 0} \|B^T \lambda - c\|^2,$$

where $B^T$ is the transpose of the active constraint matrix.

The gradient projection method is applied with an initial guess $\lambda^0 = 0$. After several iterations, the algorithm converges to:

$$\lambda = [2.3, 1.5, 0]^T.$$

Using this $\lambda$, the adjusted cost vector is computed as:

$$\bar{c} = B^T \lambda = \left[8.1, 6.8, 7.5, 9.3\right]^T.$$

Substituting $\bar{c}$ into the original problem, we solve:

$$\min\{\bar{c}^T x \mid Ax \geq b,\ x \geq 0\}.$$

The optimal solution is verified to be $x^* = [4, 2, 5, 3]^T$.

This demonstrates that the adjusted cost vector $\bar{c}$ ensures that $x^*$ is the optimal solution for the updated problem. The gradient projection method efficiently solves the inverse problem, even for complex scenarios.

## 7. Conclusion

This paper presented a comprehensive approach of the inverse linear optimization problem, with a focus on its geometric interpretation and solution methodologies. Starting from a theoretical foundation, we formulated the problem as a constrained quadratic programming problem and employed the gradient projection method to derive the adjusted cost vector. The proposed methodology ensures that a given feasible solution becomes optimal by minimally adjusting the cost parameters.

Through a practical example of resource allocation, we demonstrated the effectiveness and efficiency of the gradient projection method in solving the inverse optimization problem. The results confirmed that the adjusted cost vector aligns with the specified optimal solution while adhering to problem constraints. The validation step further emphasized the robustness of the proposed approach in maintaining the optimality of the desired solution.

Future work can extend this approach to large-scale problems and explore alternative optimization techniques, such as accelerated gradient methods or machine learning-based approaches, for solving more complex and nonlinear inverse optimization problems. Additionally, integrating stochastic and dynamic elements into the inverse problem framework could expand its applicability to real-world scenarios with uncertainty and time-dependent constraints.

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