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Beta Fractional Derivative Sturm-Liouville Problems

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Abstract. In this study, beta fractional derivative Sturm–Liouville problems are discussed. First, the existence and uniqueness problem for such equations is discussed. Then, self-adjointness is obtained with the help of boundary conditions. Eigenfunction expansion was obtained with the help of characteristic determinant and Green's function. Finally, an example is given showing the theoretical results obtained.

Keywords: Fractional differential equations, self-adjoint operators, Green's function, eigenfunction expansion.

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1. Introduction

Many important situations in science and engineering are expressed by partial differential equations. Different methods have been found to solve such equations. One of these methods is the separation into variables known as the Fourier method. When we try to solve partial differential equations using this method, we encounter the famous Sturm–Liouville equations. Sturm–Liouville problems are one of the differential equations extensively studied in the literature. Many studies have been done for various boundary conditions and situations. For example, [2, 5, 6, 9, 10, 12, 15]. For more detailed information on this, see Zettl's wonderful book [16].

Fractional derivatives and fractional differential equations have a long history. Many studies have been done on this subject, but the fact that the fractional derivative is not local and does not have some well-known properties of the ordinary derivative has led researchers to search for a new definition of fractional derivative. Recently, Khalil et al. defined a new derivative called the conformable fractional derivative, which has the important features of the ordinary derivative [11]. There are various instances where the conformable fractional derivative deviates from the classical derivative. Then Atangana et al. defined beta derivative [3, 4]. The derivative known as the beta derivative is used to adjust this concept in order to make it compatible with the classical derivative [8]. Although these definitions are not exactly fractional derivatives, they have attracted a lot of attention from researchers because they are an extension of ordinary derivatives. In [13], the authors studied the space-time generalized nonlinear Schrödinger equation involving the beta-derivative. Fadhal et al. [7] studied a nonlinear Sasa–Satsuma equation with a beta derivative. In [2], a conformable fractional Sturm–Liouville equation is studied. This study is a continuation of [2] and the fundamental properties of the beta-derived Sturm–Liouville problem are investigated. According to the authors' knowledge, there is no study on this subject in the literature. Therefore, it will contribute to researchers who want to study these issues. In [1], the authors studied the solution of ordinary fractional differential equations using Atangana's beta derivative. In [8], Iqbal et al. have obtained scores of exact wave soliton solutions of the space–time fractional foam drainage and the Boussinesq equations.

This research is organized as follows. In the first section, the solution of the problem and its uniqueness is obtained. With the help of boundary conditions, self-adjointness is examined in the second part. In the next section, the characteristic determinate and Green's function are discussed. In the fifth chapter, the eigenfunction expansion has been achieved. An example is given in the last section.

2. Preliminaries

Definition 2.1 ([3, 13])**.** Let $\beta \in (0,1]$. If $f : [0,\infty) \to \mathbb{R}$ is a function, then the β-derivative of f is defined by

$$T_\beta f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t + \frac{1}{\Gamma(\beta)})^{1-\beta}) - f(t)}{\varepsilon}, \tag{2.1}$$

and $(T_\beta f)(t) = \frac{d^\beta f(t)}{dt^\beta}$.

Definition 2.2 ([4])**.** If $f : [a,\infty) \to \mathbb{R}$ is a function, then the β-integral of f is given by the formula

$$_aI_\beta(f(t)) = \int_a^t f(x)\, d_\beta x = \int_a^t \left(x + \frac{1}{\Gamma(\beta)} \right)^{\beta-1} f(x) dx. \tag{2.2}$$

Theorem 2.3 ([13])**.** *Let f, g be β-differentiable functions for $t > 0$ and $(0 < \alpha \leq 1)$. Some properties are discussed as follows*

$$(i)\ T_\beta(\lambda f + \delta g) = \lambda T_\beta f + \delta T_\beta g,$$

for all $\lambda, \delta \in \mathbb{R}$,

$$(ii)\ T_\beta(fg) = f T_\beta(g) + g T_\beta(f),$$

$$(iii)\ T_\beta(\frac{f)}{g}) = \frac{g T_\beta(f) - f T_\beta(g)}{g^2},$$

$$(iv)\ T_\beta(f) = \left(t + \frac{1}{\Gamma(\beta)} \right)^{1-\beta} \frac{df}{dt}.$$

Theorem 2.4. *Let f, g be β-differentiable functions. Then, the following relation holds*

$$\int_a^b f(t) T_\beta\,(g)\,(t)\, d_\beta t = f\,(t)\, g\,(t)\, |_a^b - \int_a^b g\,(t)\, T_\beta\,(f)\,(t)\, d_\beta t.$$

Proof. By Theorem 3, we obtain

$$\int_a^b f(t)T_\beta\left(g\right)(t)\,d_\beta t+\int_a^b g\left(t\right)T_\beta\left(f\right)(t)\,d_\beta\left(t\right)$$

$$=\int_a^b f(t)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}g'\left(t\right)d_\beta\left(t\right)$$

$$+\int_a^b g\left(t\right)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f'\left(t\right)d_\beta\left(t\right)$$

$$=f(t)g(t)|_a^b-\int_a^b g\left(t\right)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f'\left(t\right)d_\beta\left(t\right)$$

$$+\int_a^b g\left(t\right)\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f'\left(t\right)d_\beta\left(t\right)$$

$$=f\left(b\right)g\left(b\right)-f\left(a\right)g\left(a\right).$$

□

Let

$$L_\beta^2(0,b):=\left\{\begin{array}{c} f:\left(\int_0^b\left|f\left(t\right)\right|^2 d_\beta t\right)^{1/2} \\ =\left(\int_0^b\left|f\left(t\right)\right|^2\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}dt\right)^{1/2}<\infty\end{array}\right\}.$$

Then $L_\beta^2(0,b)$ is a Hilbert space endowed with the inner product

$$\langle f,g\rangle:=\int_0^b f\left(t\right)\overline{g\left(t\right)}d_\beta t,\ \ f,g\in L_\beta^2(0,b).$$

The β-Wronskian of f and g is defined by

$$W_\beta(f,g)(t)=p(t)[f(t)T_\beta g(t)-f(t)T_\beta g(t)],\ \ t\in[0,b].$$

Theorem 2.5. *Let*

$$A\left\{t_i\right\}=\left\{x_i\right\},\ \ i\in\mathbb{N}:=\{1,2,3,...\},$$

where

$$x_i = \sum_{k=1}^{\infty} \eta_{ik} t_k, \;\; i,k \in \mathbb{N}. \tag{2.3}$$

If

$$\sum_{i,k=1}^{\infty} |\eta_{ik}|^2 < +\infty \tag{2.4}$$

then the operator A is compact in the sequence space l^2 ([14]).

3. An existence theorem

In this section, the solution of the problem and its uniqueness is obtained.

Let us obtain the existence and uniqueness of the solutions of equation defined as

$$-T_\beta \left(p\left(t\right) T_\beta y(t)\right) + q(t)y(t) = \lambda y(t), \;\; 0 \le t \le b < \infty, \tag{3.1}$$

where $\beta \in (0,1]$, $\lambda \in \mathbb{C}$, p,q are real-valued continuous and β- integrable functions on $[0,b]$ and $p\left(t\right) \neq 0$ for $t \in [0,b]$.

Theorem 3.1. *Equation (3.1) has a unique solution $y\left(x,\lambda\right)$ satisfying*

$$y(0,\lambda) = c_1, \;\; p\left(0\right) T_\beta y(0,\lambda) = c_2, \tag{3.2}$$

where $c_1, c_2 \in \mathbb{C}$.

Proof. Using Theorem 3, Eq. (3.1) can be written as the following form

$$\begin{bmatrix} y \\ T_\beta y \end{bmatrix}' = \begin{bmatrix} 0 & \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} p^{-1} \\ -\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} (q-\lambda) & 0 \end{bmatrix} \begin{bmatrix} y \\ T_\beta y \end{bmatrix}.$$

Then we get

$$Y' = MY,$$

where

$$Y = \begin{bmatrix} y \\ T_\beta y \end{bmatrix},$$

and

$$M = \begin{bmatrix} 0 & \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} p^{-1} \\ -\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} (q-\lambda) & 0 \end{bmatrix}.$$

By using Theorem 1 in §16.1 [14], we get the desired results. □

4. The self-adjoint problem

In this section, with the help of boundary conditions, self-adjointness is examined.

Consider the following equation

$$(\Lambda y)(t) := -T_\beta\left(p\left(t\right)T_\beta y(t)\right) + q(t)y(t) = \lambda y(t),\ 0 \leq t \leq b < \infty, \tag{4.1}$$

$$Z_1(y) := a_{11}y(0) + a_{12}p\left(0\right)T_\alpha y(0) = 0, \tag{4.2}$$

$$Z_2(y) := a_{21}y(b) + a_{22}p\left(b\right)T_\alpha y(b) = 0, \tag{4.3}$$

where $\beta \in (0,1]$, $\lambda \in \mathbb{C}$; p, q are real-valued continuous and β- integrable functions on $[0,b]$ and $p\left(t\right) \neq 0$ for $t \in [0,b]$; $a_{ij} \in \mathbb{R}$ $(i,j = 1,2)$, $a_{11}^2 + a_{12}^2 \neq 0$ and $a_{21}^2 + a_{22}^2 \neq 0$.

Consider the set

$$D_{\max} = \left\{ y \in L_\beta^2(0,b) : \begin{array}{c} y \text{ and } pT_\beta y \text{ are absolutely} \\ \text{continuous functions} \\ \text{on } [0,b] \text{ and } \Lambda y \in L_\beta^2(0,b) \end{array} \right\}.$$

Then we define the *maximal operator* $L_{\max}$ on $D_{\max}$ by the equality $L_{\max}y = \Lambda y$. The operator $L_{\min}$, that is the restriction of the operator $L_{\max}$ to $D_{\min}$ is called the *minimal operator,* where

$$D_{\min} = \left\{ y \in D_{\max} : y\left(0\right) = p\left(0\right)T_\beta y\left(0\right) = y\left(b\right) = p\left(b\right)T_\beta y\left(b\right) = 0 \right\}. \tag{4.4}$$

Lemma 4.1. *For arbitrary two functions $g, h \in D_{\max}$, the following relation holds*

$$\int_0^b \left[\Lambda g\left(t\right)\right]\overline{h(t)}d_\beta t - \int_0^b g\left(t\right)\left[\overline{\Lambda h(t)}\right]d_\beta t = \left[g,h\right](b) - \left[g,h\right](0), \tag{4.5}$$

where

$$\left[g,h\right](t) = p\left(t\right)\left[g(t)T_\beta\overline{h(t)} - (T_\beta g)(t)\overline{h(t)}\right],\ t \in [0,b].$$

Proof. This relation is obtained by using Theorem 4. □

Applying the preceding lemma to the problem (4.1)-(4.3) yields the following.

Theorem 4.2. *The problem defined as (4.1)-(4.3) is formally self-adjoint on* $L_\beta^2(0,b)$.

Corollary 4.3. (i) *All eigenvalues of (4.1)-(4.3) are real and simple.*
(ii) *The eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

5. The characteristic determinant and Green's function

In this section, the characteristic determinate and Green's function are discussed.

Let Θ_1 and Θ_2 be linearly independent solutions of

$$\Lambda y = \lambda y,\ 0 \leq t \leq b < \infty \tag{5.1}$$

satisfying

$$\Theta_1(0,\lambda) = 1,\ p\left(0\right)T_\beta\Theta_1(0,\lambda) = 0,$$

$$\Theta_2(0,\lambda) = 0,\ p\left(0\right)T_\beta\Theta_2(0,\lambda) = 1.$$

Then every solution of Eq. (5.1) can be written in the form

$$y(t,\lambda) = C_1\Theta_1(t,\lambda) + C_2\Theta_2(t,\lambda),$$

where C_1 and C_2 do not depend on t. If one can obtain a non-trivial solution of the following system

$$C_1 Z_1(\Theta_1) + C_2 Z_1(\Theta_2) = 0,$$

$$C_1 Z_2(\Theta_1) + C_2 Z_2(\Theta_2) = 0,$$

then the solution $y(t,\lambda)$ is called an *eigenfunction* of (5.1). Hence,

$$\lambda \in \mathbb{R} \text{ is an eigenvalue} \Leftrightarrow \Delta(\lambda) = \begin{vmatrix} Z_1(\Theta_1) & Z_1(\Theta_2) \\ Z_2(\Theta_1) & Z_2(\Theta_2) \end{vmatrix} = 0.$$

$\Delta(\lambda)$ is called the *characteristic determinant* associated with the problem (4.1)-(4.3). The eigenvalues of the problem (4.1)-(4.3) are at most countable with no finite limit points, because $\Delta(\lambda)$ is an entire function in λ.

Theorem 5.1. *All eigenvalues of the problem (4.1)-(4.3) are simple zeros of the function* $\Delta(\lambda)$.

Proof. Let

$$\Xi_1(t,\lambda) = Z_1(\Theta_2)\Theta_1(t,\lambda) - Z_1(\Theta_1)\Theta_2(t,\lambda),$$

$$\Xi_2(t,\lambda) = Z_2(\Theta_2)\Theta_1(t,\lambda) - Z_2(\Theta_1)\Theta_2(t,\lambda). \tag{5.2}$$

It is clear that $\Xi_1(t,\lambda)$ and $\Xi_2(t,\lambda)$ are solutions of (4.1) such that

$$\Xi_1(0,\lambda) = a_{12},\ p(0)\, T_\beta \Xi_1(0,\lambda) = -a_{11},$$
$$\Xi_2(b,\lambda) = a_{22},\ p(b)\, T_\beta \Xi_2(b,\lambda) = -a_{21}. \tag{5.3}$$

Furthermore, one can see that

$$W_\beta(\Xi_1,\Xi_2) = \Delta(\lambda) W_\beta(\Theta_1,\Theta_2) = \Delta(\lambda). \tag{5.4}$$

Let λ_0 be an eigenvalue of the problem (4.1)-(4.3). It follows from Corollary 9 that $\lambda_0 \in \mathbb{R}$. So $\Xi_i(t,\lambda_0)$ $(i = 1,2)$ can be taken to be real valued. From (5.4), we see that $\Xi_1(t,\lambda_0)$ and $\Xi_2(t,\lambda_0)$ are linearly dependent eigenfunctions, i.e., there exists a non-zero constant η_0 such that

$$\Xi_1(t,\lambda_0) = \eta_0 \Xi_2(t,\lambda_0).$$

By (5.2) and (5.3), we conclude that

$$\Xi_1(b,\lambda_0) = \eta_0 a_{22} = \eta_0 \Xi_2(b,\lambda),$$
$$T_\beta \Xi_1(b,\lambda_0) = -\eta_0 a_{21} = \eta_0 (T_\beta) \Xi_2(0,\lambda). \tag{5.5}$$

Writing $g(t) = \Xi_1(t, \lambda)$ and $h(t) = \Xi_1(t, \lambda_0)$ in (4.5) yields

$$(\lambda - \lambda_0) \int_0^b \Xi_1(t, \lambda)\Xi_1(t, \lambda_0) d_\beta(t)$$

$$= \Xi_1(b, \lambda) p(b) T_\beta \Xi_1(b, \lambda_0) - p(b) T_\beta \Xi_1(b, \lambda_0)\Xi_1(b, \lambda)$$

$$= \eta_0 (\Xi_1(b, \lambda) p(b) T_\beta \Xi_2(b, \lambda) - \Xi_2(b, \lambda) p(b) T_\beta \Xi_1(b, \lambda))$$

$$= \eta_0 W_\beta(\Xi_1(t, \lambda), \Xi_2(t, \lambda)) = \eta_0 \Delta(\lambda).$$

Hence

$$\Delta'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{\Delta(\lambda)}{\lambda - \lambda_0} = \frac{1}{\eta_0} \int_0^b |\Xi_1(t, \lambda_0)|^2 d_\beta(t) \neq 0.$$

due to $\Delta(\lambda)$ is an entire function in λ. □

Now, we will consider the following problem

$$-T_\beta (p(t) T_\beta y(t)) + (q(t) - \lambda) y(t) = h(t), \ 0 \leq t \leq b < \infty, \tag{5.6}$$

$$Z_1(y) = a_{11} y(0) + a_{12} p(0) T_\beta y(0) = 0, \tag{5.7}$$

$$Z_2(y) = a_{21} y(b) + a_{22} p(b) T_\beta y(b) = 0, \tag{5.8}$$

where $\beta \in (0, 1]$, $\lambda \in \mathbb{C}$; p, q are real-valued continuous and β- integrable functions on $[0, b]$ and $p(t) \neq 0$ for $t \in [0, b]$; $a_{ij} \in \mathbb{R}$ $(i, j = 1, 2)$, $a_{11}^2 + a_{12}^2 \neq 0$, $a_{21}^2 + a_{22}^2 \neq 0$; $h(.) \in L_\beta^2(0, b)$ is given.

Theorem 5.2. *Assume that λ is not an eigenvalue of (5.6)-(5.8). Let Θ satisfy the problem (5.6)-(5.8). Then we have*

$$\Theta(t, \lambda) = \int_0^b G(t, x, \lambda) h(t) d_\beta t, \ t \in (0, b), \tag{5.9}$$

where

$$G(t, x, \lambda) = \frac{-1}{\Delta(\lambda)} \begin{cases} \Xi_2(t, \lambda)\Xi_1(x, \lambda), \ 0 \leq x \leq t \\ \Xi_1(t, \lambda)\Xi_2(x, \lambda), \ t \leq x \leq b. \end{cases} \tag{5.10}$$

Conversely, the function Θ defined by (5.9) satisfies (5.6)-(5.8).

Proof. We will use a variation of constant method. Then a solution of Eq. (5.6) is given by the formula

$$\Theta(t, \lambda) = c_1(t)\Xi_1(t, \lambda) + c_2(t)\Xi_2(t, \lambda), \tag{5.11}$$

where c_1, c_2 are solutions of the following equations

$$T_\beta c_1(t) = \frac{1}{\Delta(\lambda)} \Xi_2(t, \lambda) h(t),$$

$$T_\beta c_2(t) = -\frac{1}{\Delta(\lambda)}\Xi_1(t,\lambda)h(t).$$

Thus, we obtain

$$c_1(t) = c_1(0) + \frac{1}{\Delta(\lambda)}\int_0^t \Xi_2(x,\lambda)h(x)d_\beta x,\ t\in[0,b],$$

$$c_2(t) = c_2(b) + \frac{1}{\Delta(\lambda)}\int_t^b \Xi_1(x,\lambda)h(x)d_\beta x,\ t\in[0,b].$$

This yields

$$\Theta(t,\lambda) = c_1\Xi_1(t,\lambda) + c_2\Xi_2(t,\lambda)$$

$$+\,\Xi_1(t,\lambda)\frac{1}{\Delta(\lambda)}\int_0^t \Xi_2(x,\lambda)h(x)d_\beta x$$

$$+\,\Xi_2(t,\lambda)\frac{1}{\Delta(\lambda)}\int_t^b \Xi_1(x,\lambda)h(x)d_\beta x,\ t\in[0,b] \tag{5.12}$$

where c_1, c_2 are arbitrary constants. From (5.12), we obtain

$$\Theta(0,\lambda) = c_1\Xi_1(0,\lambda) + (c_2 + \frac{1}{\Delta(\lambda)}\int_0^b \Xi_1(x,\lambda)h(x)d_\beta x)\Xi_2(0,\lambda),$$

$$T_\beta\Theta(0,\lambda) = c_1 T_\alpha\Xi_1(0,\lambda) + (c_2 + \frac{1}{\Delta(\lambda)}\int_0^b \Xi_1(x,\lambda)h(x)d_\beta x)T_\beta\Xi_2(0,\lambda).$$

By (5.7), we get

$$c_2 = -\frac{1}{\Delta(\lambda)}\int_0^b \Xi_1(x,\lambda)h(x)d_\beta x. \tag{5.13}$$

It follows from (5.12) that

$$\Theta(t,\lambda) = c_1\Xi_1(t,\lambda) + \frac{1}{\Delta(\lambda)}\int_0^t \left(\Xi_1(t,\lambda)\Xi_2(x,\lambda) - \Xi_2(t,\lambda)\Xi_1(x,\lambda)\right)h(x)d_\beta x. \tag{5.14}$$

From (5.14), we have

$$\Theta(b,\lambda) = c_1\Theta_1(b,\lambda) + \frac{1}{\Delta(\lambda)}\int_0^b \left(\Xi_1(b,\lambda)\Xi_2(x,\lambda) - \Xi_2(b,\lambda)\Xi_1(x,\lambda)\right)h(x)d_\beta x$$

and

$$T_\beta\Theta(b,\lambda) = (T_\beta\Theta_1)(b,\lambda)\left(c_1 + \frac{1}{\Delta(\lambda)}\int_0^b \Xi_2(x,\lambda)h(x)d_\beta x\right)$$
$$-\frac{1}{\Delta(\lambda)}T_\beta\Theta_2(b,\lambda)\int_0^b \Xi_1(x,\lambda)h(x)d_\beta x.$$

By (5.8), we conclude that

$$c_1 = -\frac{1}{\Delta(\lambda)}\int_0^b \Xi_2(x,\lambda)h(x)d_\alpha x.$$

So, we get

$$\Theta(t,\lambda) = -\frac{1}{\Delta(\lambda)}\Xi_2(t,\lambda)\int_0^t \Xi_1(x,\lambda)h(x)d_\beta x$$

$$-\frac{1}{\Delta(\lambda)}\Xi_1(t,\lambda)\int_t^b \Xi_2(x,\lambda)h(x)d_\beta x,$$

where $t \in [0,b]$. Conversely, it is easy to check that Θ defined by (5.9) satisfies (5.6)-(5.8). □

6. Eigenfunction Expansion

In this chapter, the eigenfunction expansion is obtained.

Let

$$D = \{y \in D_{\max} : Z_1(y) = 0,\ Z_2(y) = 0\}.$$

Then we shall define the operator $L : D \to L^2_\beta(0,b)$ by the formula $Ly = \Lambda y$. Without loss of generality we can assume that $\lambda = 0$ is not an eigenvalue. Hence $\ker L = \{0\}$. It is clear that

$$(Ly)(t) = h(t),\ h(.) \in L^2_\beta(0,b)$$

implies

$$y(t) = \int_0^b G(t,x)h(x)d_\beta x,$$

where

$$G(t,x) = -\frac{1}{W_\beta(\Xi_1,\Xi_2)}\begin{cases} \Xi_2(t)\Xi_1(x), & 0 \le x \le t \\ \Xi_1(t)\Xi_2(x), & t < x \le b. \end{cases} \tag{6.1}$$

Theorem 6.1. *$G(t,x)$ defined as (6.1) is a β-Hilbert–Schmidt kernel, i.e.,*

$$\int_0^b \int_0^b |G(t,x)|^2 d_\beta t d_\beta x < +\infty.$$

Proof. By (6.1), we deduce that

$$\int_0^b d_\beta t \int_0^t |G(t,x)|^2 d_\beta x < +\infty,\ \int_0^b d_\beta t \int_t^b |G(t,x)|^2 d_\beta x < +\infty,$$

due to $\Xi(t)\Xi(x) \in L^2_\beta(0,b) \times L^2_\beta(0,b)$. Therefore, we get

$$\int_0^b \int_0^b |G(t,x)|^2 d_\beta t d_\beta x < +\infty. \tag{6.2}$$

□

Theorem 6.2. *Let T be the integral operator*

$$\mathbf{T}\colon L^2_\beta(0,b) \to L^2_\beta(0,b),$$

$$(\mathbf{T}h)(t) = \int_0^b G(t,x)h(x)d_\beta x.$$

Then T is a self-adjoint and compact operator.

Proof. Let $\{\Phi_j\}_{j\in\mathbb{N}}$ be a complete, orthonormal basis of $L^2_\beta(0,b)$. If we set

$$t_j = (h,\Phi_j) = \int_0^b h(x)\overline{\Phi_j(x)}d_\beta x,$$

$$x_j = (g,\Phi_j) = \int_0^b g(x)\overline{\Phi_j(x)}d_\beta x,$$

$$\eta_{jk} = \int_0^b \int_0^b G(t,x)\Phi_j(x)\overline{\Phi_k(t)}d_\beta x d_\beta t,\ \ j,k \in \mathbb{N}.$$

then $L^2_\beta(0,b)$ is mapped isometrically on to l^2. By this mapping, T transforms into the operator A defined by (2.3) in l^2 and (6.2) is translated into (2.4). It follows from Theorems 5 and 12 that A and T is compact.

Let $h, g \in L^2_\beta(0,b)$. Then we have

$$(\mathbf{T}h,g) = \int_0^b (\mathbf{T}h)(t)\overline{g(t)}d_\beta t$$

$$= \int_0^b \int_0^b G(t,x)h(x)d_\beta x\overline{g(t)}d_\beta t$$

$$= \int_0^b h(x)\left(\int_0^b G(x,t)\overline{g(t)}d_\beta t\right)d_\beta x = (h,\mathbf{T}g).$$

due to $G(t,x)$ is symmetric function. □

Theorem 6.3. *The operator L has an infinite countable set $\{\lambda_n\}_{n\in\mathbb{N}}$ of real eigenvalues which can be ordered as*

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_n| < ... \to \infty \text{ as } n \to \infty.$$

The set of all normalized eigenfunctions of L forms an orthonormal basis for the space $L_\beta^2(0,b)$ *and for* $z \in \mathcal{H}$, $\mathbf{T}z = h$, $Lh = z$, $L\chi_n = \lambda_n \chi_n$ $(n \in \mathbb{N})$ *the eigenfunction expansion formula*

$$Lh = \sum_{n=1}^{\infty} \lambda_n \langle h, \chi_n \rangle \chi_n$$

is valid.

Proof. From the Hilbert–Schmidt theorem and the above theorem, we conclude that T has an infinite sequence of non-zero real eigenvalues $\{\xi_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \xi_n = 0.$$

Then,

$$|\lambda_n| = \frac{1}{|\xi_n|} \to \infty, \ n \to \infty.$$

Furthermore, let $\{\chi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\xi_n\}_{n=1}^{\infty}$. Thus we have $z \in L_\beta^2(0,b)$, $\mathbf{T}z = h$, $Lh = z$, $L\chi_n = \lambda_n \chi_n$ $(n \in \mathbb{N})$ and

$$\begin{aligned} z = Lh &= \sum_{n=1}^{\infty} \langle z, \chi_n \rangle \chi_n = \sum_{n=1}^{\infty} \langle Lh, \chi_n \rangle \chi_n \\ &= \sum_{n=1}^{\infty} \langle h, L\chi_n \rangle \chi_n = \sum_{n=1}^{\infty} \lambda_n \langle h, \chi_n \rangle \chi_n. \end{aligned}$$

□

7. Example

In this section, an example will be given for the theoretical results obtained above.

Example 7.1. Consider the following problem

$$-T_\beta^2 y(t) = \lambda y(t), \ 0 \le t \le 1, \tag{7.1}$$

$$Z_1(y) = y(0) = 0,$$
$$Z_2(y) = y(1) = 0.$$

It is clear that

$$\Theta_1(t,\lambda) = \cos\left(\int_0^t \sqrt{\lambda} d_\beta x\right),$$

and

$$\Theta_2(t,\lambda) = \frac{\sin\left(\int_0^t \sqrt{\lambda} d_\beta x\right)}{\sqrt{\lambda}}$$

are the solutions of (7.1). Since

$$\Delta(\lambda) = \begin{vmatrix} Z_1(\Theta_1) & Z_2(\Theta_1) \\ Z_1(\Theta_2) & Z_2(\Theta_2) \end{vmatrix} = \frac{1}{\sqrt{\lambda}} \sin\left(\int_0^1 \sqrt{\lambda} d_\beta x\right).$$

the eigenvalues of (7.1) are the zeros of $\sin(\int_0^1 \sqrt{\lambda} d_\beta x)$. Further

$$\left\{ \frac{1}{\sqrt{\lambda_n}} \sin(\int_0^1 \sqrt{\lambda_n} d_\beta x) \right\}_{n=1}^{\infty}$$

is an orthogonal basis of $L^2_\beta(0,1)$. Thus we get

$$G(t,x,\lambda) = -\frac{\sqrt{\lambda}}{\sin(\int_0^1 \sqrt{\lambda} d_\beta x)} \begin{cases} \Xi_2(t,\lambda)\Xi_1(x,\lambda), & 0 \le x \le t \\ \Xi_1(t,\lambda)\Xi_2(x,\lambda), & t < x \le 1, \end{cases}$$

where

$$\Xi_1(t,\lambda) = \frac{\sin(\int_0^1 \sqrt{\lambda} d_\beta x)}{\sqrt{\lambda}}$$

and

$$\Xi_2(t,\lambda) = -\frac{\sin\left(\int_0^1 \sqrt{\lambda} d_\beta x\right)}{\sqrt{\lambda}} \cos\left(\int_0^t \sqrt{\lambda} d_\beta x\right)$$

$$+ \cos\left(\int_0^1 \sqrt{\lambda} d_\beta x\right) \frac{\sin\left(\int_0^t \sqrt{\lambda} d_\beta x\right)}{\sqrt{\lambda}}.$$

8. Conclusion

Sturm–Liouville problems with beta fractional derivatives are examined in this work. Initially, the problem of existence and uniqueness for these kinds of equations is examined. We then use boundary conditions to attain self-adjointness. Using Green's function and the characteristic determinant, eigenfunction expansion was obtained. An example demonstrating the theoretical outcomes is provided at the end.

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