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Solving nonlinear partial differential equations of fractional order using an analytical method

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Abstract. In this article, we use an analytical method based on the solutions of the Riccati equation to solve nonlinear partial differential equations of fractional order. In this method, we first convert the fractional partial differential equations into an ordinary differential equation using Riemann-Liouville derivatives and a suitable transformation, then we consider the solutions of these equations as a finite series and using the solution of the equation Riccati's differential, we get the desired solutions. In this method, different types of solutions such as trigonometric, hyperbolic and exponential solutions are obtained. The results show that the method used in this article is very useful and effective for obtaining the solutions of fractional partial differential equations.

Keywords: Riemann-Liouville fractional derivative, Mittag leffler -Leffler function, Klein-Gordon equation, biological population model equation.

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1. Introduction

Mathematical models of scientific events are usually expressed by nonlinear differential equations. These equations play an important role in physics, engineering and applied mathematics to describe various types of physical mechanisms of natural phenomena in the fields of applied sciences, biochemistry, dynamic systems, etc. In recent years, a lot of research has been done to obtain analytical solutions of nonlinear differential equations, and effective and efficient methods for solving these types of equations have been presented so that we can use their solutions to investigate the behavior and properties of real phenomena.

In many of these methods, partial differential equations are converted to ordinary differential equations using a suitable transformation. Some of these methods are:

Sine-cosine method [14], Miura transform method [15], Darbox transform method [12], Hirota transform method [2], homogeneous balance method [6], homotopy perturbation method [4], $\frac{G'}{G^2}$-expansion method [8], exponential function method [10], Kudryashev method [11], Fan subequation method [17] and so on. In the last few decades, fractional differential equations have been the research of many scientists due to their many applications in science and engineering. These equations are based on integrals and derivatives of fractional order. Fractional differential equations have many applications in physics, chemistry, economics, medical engineering, biological sciences, image processing, etc. There are several definitions regarding the concepts of integral and derivative of non-integer order, the most important of which are Riemann-Liouville derivative, Caputo derivative, Hilfer derivative, and Grand-Letnikoff derivative. In this article, we examine two fractional differential equations as follows and obtain their analytical solutions.

Klein-Gordon fractional differential equation:

$$\frac{\partial^{2\ \alpha} u}{\partial t^{2\ \alpha}} - \frac{\partial^{2\ \alpha} u}{\partial x^{2\ \alpha}} + \gamma u + \beta u^3 = 0 \qquad 0 < \alpha < 1, \tag{1.1}$$

where α and β are constant numbers.

Fractional differential equation of biological population model:

$$\frac{\partial^{2\ \alpha} u}{\partial t^{2\ \alpha}} - \left(\mathrm{u}^2\right)_{xx} - \left(\mathrm{u}^2\right)_{yy} - h\left(u^2 - r\right) = 0 \qquad 0 < \alpha < 1, \tag{1.2}$$

which $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the Riemann-Liouville derivative as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\eta)^{-\alpha} \left[u\left(\eta\right) - u\left(0\right)\right] d\eta, \tag{1.3}$$

and

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left(u^{\alpha-n}\left(t\right)\right)^n \qquad n\le\alpha<n+1,\ n\ge 1. \tag{1.4}$$

Some of the useful and interesting properties of this derivative are as follows:

$$\begin{cases} \frac{\partial^\alpha t^\theta}{\partial t^\alpha}=\frac{\Gamma(\theta+1)}{\Gamma(\theta-\alpha+1)}t^{\theta-\alpha}, \\ \frac{\partial^\alpha}{\partial t^\alpha}\left[f\left(t\right)g\left(t\right)\right]=g\left(t\right)\frac{\partial^\alpha f(t)}{\partial t^\alpha}+f(t)\frac{\partial^\alpha g(t)}{\partial t^\alpha} \\ \frac{\partial^\alpha}{\partial t^\alpha}\left[f(g\left(t\right))\right]=f'_g\left[g\left(t\right)\right]\frac{\partial^\alpha}{\partial t^\alpha}g\left(t\right), \end{cases} \tag{1.5}$$

where $\Gamma(.)$ is Euler's gamma function, which is defined as follows:

$$\Gamma\left(\mathrm{z}\right)=\int_0^\infty \mathrm{e}^{-\mathrm{t}}\mathrm{t}^{\mathrm{z}-1}\mathrm{dt}\ . \tag{1.6}$$

In the Klein-Gordon equation, the fraction is one of the most important mathematical models in quantum. This equation appears in relativity physics and describes the phenomenon of wave propagation. Also, this equation has many uses in the fields of plasma physics and optics. The fractional Klein-Gordon equation has been investigated by many researchers. Among others, we can refer to the fractional equation method [5], Kordyashev's improved method [3], and the variation repetition method [16].

The equation of the biological population model has also been studied by many scientists with methods such as the fractional sub-equation method [9], the method of repeating changes [17], and the homotopy analysis method [13].

This article is written as follows:

1. In section 2, we describe the analytical method presented for solving non-linear partial differential equations of fractional order.

2. In section 3, we solve the given fractional partial differential equations with the described method.

3. In section 4, the conclusion of the article is given.

2. Description of the analytical method for solving nonlinear partial differential equations of fractional order

In order to obtain the solutions of nonlinear partial differential equations of fractional order, we act as follows:

Step 1- Suppose the fractional differential equation is as follows:

$$P\left(u,\frac{\partial^\alpha u}{\partial x^\alpha},\frac{\partial^\alpha u}{\partial t^\alpha},\frac{\partial^{2\ \alpha} u}{\partial x^{2\ \alpha}},\frac{\partial^{2\ \alpha} u}{\partial t^{2\ \alpha}},\cdots\right)=0 \tag{2.1}$$

With a suitable transformation, this equation becomes an ordinary differential equation in terms of $u = u(\xi)$ as follows:

$$Q\left(u, u', u'', u''', \cdots\right) = 0. \tag{2.2}$$

Step 2- We consider the solutions of equation (2.2) as follows:

$$u(\xi) = \sum_{i=0}^{n} a_i (\varphi(\xi))^i \qquad a_n \neq 0 \tag{2.3}$$

where a_i's are the constants that we have to calculate and $\varphi(\xi)$ applies in the Riccati equation as follows:

$$r_1 \varphi'(\xi) - r_2 \varphi^2(\xi) - r_3 \varphi(\xi) - r_4 = 0. \tag{2.4}$$

The solutions to equation (2.4) are as follows:

Case 1. For $r = r_3^2 - 4r_2r_4 > 0$ and $r_3 \neq 0$ we have:

$$\varphi(\xi) = -\frac{r_3}{2r_2} - \frac{\sqrt{r}}{2r_2} \frac{c_1 \sinh\left(\frac{\sqrt{r}}{2r_1}\xi\right) + c_2 \cosh\left(\frac{\sqrt{r}}{2r_1}\xi\right)}{c_1 \cosh\left(\frac{\sqrt{r}}{2r_1}\xi\right) + c_2 \sinh\left(\frac{\sqrt{r}}{2r_1}\xi\right)}.$$

Case 2. For $r = r_3^2 - 4r_2r_4 < 0$ and $r_3 \neq 0$ we have:

$$\varphi(\xi) = -\frac{r_3}{2r_2} - \frac{\sqrt{-r}}{2r_2} \frac{c_2 \cos\left(\frac{\sqrt{-r}}{2r_1}\xi\right) - c_1 \sin\left(\frac{\sqrt{-r}}{2r_1}\xi\right)}{c_1 \cos\left(\frac{\sqrt{-r}}{2r_1}\xi\right) + c_2 \sin\left(\frac{\sqrt{-r}}{2r_1}\xi\right)}.$$

Case 3. For $r = r_3^2 - 4r_2r_4 = 0$ and $r_3 \neq 0$ we have:

$$\varphi(\xi) = -\frac{r_3}{2r_2} + \frac{c_2}{c_1 + c_2\xi}.$$

Case 4. For $t = r_2r_4 < 0$ and $r_3 = 0$ we have:

$$\varphi(\xi) = -\frac{\sqrt{-t}}{r_2} \frac{c_1 \sinh\left(\frac{\sqrt{-t}}{r_1}\xi\right) + c_2 \cosh\left(\frac{\sqrt{-t}}{r_1}\xi\right)}{c_1 \cosh\left(\frac{\sqrt{-t}}{r_1}\xi\right) + c_2 \sinh\left(\frac{\sqrt{-t}}{r_1}\xi\right)}.$$

Case 5. For $t = r_2r_4 > 0$ and $r_3 = 0$ we have:

$$\varphi(\xi) = -\frac{\sqrt{t}}{r_2} \frac{c_2 \cos\left(\frac{\sqrt{t}}{r_1}\xi\right) - c_1 \sin\left(\frac{\sqrt{t}}{r_1}\xi\right)}{c_1 \cos\left(\frac{\sqrt{t}}{r_1}\xi\right) + c_2 \sin\left(\frac{\sqrt{t}}{r_1}\xi\right)}.$$

Case 6. For $r_4 = 0$ we have:

$$\varphi(\xi) = \frac{c_1 r_3^2 \exp\left(-\frac{r_3}{r_1}\xi\right)}{-r_1 r_2 + c_1 r_1 r_3 \exp\left(-\frac{r_3}{r_1}\xi\right)},$$

Case 7. For $r_3 \neq 0$ and $r_2 = 0$ we have:

$$\varphi(\xi) = -\frac{r_4}{r_3} + c_1 \exp\left(-\frac{r_3}{r_1}\xi\right).$$

Case 8. For $r_2 = r_3 = 0$ we have:

$$\varphi(\xi) = c_1 + \frac{r_4}{r_1}\xi.$$

Case 9. For $r_1 = r_2$ and $r_3 = r_4 = 0$ we have:

$$\varphi(\xi) = -\frac{1}{c_1 + \xi}.$$

The number n in equation (2.3) is obtained by balancing between the highest order of the derivative and the highest nonlinear term in equation (2.2).

Step 3. By inserting relation (2.3) in equation (2.2) with the value of n obtained in step 2, equation (2.2) is obtained as a polynomial in terms of powers of $\varphi(\xi)$. By setting the power coefficients of $\varphi(\xi)$ equal to zero, a system is obtained in terms of the constants $a_i (1 \leq i \leq n)$ and $r_i (1 \leq i \leq 4)$.

Step 4. By solving the algebraic system obtained in step 3 with the help of Mathematica software, we place the obtained answers in equation (2.3).

3. Applications of the presented method for solving nonlinear partial differential equations of fractional order

3.1. **Klein-Gardan partial differential equation of fractional order.** In this section, we consider the Klein-Gardan differential equation as follows:

$$\frac{\partial^{2\ \alpha} u}{\partial t^{2\ \alpha}} - \frac{\partial^{2\ \alpha} u}{\partial x^{2\ \alpha}} + \gamma u + \beta u^3 = 0 \qquad 0 < \alpha < 1, \tag{3.1}$$

where γ and β are fixed numbers. Using conversion

$$u(x,t) = u(\xi); \qquad \xi = \frac{k x^{\alpha}}{\Gamma(\alpha+1)} - \frac{c\ t^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.2}$$

where c is the wave speed and based on relations (1.3) to (1.6) of equation (3.1) it becomes the following ordinary differential equation:

$$\left(\mathrm{c}^2 - k^2\right) u''(\xi) + \gamma u + \beta u^3 \ = 0. \tag{3.3}$$

By balancing between the sentences u^3 and u'', we can write:

$$3\ n = n + 2 \ \Rightarrow n = 1.$$

Therefore from (2.3) we have:

$$u(\xi) = a_0 + a_1 \varphi(\xi). \tag{3.4}$$

By placing relation (3.4) in (3.3) based on step 3, we have the following equations:

$$\begin{cases} \gamma\ a_0 + \beta\ a_0^3 + \frac{\left(c^2-k^2\right) r_3\ \ r_4\ a_1}{r_1^2} = 0, \\ \gamma\ a_1 + 3\ \beta\ a_0^2\ a_1 + \frac{\left(c^2-k^2\right) r_3^2\ a_1}{r_1^2} + \frac{2\ \left(c^2-k^2\right)\ r_2\ \ r_4\ \ a_1}{r_1^2} = 0, \\ 3\ \beta\ a_0\ a_1^2 + \frac{3\ \left(c^2-k^2\right) r_2\ \ r_3\ a_1}{r_1^2} = 0, \\ \beta\ \ a_1^3 + \frac{2\ \left(c^2-k^2\right)\ r_2^2\ a_1}{r_1^2} = 0, \end{cases}$$

By solving this system, we get the following solutions for equation (3.3).

a) The first set of answers is:

$$r_1 \neq 0,\ \ r_2 \neq 0,\ \ r_3 = 0,\ \ r_4 \neq 0,\ \ \ c = \sqrt{k^2 - \frac{r_1^2 \gamma}{2 r_2 r_4}},\ \ \ a_0 = 0,\ \ \ a_1 = \sqrt{\frac{r_2 \gamma}{r_4 \beta}}, \tag{3.5}$$

According to the conditions obtained for $r_i(1i4)$, according to the fourth and fifth states of the solutions of equation (2.4), the solutions of equation (3.1) based on equation (3.4) are obtained as follows:

$$u_1(x,t) = \sqrt{-\frac{\gamma}{\beta}}\ \frac{c_1 \sinh\left[\frac{\sqrt{-r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ + c_2 \cosh\left[\frac{\sqrt{-r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}{c_1 \cosh\left[\frac{\sqrt{-r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ + c_2\ \sinh\left[\frac{\sqrt{-r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}. \tag{3.6}$$

By arbitrary selection of coefficients, some special solutions can be obtained as follows:

$$u_2(x,t) = \sqrt{-\frac{\gamma}{\beta}}\ \tanh\left[\frac{\sqrt{-r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ ;\ c_1 \neq 0.\ c_2 = 0, \tag{3.7}$$

$$u_3(x,t) = \sqrt{-\frac{\gamma}{\beta}}\ \coth\left[\frac{\sqrt{-r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ ;\ c_1 = 0.\ c_2 \neq 0, \tag{3.8}$$

$$u_4(x,t) = -\sqrt{\frac{\gamma}{\beta}}\ \frac{-c_1 \sin\left[\frac{\sqrt{r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ + c_2 \cos\left[\frac{\sqrt{r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}{c_1 \cos\left[\frac{\sqrt{r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ + c_2 \sin\left[\frac{\sqrt{r_2 r_4}}{r_1}\left(\frac{(kx^\alpha - ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}. \tag{3.9}$$

In this case, the following solutions can be obtained by choosing the coefficients as desired.

$$u_5(x,t)=\sqrt{\frac{\gamma}{\beta}}\ \tan\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right];\ c_2=0,\ c_1\neq 0, \quad (3.10)$$

$$u_6(x,t)=-\sqrt{\frac{\gamma}{\beta}}\ \cot\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right];\ c_1=0,\ c_2\neq 0. \quad (3.11)$$

b) The second set of answers is:

$$r_1\neq 0,\ r_2\neq 0,\ r_3\neq 0, r_4=r_4,\ \ r_3^2-4_{r_2r_4}\neq 0,$$

$$\begin{aligned} c&=\sqrt{k^2+\frac{2r_1^2\gamma}{r_3^2-4r_2r_4}},\ a_0=\frac{\sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1\sqrt{\beta}},\\ a_1&=\frac{2r_2\ \sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}, \end{aligned} \quad (3.12)$$

According to the first, second and sixth cases, from the solutions of equation (2.4), the solutions of equation (3.1) are obtained as follows:

$$\begin{aligned} u_7(x,t)&=\frac{\sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1\sqrt{\beta}}+\frac{2r_2\sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}\\ &\times\left[-\frac{r_3}{2r_2}-\frac{\sqrt{r}}{2r_2}\frac{c_1\sinh\left[\frac{\sqrt{r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ +c_2\cosh\left[\frac{\sqrt{r}}{2r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}{c_1\cosh\left[\frac{\sqrt{r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ +c_2\sinh\left[\frac{\sqrt{r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}\right] \end{aligned} \quad (3.13)$$

Our special solutions in this case are:

$$\begin{aligned} u_8(x,t)&=\frac{\sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1\sqrt{\beta}}+\frac{2r_2\sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}\\ &\times\left[-\frac{r_3}{2r_2}-\frac{\sqrt{r}}{2r_2}\tanh\left[\frac{\sqrt{r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\right];\ c_2=0,\ c_1\neq 0, \end{aligned} \quad (3.14)$$

$$\begin{aligned} u_9(x,t)&=\frac{\sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1\sqrt{\beta}}+\frac{2r_2\sqrt{2r_2r_4(k^2-c^2)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}\\ &\times\left[-\frac{r_3}{2r_2}-\frac{\sqrt{r}}{2r_2}\ \coth\left[\frac{\sqrt{r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\right];\ c_1=0,\ c_2\neq 0, \end{aligned} \quad (3.15)$$

$$u_{10}(x,t)=\frac{\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1\sqrt{\beta}}+\frac{2r_2\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}$$
$$\times\left[-\frac{r_3}{2r_2}-\frac{\sqrt{-r}}{2r_2}\frac{-c_1\sin\left[\frac{\sqrt{-r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ +c_2\cos\left[\frac{\sqrt{-r}}{2r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}{c_1\cos\left[\frac{\sqrt{-r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\ +c_2\sin\left[\frac{\sqrt{-r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}\right], \tag{3.16}$$

$$u_{11}(x,t)=\frac{\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1\sqrt{\beta}}+\frac{2r_2\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}$$
$$\times\left[-\frac{r_3}{2r_2}+\frac{\sqrt{-r}}{2r_2}\tan\left[\frac{\sqrt{-r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\right];\ c_2=0,\ c_1\neq 0, \tag{3.17}$$

$$u_{12}(x,t)=\frac{\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1\sqrt{\beta}}+\frac{2r_2\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}$$
$$\times\left[-\frac{r_3}{2r_2}-\frac{\sqrt{-r}}{2r_2}\cot\left[\frac{\sqrt{-r}}{2\ r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]\right];\ c_1=0,c_2\neq 0, \tag{3.18}$$

$$u_{13}(x,t)=\frac{\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1\sqrt{\beta}}+\frac{2r_2\sqrt{2r_2r_4\left(k^2-c^2\right)-r_1^2\gamma}}{r_1r_3\sqrt{\beta}}$$
$$\times\left[\frac{c_1r_3^2\exp\left[\frac{-r_3}{r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}{-r_1r_2+c_1r_1r_3\exp\left[\frac{-r_3}{r_1}\left(\frac{(kx^\alpha-ct^\alpha)}{\Gamma(\alpha+1)}\right)\right]}\right]. \tag{3.19}$$

3.2. **Fractional order biological population model equation.** In this section, we consider the equation of the biological population model as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left(u^2\right)_{xx}+\left(u^2\right)_{yy}+h\left(u^2-r\right) \tag{3.20}$$

where h and r are real numbers. By using the following transformation

$$u(x,y,t)=u(\xi);\qquad \xi=\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)} \tag{3.21}$$

on which v,c are constant and $i^2=-1$, equation (3.20) is transformed as follows:

$$cu'+hu^2-hr=0. \tag{3.22}$$

By balancing and using relation (2.3) we can write:

$$u(\xi)=a_0+a_1\varphi(\xi), \tag{3.23}$$

Now, by inserting (3.23) into equation (3.22) based on step 3 of the system, we have the following equations:

$$-h\ r+h\ a_0^2+\frac{c\ r_4\ a_1}{r_1}=0,$$
$$\frac{c\ r_3\ a_1}{r_1}+2\ h\ a_0\ a_1=0,$$
$$\frac{c\ r_2\ a_1}{r_1}+h\ a_1^2=0.$$

By solving this system, we have the following solutions for equation (3.22): **a)** The first set of solution is:

$$r_1=r_2,\quad r_2\neq 0,\quad r_3=0,\quad r_4\neq 0,\ c=\frac{i\ hr_1\sqrt{r}}{\sqrt{r_2r_4}},\ a_0=0,\quad a_1=\frac{hr_1r}{cr_4}. \tag{3.24}$$

Therefore, the solutions of equation (2.4) can be obtained:

$$u_1(x,y,t)=-\frac{hr_1r\sqrt{-r_2r_4}}{cr_2r_4}$$
$$\frac{c_1\sinh\left[\frac{\sqrt{-r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]+c_2\cosh\left[\frac{\sqrt{-r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]}{c_1\cosh\left[\frac{\sqrt{-r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]+c_2\sinh\left[\frac{\sqrt{-r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]} \tag{3.25}$$

Using arbitrary selection of coefficients, special solutions can be obtained as follows:

$$u_2(x,y,t)=-\frac{hr_1r\sqrt{-r_2r_4}}{cr_2r_4}\ \tanh\left[\frac{\sqrt{-r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right],$$
$$c_1\neq 0,\ c_2=0, \tag{3.26}$$

$$u_3(x,y,t)=-\frac{hr_1r\sqrt{-r_2r_4}}{cr_2r_4}\ \coth\left[\frac{\sqrt{-r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right],$$
$$c_1=0,\ c_2\neq 0, \tag{3.27}$$

$$u_4(x.y.t)=-\frac{hr_1r\sqrt{r_2r_4}}{cr_2r_4}$$
$$\frac{-c_1\sin\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]+c_2\cos\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]}{c_1\cos\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]+c_2\sin\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]} \tag{3.28}$$

With arbitrary selection of coefficients, we have

$$u_5\left(x,y,t\right)=\frac{hr_1r\sqrt{r_2r_4}}{cr_2r_4}\tan\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\right],$$
$$c_1\neq 0,\ c_2=0, \tag{3.29}$$
$$u_6\left(x.y.t\right)=-\frac{hr_1r\sqrt{r_2r_4}}{cr_2r_4}\cot\left[\frac{\sqrt{r_2r_4}}{r_1}\left(\nu x+i\nu y-\frac{ct^{\alpha}}{\Gamma\left(\alpha+1\right)}\right)\right],$$
$$c_1=0,\ c_2\neq 0. \tag{3.30}$$

b) The second set of solutions is:

$$r_1\neq 0,\ r_2\neq 0,\quad r_3=0,\quad r_4=0,\ h\neq 0,\ r=0,\quad a_0=0,\quad a_1=\frac{cr_2}{hr_1}. \tag{3.31}$$

Therefore, from the ninth state, we have the following solution to equation (2.3):

$$u_7\left(x,y,t\right)=\frac{cr_2}{hr_1}\frac{1}{c_1+\ \left(\nu x+i\nu y-\frac{ct^{\alpha}}{\Gamma(\alpha+1)}\right)}. \tag{3.32}$$

c) The third set of solutions to (3.22) is:

$$r_1\neq 0,\ r_2\neq 0,\ \ r_3\neq 0,\ \ r_4\neq r_4,\ r_3^2-4r_2r_4\neq 0,\ a_0=-\frac{cr_3}{2\ hr_1},$$
$$a_1=-\frac{cr_2}{hr_1}. \tag{3.33}$$

Therefore, from the first and second cases, we have the solutions of equation (2.3) as follows:

$$u_8\left(x,y,t\right)=\ -\frac{cr_3}{2\ hr_1}+\frac{cr_2}{hr_1}$$
$$\left[\frac{r_3}{2r_2}+\frac{\sqrt{r}}{2r_2}\right.$$
$$\left.\frac{c_1\sinh\left[\frac{\sqrt{r}}{2\ r_1}\left(\nu x+i\nu y-\frac{ct^{\alpha}}{\Gamma(\alpha+1)}\right)\right]\ +c_2\cosh\left[\frac{\sqrt{r}}{2\ r_1}\left(\nu x+i\nu y-\frac{ct^{\alpha}}{\Gamma(\alpha+1)}\right)\right]}{c_1\cosh\left[\frac{\sqrt{r}}{2\ r_1}\left(\nu x+i\nu y-\frac{ct^{\alpha}}{\Gamma(\alpha+1)}\right)\right]\ +c_2\sinh\left[\frac{\sqrt{r}}{2\ r_1}\left(\nu x+i\nu y-\frac{ct^{\alpha}}{\Gamma(\alpha+1)}\right)\right]}\right]. \tag{3.34}$$

Special solutions in this case are:

$$u_9(x,y,t) = -\frac{cr_3}{2\ hr_1} + \frac{cr_2}{hr_1}[\frac{r_3}{2r_2} + \frac{\sqrt{r}}{2r_2}\tanh\left[\frac{\sqrt{r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]],$$

$$c_1 \neq 0,\ c_2 = 0, \tag{3.35}$$

$$u_{10}(x,y,t) = -\frac{cr_3}{2\ hr_1} + \frac{cr_2}{hr_1}[\frac{r_3}{2r_2} + \frac{\sqrt{r}}{2r_2}\ \coth\left[\frac{\sqrt{r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]\ ],$$

$$c_1 = 0,\ c_2 \neq 0, \tag{3.36}$$

$$u_{11}(x,y,t) = -\frac{cr_3}{2\ hr_1} + \frac{cr_2}{hr_1}$$

$$\left[\frac{r_3}{2r_2} + \frac{\sqrt{-r}}{2r_2}\right.$$

$$\left.\frac{-c_1\sin\left[\frac{\sqrt{-r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]\ + c_2\cos\left[\frac{\sqrt{-r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]}{c_1\cos\left[\frac{\sqrt{-r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]\ + c_2\sin\left[\frac{\sqrt{-r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]}\right] \tag{3.37}$$

In this case, the following special solutions can be obtained by choosing the coefficients as desired.

$$u_{12}(x,y,t) = -\frac{cr_3}{2\ hr_1} + \frac{cr_2}{hr_1}[\frac{r_3}{2r_2} - \frac{\sqrt{-r}}{2r_2}\ \tan\left[\frac{\sqrt{-r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]\ ],$$

$$c_1 \neq 0,\ c_2 = 0, \tag{3.38}$$

$$u_{13}(x,y,t) = -\frac{cr_3}{2\ hr_1} + \frac{cr_2}{hr_1}[\frac{r_3}{2r_2} + \frac{\sqrt{-r}}{2r_2}\ \cot\left[\frac{\sqrt{-r}}{2\ r_1}\left(\nu x + i\nu y - \frac{ct^\alpha}{\Gamma(\alpha+1)}\right)\right]\ ],$$

$$c_1 \neq 0,\ c_2 = 0. \tag{3.39}$$

4. Conclusion

In this article, an analytical approach based on the solution of the Riccati equation is used to solve the nonlinear partial differential equation with fractional order. In this approach, we first convert the fractional partial differential equation into an ordinary differential equation using a suitable transformation; Then we consider its answers as a series. In this method, different forms of solutions are obtained, including trigonometric, hyperbolic and exponential solutions. The results indicate that the approach used in this article is very useful and effective for determining the solution of fractional partial differential equations.

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