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Intuitionistic L-fuzzy Ideals and Subalgebras of Novikov Algebras

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Abstract. In this paper, a new concept of intuitionistic L-fuzzy sets (ILFSs) is applied to Novikov algebras and some fundamental notions including ILF subspace, ILF ideal and ILF subalgebra of Novikov algebras are introduced alongside operations such as multiplication, union, and intersection on ILFSs. Additionally, throughout this paper, some necessary and sufficient conditions including an ILF subspace is an ILF ideal are obtained and key structural properties are established. Moreover, it is demonstrated that a quotient algebra X/A of a ILF ideal $A = (\mu_A, \nu_A)$ is isomorphic to X/X_A of a non-ILF ideal X_A. Furthermore, the algebraic properties of the surjective homomorphic image and preimage of an ILF ideal is explored which helps reach deeper insights into the algebraic structure of Novikov algebras and further research of ILFSs.

Keywords: Intuitionistic L-fuzzy ideals, intuitionistic L-fuzzy subalgebras, intuitionistic L-fuzzy subspaces, Novikov algebra, lattice.
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1. Introduction

The concept of Intuitionistic Fuzzy Sets (IFSs, for short) was introduced by Atanassov (in 1983) as an extension of Zadeh's classical fuzzy set theory, allowing for better representation of uncertainty by using both membership and the non-membership value. IFSs are known as the complement of the membership degree, offering a more flexible model of vagueness. They are widely applied in various branches of mathematics, including algebra and lattice theory, graph theory, network analysis, etc. The notions of lattice and complete lattice were first presented by Garrett Birkhoff, who also developed the fundamental properties of lattice in Birkhoff's book "Lattice Theory" (1940), and Øystein Ore in 1930. Later in 1983 and 1986, Atassanov and Stoeva integrated two concepts of IFSs and lattice theory, named ILFSs, which provided a more generalized framework for handling uncertainty by including lattice-based membership and non-membership functions. In this paper, the concept of an ILF subspace is extended to Novikov algebras. In section 2, ILF ideals and ILF subalgebras of Novikov algebras are defined, and some fundamental properties are discussed. In section 3, we show that addition, product and intersection of ILF ideals are ILF ideals [resp. ILF subalgebras], but the union of ILF ideals may not be an ILF ideal. In section 4, we show that the quotient algebra X/A of a ILF ideal A is isomorphic to the algebra X/X_A of a non-ILF ideal X_A. In section 5, it is showed that if $f : X_1 \longrightarrow X_2$ is an ILF Novikov algebra homomorphism, then the preimage of an ILF ideal is an ILF ideal [resp. ILF subalgebra]. When f is surjective, a homomorphic image is an ILF ideal. Moreover, the addition, product and intersection of ILF ideals in X_1 are preserved by f.

2. preliminaries

In this part, some essential definitions and notions of ILFSs are presented.

Definition 2.1. A *fuzzy set* (or *fuzzy subset*) is a pair (S, μ) where S is a non-empty set and $\mu : S \to [0,1]$ is a membership function. The set of all fuzzy subsets of S is denoted by $[0,1]^S$.

Definition 2.2. A *bounded lattice* $L = (L, \wedge, \vee, 0, 1)$ is an algebraic structure such that the symbol $\vee$ denotes maximum and $\wedge$ denotes minimum and for every $x \in L$, the conditions $x \wedge 1 = x$, $x \vee 1 = 1$, $x \wedge 0 = 0$ and $x \vee 0 = x$ satisfy, where the constants 0, 1 of the lattice L represent the upper bound (top) and the lower bound (bottom).

Definition 2.3. Let S be a non-empty set and L be a non-trivial complete distributive lattice (in particular $L = [0,1]$), then an *L-fuzzy set*

μ in S is characterised by a map $\mu : S \to L$ and the set of all L-fuzzy subsets in S is denoted by L^S.

Definition 2.4. A *pre-Lie algebra* X is a vector space with a binary operation $(x, y) \mapsto x.y$ satisfying $(x.y).z - x.(y.z) = (y.x).z - y.(x.z)$ for all $x, y, z \in X$. The algebra X is called *Novikov*, if $(x.y).z = (x.z).y$ for all $x, y, z \in X$. Throughout this paper X is a Novikov algebra over a field F, unless explicitly stated otherwise.

Definition 2.5. An ILF set (or subset), briefly an ILFS of a non-void set S is the form $A = \{(x, \mu_A(x), \nu_A(x))|x \in S\}$ (shortly $A = (\mu_A, \nu_A)$ or A), where the maps $\mu_A : S \to [0, 1]$ and $\nu_A : S \to [0, 1]$ are L-fuzzy subsets of S such that $\mu_A(x)$ denotes the membership degree and $\nu_A(x)$ denotes the non-membership degree and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for every $x \in S$. The set of all $ILFSs$ of S is denoted by $ILFS(S)$.

Definition 2.6. The addition and multiplication of two ILF set $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are extented to two operations on L^X, denoted by $+$ and $\times$ as follows:

$$(i)(\mu_A + \mu_B)(x) = sup\{\mu_A(a) \wedge \mu_A(b) : a + b = x\},$$

$$(ii)(\mu_A \times \mu_B)(x) = sup\{\mu_A(a) \wedge \mu_A(b) : a.b = x\},$$

and

$$(i)(\nu_A + \nu_B)(x) = inf\{\nu_A(a) \vee \nu_A(b) : a + b = x\},$$

$$(ii)(\nu_A \times \nu_B)(x) = inf\{\nu_A(a) \vee \nu_A(b) : a.b = x\}.$$

for all $A, B \in L^X$ and $x, a, b \in X$. The scalar multiplication kx for $k \in F$ and $x \in X$ is extended to an action of feild F on L^X as follows:

$$(k\mu_A)(x) = \begin{cases} \mu_A(k^{-1}x) \; if \; k \neq 0 \\ 1 \; if \; k = 0, x = 0 \\ 0 \; if \; k = 0, x \neq 0 \end{cases}$$

and

$$(k\nu_A)(x) = \begin{cases} \nu_A(k^{-1}x) \; if \; k \neq 0 \\ 0 \; if \; k = 0, x = 0 \\ 1 \; if \; k = 0, x \neq 0 \end{cases}$$

Definition 2.7. Let S be a nonempty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ be two $ILFSs$ of S, then for every $x \in S$ we have:
(i) $A \subseteq B \iff \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;
(ii) $A = B \iff \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$;
(iii) $A^c = (\nu_A, \mu_A)$;
(iv) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x))|x \in S\}$;
(v) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x))|x \in S\}$.

Definition 2.8. Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i\in I}$ be a family of $ILFSs$ of S, then $\bigcap_{i\in I} A_i = (\mu_{\cap_{i\in I} A_i}, \nu_{\cap_{i\in I} A_i}) = \{(x, \bigwedge_{i\in I} \mu_{A_i}(x), \bigvee_{i\in I} \nu_{A_i}(x))|x \in S\}$ and $\bigcup_{i\in I} A_i = (\mu_{\cup_{i\in I} A_i}, \nu_{\cup_{i\in I} A_i}) = \{(x, \bigvee_{i\in I} \mu_{A_i}(x), \bigwedge_{i\in I} \nu_{A_i}(x))|x \in S\}$. By $A(x) = t$, we mean $\mu_A = t$ and $\nu_A = 1 - t$.

Definition 2.9. Let V be a vecter space over a field F. An ILFS $A = (\mu_A, \nu_A)$ of V is called an *ILF subspace* satisfying:

(i) $\mu_A(0) = 1$, $\nu_A(0) = 0$;

(ii) $\begin{cases} \mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y); \\ \nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y), \; for\ every\ x, y \in V. \end{cases}$

(iii) $\begin{cases} \mu_A(rx) \geq \mu_A(x); \\ \nu_A(rx) \leq \nu_A(x), \; for\ every\ x \in V\ and\ r \in F. \end{cases}$

Definition 2.10. Let G be a group. An ILFS $A = (\mu_A, \nu_A)$ of G is called an *ILF subgroup* of G if the following conditions hold for every $x, y \in G$:

(i) $\begin{cases} \mu_A(xy) \leq \mu_A(x) \wedge \mu_A(y); \\ \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y); \end{cases}$

(ii) $\begin{cases} \mu_A(x^{-1}) \geq \mu_A(x) \text{ (consequently } \mu_A(x^{-1}) = \mu_A(x); \\ \nu_A(x^{-1}) \geq \nu_A(x) \text{ (consequently } \nu_A(x^{-1}) = \nu_A(x)). \end{cases}$

Definition 2.11. Let S be a non-empty set. An ILF subspace $A = (\mu_A, \nu_A)$ is called an *ILF ideal* of S, if it satisfies the following properties for every $x, y \in S$:

(1) $\begin{cases} \mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y); \\ \nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y). \end{cases}$

(2) $\begin{cases} \mu_A(xy) \geq \mu_A(x) \vee \mu_A(y); \\ \nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y). \end{cases}$

Definition 2.12. Let $A = (\mu_A, \nu_A)$ be an ILF ideal. For every $x \in X$, the ILF subset $x + A : X \to L$, defined by $(x + A)(y) = A(y - x)$ is called $coset\ of\ the\ ILF\ ideal\ A$.

Definition 2.13. An ILF subspace $A = (\mu_A, \nu_A)$ is called an *ILF subalgebra* of S, if it satisfies the following properties for every $x, y \in S$:

(i) $\begin{cases} \mu_A(x-y) \leq \mu_A(x) \vee \mu_A(y); \\ \nu_A(x-y) \geq \nu_A(x) \wedge \nu_A(y). \end{cases}$

(ii) $\begin{cases} \mu_A(xy) \leq \mu_A(x) \wedge \mu_A(y); \\ \nu_A(xy) \geq \nu_A(x) \vee \nu_A(y). \end{cases}$

Definition 2.14. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFLSs, then $(A \oplus B)(x) = (\mu_{A\oplus B}(x), \nu_{A\oplus B}(x))$ where $\mu_{A\oplus B}(x) = \bigvee\{\mu_A(a) \wedge \mu_B(b)|a+b = x\}$ and $\nu_{A\oplus B}(x) = \bigwedge\{\nu_A(a) \vee \nu_B(b)|a+b = x\}$ for all $x \in S$. Also $(A \otimes B)(x) = (\mu_{A\otimes B}(x), \nu_{A\otimes B}(x))$ where $\mu_{A\otimes B}(x) = \bigvee\{\mu_A(a) \wedge \mu_B(b)|a.b = x\}$ and $\nu_{A\otimes B}(x) = \bigwedge\{\nu_A(a) \vee \nu_B(b)|a.b = x\}$ for all $x \in S$.

Definition 2.15. The algebra X/A of an ILF ideal A is called the quotient algebra of Novikov algebra X. An addition, a scalar multiplication and a multiplication operations of the cosets are defined as follows:
$(i)(x+A)\oplus(y+A)=(x+y)+A$
$(ii)k\odot(x+A)=kx+A$
$(iii)(x+A)\otimes(y+A)=(x.y)+A$ for all $k\in F$, $x,y\in A$.
The addition, the scalar multiplication and the multiplication operation of the cosets in Definition 2.15 are well defined by Definition 2.14.

Definition 2.16. Let X_1, X_2 be two Novikov algebras and $A=(\mu_A,\nu_A)$ be an ILFS of X_1. A map $f:X_1\longrightarrow X_2$ has a natural extention $\tilde{f}:ILF^{X_1}\longrightarrow ILF^{X_2}, \tilde{f}(A)=(\tilde{\mu},\tilde{\nu})$ s.t

$$\tilde{\mu}(y)=\begin{cases} sup\{\mu_A(x)\mid x\in f^{-1}(y)\},\ f^{-1}(y)\neq\emptyset \\ 0\ ,\ f^{-1}(y)=\emptyset \end{cases}$$

and

$$\tilde{\nu}(y)=\begin{cases} inf\{\nu_A(x)\mid x\in f^{-1}(y)\},\ f^{-1}(y)\neq\emptyset \\ 1\ ,\ f^{-1}(y)=\emptyset \end{cases}$$

for all $A\in ILF^{X_1}$ and $y\in X_2$. $\tilde{f}(A)$ is called the *homomorphic image* of A.

Definition 2.17. Let X_1 and X_2 be two Novikov algebras and $f: X_1\longrightarrow X_2$ be an algebra homomorphism. The *preimage* of B, denoted by $f^{-1}(B)$, is the form of $f^{-1}(B)=(\mu_B^{-1},\nu_B^{-1})\in L^{X_1}$, where $\mu_B^{-1}(x)=\mu_B(f(x))$ and $\nu_B^{-1}(x)=\nu_B(f(x))$ for all $x\in X_1$.

Definition 2.18. Let $f:X_1\longrightarrow X_2$ be an algebra homomorphism. An ILF subset $A=(\mu_A,\nu_A)$ in X_1 is called *f-variant* if for any $x,\ y\in X_1$, $f(x)=f(y)$ implies $A(x)=A(y)$ (i.e. $\mu_A(x)=\mu_A(y)$ and $\nu_A(x)=\nu_A(y)$).

3. MAIN RESULT

In the following part, we presented several theorems and lemmas that play a crucial role in establishing the theoretical foundations of our study.

Lemma 3.1. Let V be a vecter space over a field F. An ILFS $A=(\mu_A,\nu_A)$ of V is an ILF subspace if and only if
(i) $\mu_A(0)=1,\ \nu_A(0)=0;$
(ii) $\begin{cases} \mu_A(kx+ly)\geq\mu_A(x)\wedge\nu_A(y); \\ \nu_A(kx+ly)\leq\nu_A(x)\vee\nu_A(y). \end{cases}$

Remark 3.2. Let V be a vecter space over a field F. For every $k \in F, x \in V$. We have $A(kx) = A(x)$ whenever A is an ILF ideal (or ILF subalgebra) of X.

Lemma 3.3. Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i \in I}$ be a family of ILF subspaces of X, then $\bigcap_{i\in I} A_i$ is an ILF subspace of X.

Proof. Let $A = \bigcap_{i\in I} A_i = (\mu_A, \nu_A)$. Obviously $\mu_A(0) = \bigwedge_{i\in I} \mu_{A_i}(0) = 1$ and $\nu_A(0) = \bigvee_{i\in I} \nu_{A_i}(0) = 0$. Now let $x, y \in X$ and $k, l \in F$. Then $\mu_A(kx+ly) = \bigwedge_{i\in I} \mu_{A_i}(kx+ly) \geq \bigwedge_{i\in I}(\mu_{A_i}(x) \wedge \mu_{A_i}(y)) = (\bigwedge_{i\in I} \mu_{A_i}(x)) \wedge (\bigwedge_{i\in I} \mu_{A_i}(y)) = \mu_A(x) \wedge \mu_A(y)$. Similary $\nu_A(kx + ly) \leq \nu_A(x) \vee \nu_A(y)$. So by Lemma 3.1, $A = \bigcap_{i\in I} A_i$ is an ILF subspace of X. □

Lemma 3.4. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two ILF subspaces of X. Then $A \oplus B$ is also an ILF subspace of X.

Proof. $\mu_{A\oplus B}(0) = \bigvee\{\mu_A(x) \wedge \mu_B(y) | x + y = 0\} \geq \mu_A(0) \wedge \mu_b(0) = 1$. So $\mu_{A\oplus B}(0) = 1$. Also $\nu_{A\oplus B}(0) = \bigwedge\{\nu_A(x) \vee \nu_B(y) | x + y = 0\} \leq \nu_A(0) \vee \nu_B(0) = 0$ and so $\nu_{A\oplus B}(0) = 0$. Now let $x, y \in X$ and $k, l \in F$. Then $\mu_{A\oplus B}(kx + ly) = \bigvee\{\mu_A(t) \wedge \mu_B(u) | t + u = kx + ly\} \geq (\bigvee\{\mu_A(t_1) \wedge \mu_A(u_1) | t_1 + u_1 = x\}) \wedge (\bigvee\{\mu_A(t_2 \wedge \mu_B(u_2 | t_2 + u_2 = y)\}) = \mu_{A\oplus B}(x) \wedge \mu_{A\oplus B}(y)$. Also $\nu_{A\oplus B}(kx + ly) = \bigwedge\{\mu_A(t) \vee \nu_B(u) | t + u = kx + ly\} \leq (\bigwedge\{\nu_A(t_1) \vee \nu_A(u_1) | t_1 + u_1 = x\}) \vee (\bigwedge\{\nu_A(t_2 \vee \nu_B(u_2 | t_2 + u_2 = y)\}) = \nu_{A\oplus B}(x) \vee \nu_{A\oplus B}(y)$. So by Lemma 3.1, $A \oplus B$ is an ILF subspace of X. □

Theorem 3.5.

(i) Let $A = (\mu_A, \nu_A)$ be an ILF ideal and $B = (\mu_B, \nu_B)$ be an ILF subalgebra of X. Then $A \oplus B = (\mu_{A\oplus B}, \nu_{A\oplus B})$ is also an ILF subalgebra of X.

(ii) Let $\{A_i = (\mu_{A_i}, \nu_{A_i}) | i \in I\}$ be a set of ILF subalgebras of X. Then the $\bigcap_{i\in I} A_i$ of X is also an ILF subalgebra of X.

Proof.

(i) $A \oplus B$ is an ILF subspace of X by Lemma 3.4. Let $x, y \in X$, then
$\mu_{A\oplus B}(x.y) \geq sup\{\mu_A(x_1.y) \wedge \mu_B(x_2.y)\} : x_1 + x_2 = x\}$
$\geq sup\{(\mu_A(x_1) \vee \mu_A(y)) \wedge (\mu_B(x_2) \wedge \mu_B(y)) : x_1 + x_2 = x\}$
$= sup\{(\mu_A(x_1) \wedge \mu_B(x_2) \wedge \mu_B(y)) \vee (\mu_A(y) \wedge \mu_B(x_2) \wedge \mu_B(y)) : x_1 + x_2 = x\}$
$\geq sup\{(\mu_A(x_1) \wedge \mu_B(x_2) \wedge \mu_B(y)) : x_1 + x_2 = x\}$
$= (\mu_{A\oplus B})(x) \wedge \mu_B(y) \geq \mu_{A\oplus B}(x) \wedge \mu_{A\oplus B}(y)$. Also
$\nu_{A\oplus B}(x.y) \leq inf\{\nu_A(x_1.y) \vee \nu_B(x_2.y)\} : x_1 + x_2 = x\}$
$\leq inf\{(\nu_A(x_1) \vee \nu_A(y)) \vee (\nu_B(x_2) \vee \nu_B(y)) : x_1 + x_2 = x\}$
$= inf\{(\nu_A(x_1) \vee \nu_B(x_2) \vee \nu_B(y)) \wedge (\nu_A(y) \vee \nu_B(x_2) \vee \nu_B(y)) : x_1 + x_2 = x\}$
$\leq inf\{(\nu_A(x_1) \vee \nu_B(x_2) \vee \nu_B(y)) : x_1 + x_2 = x\} = (\nu_{A\oplus B})(x) \vee \nu_B(y) \leq \nu_{A\oplus B}(x) \vee \nu_{A\oplus B}(y)$. Thus $A \oplus B$ is an ILF subalgebra of X.

(ii) $\bigcap A_i$ is an ILF subspace by lemma 3.3. Let $x, y \in X$, then

$$(\mu_{A_i})(x.y) = inf_{i\in I}\{\mu_{A_i}(x.y)\} \geq inf\{\mu_{A_i}(x) \wedge \mu_{A_i}(y)\}$$

$$= (\bigcap_{i\in i}\mu_{A_i})(x) \wedge (\bigcap_{i\in I}\mu_{A_i})(y).$$

Also

$$(\nu_{A_i})(x.y) = sup_{i\in I}\{\nu_{A_i}(x.y)\} \leq inf\{\nu_{A_i}(x) \vee \nu_{A_i}(y)\}$$

$$= (\bigcap_{i\in i}\nu_{A_i})(x) \vee (\bigcap_{i\in I}\nu_{A_i})(y).$$

Thus $\bigcap A_i$ is an ILF subalgebra of X.

□

Theorem 3.6.
(i) Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be ILF ideals. Then $A \oplus B = (\mu_{A\oplus B}, \nu_{A\oplus B})$ is also an ILF ideal of X.
(ii) Let $\{A_i = (\mu_{A_i}, \nu_{A_i})|i \in I\}$ be a set of ILF ideals of X. Then the $\bigcap_{i\in I} A_i$ of X is also an ILF ideal of X.
Proof.
(i) $A \oplus B$ is an L-fuzzy subspace by Lemma 3.4. Let $x, y \in X$, then

$$(\mu_{A\oplus B})(x.y) \geq sup\{\mu_A(x_1.y) \wedge \mu_B(x_2.y)|x_1 + x_2 = x\}$$

$$\geq sup\{(\mu_A(x_1) \vee \mu_A(y)) \wedge (\mu_B(x_2) \vee \mu_B(y)) : x_1 + x_2 = x\}$$

$$\geq sup\{[(\mu_A(x_1) \vee \mu_A(y)) \wedge (\mu_B(x_2)] \vee [(\mu_A(y)) \wedge \mu_B(y)] : x_1 + x_2 = x\}$$

$$\geq sup\{(\mu_A(x_1) \wedge \mu_B(x_2)) \vee (\mu_A(y)) \wedge \mu_B(y)) : x_1 + x_2 = x\} \geq$$

$$\mu_{A\oplus B}(x) \vee (\mu_A(y) \wedge \mu_B(y)) \geq \mu_{A\oplus B}(x)$$

for $x_1, x_2 \in X$. Similarly, we can prove that $\mu_{A\oplus B}(x.y) \geq \mu_{A\oplus B}(y)$. Thus $\mu_{A\oplus B}(x.y) \geq \mu_{A\oplus B}(x) \vee \mu_{A\oplus B}(y)$. Also

$$(\nu_{A\oplus B})(x.y) \leq inf\{\nu_A(x_1.y) \vee \nu_B(x_2.y)|x_1 + x_2 = x\}$$

$$\leq inf\{(\nu_A(x_1) \wedge \nu_A(y)) \vee (\nu_B(x_2) \wedge \nu_B(y)) : x_1 + x_2 = x\}$$

$$\leq inf\{[(\nu_A(x_1) \wedge \nu_A(y)) \vee (\nu_B(x_2)] \wedge [(\nu_A(y)) \vee \nu_B(y)] : x_1 + x_2 = x\}$$

$$\leq inf\{(\nu_A(x_1) \vee \nu_B(x_2)) \wedge (\nu_A(y)) \vee \nu_B(y)) : x_1 + x_2 = x\} \leq$$

$$\nu_{A\oplus B}(x) \wedge (\nu_A(y) \vee \nu_B(y)) \leq \nu_{A\oplus B}(x)$$

for $x_1, x_2 \in X$. Similarly, we can prove that $\nu_{A\oplus B}(x.y) \leq \nu_{A\oplus B}(y)$. Thus $\nu_{A\oplus B}(x.y) \leq \nu_{A\oplus B}(x) \wedge \nu_{A\oplus B}(y)$. Thus $A \oplus B$ is an ILF ideal of X.

(ii) $\bigcap_{i\in I} A_i$ is an ILF subspace by Lemma 3.3. Let $x, y \in X$,then

$$(\bigcap_{i\in I}\mu_{A_i})(x.y) = inf_{i\in I}\{\mu_{A_i}(x.y)\} \geq \vee\{inf_{i\in I}\mu_{A_i}(x), inf_{i\in I}\mu_{A_i}(y)\}$$

$$= (\bigcap_{i\in I}\mu_{A_i})(x) \vee (\bigcap_{i\in I}\mu_{A_i})(y).$$

and $(\bigcap_{i\in I}\nu_{A_i})(x.y) = sup_{i\in I}\{\nu_{A_i}(x.y)\} \leq \wedge\{sup_{i\in I}\nu_{A_i}(x), sup_{i\in I}\nu_{A_i}(y)\}$

$$= (\bigcap_{i\in I}\nu_{A_i})(x) \wedge (\bigcap_{i\in I}\nu_{A_i})(y).$$

□

Proposition 3.7. Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B), C = (\mu_C, \nu_C)$ be three ILF ideals of X, then:
$(i)(A \otimes B) \otimes C = (A \otimes C) \otimes B$.
$(ii)A \otimes (B \otimes C) \subseteq ((A \otimes B) \otimes C) \oplus (B \otimes (A \otimes C)) \oplus ((B \otimes A) \otimes C)$.

Proof. (i) Let $x \in X$, then

$$\mu_{(A\otimes B)\otimes C}(x) = sup\{\mu_{A\otimes B}(m) \wedge \mu_C(n)\}|m.n = x\}$$
$$= sup\{sup\{\mu_A(a) \wedge \mu_B(b) : a.b = m\} \wedge \mu_C(n)|m.n = x\}$$
$$= sup\{(\mu_A(a) \wedge \mu_B(b)) \wedge \mu_C(n)|(a.b).n = x\})$$
$$= sup\{(\mu_A(a) \wedge \mu_C(n)) \wedge \mu_B(b)|(a.n).b = x\}$$
$$= sup\{sup\{\mu_A(a) \wedge \mu_C(n) : a.n = c\} \wedge \mu_B(b)|c.b = x\}$$
$$= sup\{\mu_{A\otimes C}(c) \wedge \mu_B(b)|c.b = x\} = \mu_{(A\otimes C)\otimes B}(x).$$

Also

$$\nu_{(A\otimes B)\otimes C}(x) = inf\{\nu_{A\otimes B}(m) \vee \nu_C(n)\}|m.n = x\}$$
$$= inf\{inf\{\nu_A(a) \vee \nu_B(b) : a.b = m\} \vee \nu_C(n)|m.n = x\}$$
$$= inf\{(\nu_A(a) \vee \nu_B(b)) \vee \nu_C(n)|(a.b).n = x\})$$
$$= inf\{(\nu_A(a) \vee \nu_C(n)) \vee \nu_B(b)|(a.n).b = x\}$$
$$= inf\{inf\{\nu_A(a) \vee \nu_C(n) : a.n = c\} \vee \nu_B(b)|c.b = x\}$$
$$= inf\{\nu_{A\otimes C}(c) \vee \nu_B(b)|c.b = x\} = \nu_{(A\otimes C)\otimes B}(x).$$

for $a, b, c, m, n \in X$.

(ii) Let $x \in X$, then

$$\mu_{A\otimes(B\otimes C)}(x) = sup\{\mu_A(m) \wedge \mu_{B\otimes C}(n)|m.n = x\}$$
$$= sup\{\mu_A(m) \wedge sup\{\mu_B(b) \wedge \mu_C(c)|b.c = n\}|m.n = x\}$$
$$= sup\{\mu_A(m) \wedge (\mu_B(b) \wedge \mu_C(c))|m.(b.c) = x\}$$
$$= sup\{((\mu_A(m) \wedge \mu_B(b)) \wedge \mu_C(c)) \wedge (\mu_B(b) \wedge (\mu_A(m) \wedge \mu_C(c)))$$
$$\wedge((\mu_B(b) \wedge \mu_A(m)) \wedge \mu_C(c))|(m.b).c + b.(m.c) - (b.m).c = x\}$$
$$\leq sup\{(sup(\mu_A(m) \wedge \mu_B(b)) \wedge \mu_C(c)) \wedge (\mu_B(b) \wedge sup(\mu_A(m))$$

$\wedge\mu_C(c)))\wedge(sup(\mu_B(b)\wedge\mu_A(m)))\wedge\mu_C(c))|(m.b).c+b.(m.c)-(b.m).c=x\}$
$= sup\{((\mu_{A\otimes B})(r)\wedge\mu_C(c))\wedge(\mu_B(b)\wedge(\mu_{A\otimes C})(s))\wedge((\mu_{B\otimes A})(t)\wedge\mu_C(c))|r.c+b.s+t.c=x,$
$r=m.b, s=m.c, t=-b.m\} \leq sup\{sup((\mu_{A\otimes\mu B})(r)\wedge\mu_C))\wedge sup(\mu_B(b)\wedge(\mu_{A\otimes C}(s))$
$\wedge sup((\mu_{B\otimes\mu A})(t)\wedge\mu_C(c))|r.c+b.s+t.c=x, r=m.b, t=-b.m\}$
$= sup\{(\mu_{A\otimes B}\wedge\mu_C)(u)\wedge(\mu_B\wedge\mu_{A\otimes C})(v)\wedge(\mu_{B\otimes A}\wedge\mu_C)(w)|u+v+w=x,$
$u=r.c, v=b.s, w=t.c\} = (\mu_{(A\otimes B)\otimes C}\oplus\mu_{B\otimes(A\otimes C)}\oplus\mu_{(B\otimes A)\otimes C})(x)$
for $a,b,c,m,n,r,s,t,u,v,w\in X$. Now, let $x\in X$, then

$$\nu_{A\otimes(B\otimes C)}(x) = inf\{\nu_A(m)\vee\nu_{B\otimes C}(n)|m.n=x\}$$
$$= inf\{\nu_A(m)\vee inf\{\nu_B(b)\vee\nu_C(c)|b.c=n\}|m.n=x\}$$
$$= inf\{\nu_A(m)\vee(\nu_B(b)\vee\nu_C(c))|m.(b.c)=x\}$$
$$= inf\{((\nu_A(m)\vee\nu_B(b))\vee\nu_C(c))\vee(\nu_B(b)\vee(\nu_A(m)\vee\nu_C(c)))\vee((\nu_B(b)\vee\nu_A(m))\vee\nu_C(c)) |(m.b).c+b.(m.c)-(b.m).c=x\}$$
$$\geq inf\{(inf(\nu_A(m)\vee\nu_B(b))\vee\nu_C(c))\vee(\nu_B(b)\vee inf(\nu_A(m))\vee\nu_C(c)))\vee((\nu_B(b)\vee\nu_A(m)))\vee\nu_C(c)) |(m.b).c+b.(m.c)-(b.m).c=x\}$$
$$= inf\{((\nu_{A\otimes B})(r)\vee\nu_C(c))\vee(\nu_B(b)\vee(\nu_{A\otimes C})(s))\vee((\nu_{B\otimes A})(t)\vee\nu_C(c)) |r.c+b.s+t.c=x, r=m.b, s=m.c, t=-b.m\}$$
$$\geq inf\{inf((\nu_{A\otimes B})(r)\vee\nu_C))\vee inf(\nu_B(b)\vee(\nu_{A\otimes C}(s))\vee inf((\nu_{B\otimes A})(t)\vee\nu_C(c)) |r.c+b.s+t.c=x, r=m.b, t=-b.m\}$$
$$= inf\{(\nu_{A\otimes B}\vee\nu_C)(u)\vee(\nu_B\vee\nu_{A\otimes C})(v)\vee(\nu_{B\otimes A}\vee\nu_C)(w) |u+v+w=x, u=r.c, v=b.s, w=t.c\}$$
$$= (\nu_{(A\otimes B)\otimes C}\oplus\nu_{B\otimes(A\otimes C)}\oplus\nu_{(B\otimes A)\otimes C})(x)$$

for $a,b,c,m,n,r,s,t,u,v,w\in X$. □

Theorem 3.8. Let $A=(\mu_A,\nu_A)$ be an ILF subspace of X and $\chi_X^{ILF}=(\chi_X,\chi_X^c)$. Then A is an ILF ideal of X if and only if $\chi_X^{ILF}\otimes A\subseteq A$ and $A\otimes\chi_X^{ILF}\subseteq A$ for all $x\in X$.

Proof. $(\Leftarrow)$: Suppose that $\chi_X^{ILF}\otimes A\subseteq A$. Let $x,y\in A$. Then $\mu_A(x.y)\geq(\chi_X\otimes\mu_A)(x.y)=sup\{\chi_X(a)\wedge\mu_A(b):a.b=x.y\})\geq\chi_X(x)\wedge\mu_A(y)\geq\mu_A(y)$. Also $\nu_A(x.y)\leq(\chi_X^c\otimes\nu_A)(x.y)=inf(\chi_X^c(a)\vee\nu_A(b)|a.b=x.y\})$
$\leq\chi_X^c(x)\vee\nu_A(y)\leq\nu_A(y)$.
Thus A is an ILF ideal of X.
$(\Rightarrow)$: Suppose $A=(\mu_A,\nu_A)$ is an ILF ideal of X. Put $\chi_X^{ILF}\otimes A=(\mu_1,\nu_1)$ and $A\otimes\chi_X^{ILF}=(\mu_2,\nu_2)$. For $x,y\in X$ we have $\mu_1(x)=sup\{\chi_X(a)\wedge\mu_A(b)|a.b=x\}=sup\{\mu_A(b)|a.b=x\}\leq\mu_A(x)$. Similary $\mu_2(x)\leq\mu_{A(x)}$. Also $\nu_1(x)=inf\{\chi_X^c(a)\vee\nu_A(b)|a.b=x\}=inf\{\nu_A(b)|a.b=x\}\geq\nu_A(x)$. Similary $\nu_2(x)\geq\nu_A(x)$, so $\chi_X^{ILF}\otimes A\subseteq A$ and $A\otimes\chi_X^{ILF}\subseteq A$. □

Theorem 3.9. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two ILF ideals of X. Then $A \otimes B$ is also an ILF ideal of X.

Proof. By Proposition 3.7(2), we have

$$\chi_X^{ILF} \otimes (\mu_A \otimes \mu_B) \subseteq ((\chi_X^{ILF} \otimes \mu_A) \otimes \mu_B) \oplus (\mu_A \otimes (\chi_X^{ILF} \otimes \mu_B)) \oplus ((\mu_A \otimes \chi_X^{ILF})\mu_B)$$
$$\subseteq (\mu_A \otimes \mu_B) \oplus (\mu_A \otimes \mu_B) \oplus (\mu_A \otimes \mu_B) \subseteq (\mu_A \otimes \mu_B).$$

and

$$\chi_X^{ILF} \otimes (\nu_A \otimes \nu_B) \supseteq ((\chi_X^{ILF} \otimes \nu_A) \otimes \nu_B) \oplus (\nu_A \otimes (\chi_X^{ILF} \otimes \nu_B)) \oplus ((\nu_A \otimes \chi_X^{ILF})\nu_B)$$
$$\supseteq (\nu_A \otimes \nu_B) \oplus (\nu_A \otimes \nu_B) \oplus (\nu_A \otimes \nu_B) \supseteq (\nu_A \otimes \nu_B).$$

By Proposition 3.7(1), it is obvious that

$$(\mu_A \otimes \mu_B) \otimes \chi_X^{ILF} = (\mu_A \otimes \chi_X^{ILF}) \otimes \mu_B \subseteq \mu_A \otimes \mu_B.$$

and

$$(\nu_A \otimes \nu_B) \otimes \chi_X^{ILF} = (\nu_A \otimes \chi_X^{ILF}) \otimes \nu_B \supseteq \nu_A \otimes \nu_B.$$

By Theorem 3.8, $A \otimes B$ is an ILF ideal of X. □

Proposition 3.10. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ br two proper ILF subspaces of X. Then the union of A and B cannot be an ILF subspace.

Proof. Let $A = (\mu_A . \nu_A)$ and $B = (\mu_B, \nu_B)$ be proper ILF subspaces of X such that $A(x) = 1$ or $B(x) = 1$ for all $x \in X$. Let $u, v \in X$ be such that $A(u) = 1, A(v) < 1, B(v) < 1, B(v) = 1$, and consider uv. If $A(uv) = 1$, then since $A(u^{-1}) = 1$ we would have $A(v) = A(u^{-1}(uv)) \geq min(A(u^{-1}), A(uv)) = 1$, contradition; A similar contradition is obtained if $B(uv) = 1$. □

Remark 3.11. Let $\{A_i = (\mu_{A_i}, \nu_{A_i}) | i \in I\}$ be a set of ILF ideals in X. Then the $\cup_{i \in I} A_i$ may not be an ILF ideal [resp. ILF subalgebra]. It can be proved by the same method as the proof of Proposition 3.10.

Theorem 3.12. Let $A = (\mu_A, \nu_A)$ be an ILF ideal of X, then $x + A = y + A$ if and only if $A(x - y) = A(0)$, for $x, y \in X$. In this case $A(x) = A(y)$.

Proof. If $x + \mu_A = y + \mu_A$ and $x + \nu_A = y + \nu_A$, then evaluating both side of this equation at x we get $\mu_A(x - y) = \mu_A(x - x) = \mu_A(0)$ and $\nu_A(x - y) = \nu_A(x - x) = \nu_A(0)$, thus we have $A(x - y) = A(x - x) = A(0)$ for $x, y \in X$. Conversely, If $A(x - y) = A(0)$, then

$$(x + \mu_A)(z) = \mu_A(z - x) = \mu_A(z - y + y - x)$$
$$\geq \mu_A(z - y) \wedge \mu_A(y - x) = \mu_A(z - y) = (y + \mu_A)(z)$$

for all $z \in X$. Thus $x+\mu_A \geq y+\mu_A$. By similar way $y+\mu_A \geq x+\mu_A$. So $x+\mu_A = y+\mu_A$. In the other hand

$$(x+\nu_A)(z) = \nu_A(z-x) = \nu_A(z-y+y-x)$$
$$\leq \mu_A(z-y) \vee \nu_A(y-x) = \nu_A(z-y) = (y+\nu_A)(z),$$

for all $z \in X$. Thus $x+\nu_A \leq y+\nu_A$. Similary $y+\nu_A \leq x+\nu_A$. So $x+\nu_A = y+\nu_A$. □

Remark 3.13. Let $A = (\mu_A, \nu_A)$ be an ILF of X, then $X_A = \{x \in X | A(x) = 1; (\mu_A(x) = 1, \nu_A(x) = 0)\}$ is an ILF ideal of X.

Remark 3.14. If A is an ILF ideal of X, then $(x+A)(z) = A(y-x)$ for all $z \in y+X_A$. In particular, $(x+A)(z) = A(x)$.

Proposition 3.15. Let $A = (\mu_A, \nu_A)$ be an ILF ideal, then $A(0) \geq A(x) \geq A(1)$ for all $a \in X$.

Proposition 3.16. Let $A = (\mu_A, \nu_A)$ be an ILF ideal of X and x, y, u, v be any elements in R. If $x+A = u+A$ and $y+A = v+A$, then
(i)$(x+y)+A = (u+v)+A$;
(ii)$(x.y)+A = (u.v)+A$.

Proof. (i) Since by the Proposition 3.12, $A(x-u) = A(y-v) = A(0)$, we get $\mu_A(x+y-u-v) = \mu_A(x-u+y-v) \geq \mu_A(x-u) \wedge \mu_A(y-v) = \mu_A(0)$ and $\nu_A(x-u) = \nu_A(y-v) = \nu_A(0)$, we get $\nu_A(x+y-u-v) = \nu_A(x-u+y-v) \leq \nu_A(x-u) \vee \nu_A(y-v) = \nu_A(0)$. Hence $\mu_A(x+y-u-v) = \mu_A(0)$, and $\nu_A(x+y-u-v) = \nu_A(0)$, therefore $(x+y)+A = (u+v)+A$.
(ii) $\mu_A(uv-xy) = \mu_A(uv-uy+uy-xy) \geq \mu_A[u(v-y)] \wedge \mu_A[(u-x)y]$

$$\geq [\mu_A(u) \vee \mu_A(v-y)] \wedge [\mu_A(u-x) \wedge \mu_A(y)]$$
$$= [\mu_A(u) \vee \mu_A(0)] \wedge [\mu_A(0) \vee \mu_A(y)] = \mu_A(0).$$

and

$$\nu_A(uv-xy) = \nu_A(uv-uy+uy-xy)$$
$$\leq \nu_A[u(v-y)] \vee \nu_A[(u-x)y]$$
$$\leq [\nu_A(u) \wedge \nu_A(v-y)] \vee [\nu_A(u-x) \vee \nu_A(y)]$$
$$= [\nu_A(u) \wedge \nu_A(0)] \vee [\nu_A(0) \wedge \nu_A(y)] = \nu_A(0).$$

Therefore $A(uv-xy) = A(0)$ and $xy+A = uv+A$. □

Proposition 3.17. Let $A = (\mu_A, \nu_A)$ be an ILF ideal and x_1, x_2, y_1, y_2, k be any elements in X. If $x_1+A = y_1+A$ and $x_2+A = y_2+A$, then
(i)$(x_1+x_2)+A = (y_1+y_2)+A$;
(ii)$(x_1.x_2)+A = (y_1.y_2)+A$;
(iii)$kx_1+A = ky_1+A$, for all $k \in F$.

Proof. The proof of (i) and (ii) by proposition 3.16. it is sufficient to prove (iii). Since $\mu_A(x_1 - x_2) = \mu_A(k(x_1 - x_2)) = 0$, we get that $kx_1 + \mu_A = ky_1 + \mu_A$. Similary since $\nu_A(x_1 - x_2) = \nu_A(k(x_1 - x_2)) = 1$, we get that $kx_1 + \nu_A = ky_1 + \nu_A$.

□

Theorem 3.18. Let A be a ILF ideal of X. The Novikov quotient algebra X/A is isomorphic to the algebra X/X_A.

Proof. Consider the surjective algebra homomorphism $\pi : X \longrightarrow X/A$ defines by $\pi(x) : x + A$. By Theorem 3.12, $Ker(\pi) = X_A$. By the fundamental theorem of homomorphisms, there exists an isomorphism from X/X_A to X/A. The isomorphic correspondence is given by $x + A = x + X_A$ for $x \in X$.

□

4. ILF IDEALS ON HOMOMORPHISM

Example 4.1. Let $f : \mathbb{Z} \to 2\mathbb{Z}$ be a map such that $f(x) = 2x$ and $A = (\mu_A, \nu_A)$ be an ILF set of $\mathbb{Z}$ such that

$$\mu_A(x) = \begin{cases} 0 \ , \ x \in 2\mathbb{Z} \\ 1/2 \ , \ x \in 2\mathbb{Z} + 1 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 2/3 \ , \ x \in 2\mathbb{Z} \\ 2/5 \ , \ x \in 2\mathbb{Z} + 1 \end{cases}$$

Then the homomorphic image of A is $\tilde{f}(A) = (\tilde{\mu}, \tilde{\nu})$ such that

$$\tilde{\mu}(y) = \begin{cases} 0 \ , \ y \in 4\mathbb{Z} \\ 1/2 \ , \ y \in 4\mathbb{Z} + 2 \end{cases}$$

and

$$\tilde{\nu}(y) = \begin{cases} 2/3 \ , \ y \in 4\mathbb{Z} \\ 2/5 \ , \ y \in 4\mathbb{Z} + 2 \end{cases}$$

Example 4.2. Let $f : \mathbb{Z} \to 2\mathbb{Z}$ again be a map that $f(x) = 2x$ and $B = (\mu_B, \nu_B) \subseteq ILF^{2\mathbb{Z}}$ such that

$$\mu_B(y) = \begin{cases} 0 \ , \ y \in 4\mathbb{Z} \\ 1/2 \ , \ y \in 4\mathbb{Z} + 2 \end{cases}$$

and

$$\nu_B(y) = \begin{cases} 2/3 \ , \ y \in 4\mathbb{Z} \\ 2/5 \ , \ y \in 4\mathbb{Z} + 2 \end{cases}$$

Then $f^{-1}(B) = (\hat{\mu}, \hat{\nu})$ such that $\hat{\mu}(x) = \mu_B(2x)\begin{cases} 0\ ,\ x \in 2\mathbb{Z} \\ 1/2\ ,\ x \notin 2\mathbb{Z} \end{cases}$ and $\hat{\nu}(y) = \nu_B(2x) = \begin{cases} 2/3\ ,\ y \in 2\mathbb{Z} \\ 2/5\ ,\ y \notin 2\mathbb{Z} \end{cases}$ is the preimage of B.

Example 4.3. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f : 2x^2 + 1$ and $A = (\mu_A, \nu_A) \subseteq ILF^{\mathbb{R}}$ such that

$$\mu_A(x) = \begin{cases} 1/4\ ,\ x \in \mathbb{Z} \\ 2/3\ ,\ x \notin \mathbb{Z} \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 1/3\ ,\ x \in \mathbb{Z} \\ 1/2\ ,\ x \notin \mathbb{Z} \end{cases}$$

Vividly for all $x, y \in \mathbb{R}$, $f(x) = f(y)$ implies $x = y$, hence A is $f-$ *variant*.

Remark 4.4. In case $f : X_1 \to X_2$ be monomorphism, it's trivial that every choosen ILF subset A on X_1 is f-variant , otherwise A can either be f-variant or not.

Example 4.5. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f : x^2 + 1$ and $A = (\mu_A, \nu_A) \subseteq ILF^{\mathbb{R}}$ such that

$$\mu_A(x) = \begin{cases} 1\ ,\ x \geq 1 \\ 1/3\ ,\ x < 1 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0\ ,\ x \geq 1 \\ 1/2\ ,\ x < 1 \end{cases}$$

Put $x = 2$ and $y = -2$ then $f(2) = f(-2)$ though $\mu_A(2) \neq \mu_A(-2)$. Hence $A(2) \neq A(-2)$ and A is not f-variant.

Remark 4.6. Let $f : X_1 \longrightarrow X_2$ be an algebra homomorphism and $A = (\mu_A, \nu_A)$ be an ILF subspace of X_1, then $f(A)$ is not necessarily an ILF subspace of X_2.

Example 4.7. Let $f : \mathbb{Z} \to \mathbb{Z}$ be an algebra homomorphism definded as $f(x) = 2x$ for all $x \in \mathbb{Z}$ and $A = (\mu_A, \nu_A)$ be an ILF subspace of $\mathbb{Z}$ such that
$\mu_A(x) = \begin{cases} 1\ ,\ x \in 2\mathbb{Z} \\ 1/3\ ,\ x \in 2\mathbb{Z}+1 \end{cases}$ and $\nu_A(x) = \begin{cases} 0\ ,\ x \in 2\mathbb{Z} \\ 1/2\ ,\ x \in 2\mathbb{Z}+1 \end{cases}$

Then $f(A) = (\hat{\mu}, \hat{\nu})$ is definded such that $\hat{\mu}(y) = \begin{cases} 1\ ,\ y \in 4\mathbb{Z} \\ 1/3\ ,\ y \in 4\mathbb{Z}+2 \end{cases}$

and $\hat{\nu}(y) = \begin{cases} 0\ ,\ y \in 4\mathbb{Z} \\ 1/2\ ,\ y \in 4\mathbb{Z}+2 \end{cases}$
which is not an ILF subspace of $\mathbb{Z}$. For instance, if we choose two different variables $y,\ z \in 4\mathbb{Z}+2$ obviously $\hat{\mu}(y+z) \not\leq \hat{\mu}(y) \wedge \hat{\mu}(z)$.

Remark 4.8. In case $f : X_1 \to X_2$ be a algebra homomorphism and $A = (\mu_A, \nu_A)$ be an ILF ideal [resp. ILF subalgebra] of X_1, $f(A)$ is not necessarily an ILF ideal (ILF subalgebra) of X_2.

Theorem 4.9. Let F be a field and $f : X_1 \longrightarrow X_2$ be an algebra homomorphism. If $B = (\mu_B, \nu_B)$ is an ILF subalgebra of X_2, then $\tilde{f}^{-1}(B) = (\tilde{\mu}_B^{-1}, \tilde{\nu}_B^{-1})$ is also an ILF subalgebra of X_1.

Proof. By Definition 2.16, for all $k \in F$ and $x,\ y \in X_1$ we have
(i)$\tilde{\mu}_B^{-1}(x+y) = \mu_B(f(x+y)) = \mu_B(f(x)+f(y)) \geq \mu_B(f(x)) \wedge \mu_B(f(y))$

$$\geq \tilde{\mu}_B^{-1}(x) \wedge \tilde{\mu}_B^{-1}(y),$$

(ii)$\tilde{\mu}_B^{-1}(kx) = \mu_B(f(kx)) = \mu_B(kf(x)) \geq \mu_B(f(x)) = \tilde{\mu}_B^{-1}(x)$,
(iii)$\tilde{\mu}_B^{-1}(x.y) = \mu_B(f(x.y)) = \mu_B(f(x).f(y)) \geq \mu_B(f(x)) \wedge \mu_B(f(y))$

$$\geq \tilde{\mu}_B^{-1}(x) \wedge \tilde{\mu}_B^{-1}(y)$$

(iv) $\tilde{\nu}_B^{-1}(x+y) = \nu_B(f(x+y)) = \nu_B(f(x)+f(y)) \leq \nu_B(f(x)) \vee \nu_B(f(y))$

$$\leq \tilde{\nu}_B^{-1}(x) \vee \tilde{\nu}_B^{-1}(y),$$

(v)$\tilde{\nu}_B^{-1}(kx) = \nu_B(f(kx)) = \nu_B(kf(x)) \leq \nu_B(f(x)) = \tilde{\nu}_B^{-1}(x)$,
(vi)$\tilde{\nu}_B^{-1}(x.y) = \nu_B(f(x.y)) = \nu_B(f(x).f(y)) \leq \nu_B(f(x)) \vee \nu_B(f(y))$

$$\leq \tilde{\nu}_B^{-1}(x) \vee \tilde{\nu}_B^{-1}(y).$$

□

Theorem 4.10. Let F be a field and $f : X_1 \longrightarrow X_2$ be an algebra homomorphism. If $B = (\mu_B, \nu_B)$ is an ILF ideal of X_2 , then $\tilde{f}^{-1}(B) = (\tilde{\mu}_B^{-1}, \tilde{\nu}_B^{-1})$ is also an ILF ideal of X_1.

Proof. By Definition 2.16, for all $k \in F$ and $x,\ y \in X_1$ we have
(i)$\tilde{\mu}_B^{-1}(x+y) = \mu_B(f(x+y)) = \mu_B(f(x)+f(y)) \geq \mu_B(f(x)) \wedge \mu_B(f(y))$

$$\geq \tilde{\mu}_B^{-1}(x) \wedge \tilde{\mu}_B^{-1}(y),$$

(ii)$\tilde{\mu}_B^{-1}(kx) = \mu_B(f(kx)) = \mu_B(kf(x)) \geq \mu_B(f(x)) = \tilde{\mu}_B^{-1}(x)$.
(iii)$\tilde{\mu}_B^{-1}(x.y) = \mu_B(f(x.y)) = \mu_B(f(x).f(y)) \geq \mu_B(f(x)) \vee \mu_B(f(y))$

$$\geq \tilde{\mu}_B^{-1}(x) \vee \tilde{\mu}_B^{-1}(y)$$

(iv) $\tilde{\nu}_B^{-1}(x+y) = \nu_B(f(x+y)) = \nu_B(f(x)+f(y)) \leq \nu_B(f(x)) \vee \nu_B(f(y))$

$$\leq \tilde{\nu}_B^{-1}(x) \vee \tilde{\nu}_B^{-1}(y),$$

$(v)\tilde{\nu}_B^{-1}(kx) = \nu_B(f(kx)) = \nu_B(kf(x)) \leq \nu_B(f(x)) = \tilde{\nu}_B^{-1}(x),$
(vi)$\tilde{\nu}_B^{-1}(x.y) = \nu_B(f(x.y)) = \nu_B(f(x).f(y)) \leq \nu_B(f(x)) \wedge \nu_B(f(y))$

$$\leq \tilde{\nu}_B^{-1}(x) \wedge \tilde{\nu}_B^{-1}(y)$$

□

Proposition 4.11. Let $A_1,\ A_2, ..., A_n$ be ILFSs of X and $\lambda_1,\ \lambda_2, ..., \lambda_n$ be scalares. The following assertions are equivalent.
(i)$\lambda_1 A_1 + \lambda_2 A_2 + ...\lambda_n A_n \subseteq A$.
(ii) For all $x_1,\ x_2, ..., x_n \in X$, we have

$$\mu_A(\lambda_1 x_1 + ... + \lambda_n x_n) \geq min\{\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)\}.$$

Proof. $(i) \Rightarrow (ii)$

$$\begin{aligned}\mu_A(\lambda_1 x_1 + ... + \lambda_n x_n) &\geq \mu_{\lambda_1 A_1 + ... + \lambda_n A_n}(\lambda_1 x_1 + ... + \lambda_n x_n)\\ &\geq min\{\mu_{\lambda_1 A_1}(\lambda_1 x_1), ..., \mu_{\lambda_n A_n}(\lambda_n x_n)\}\\ &\geq min\{\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)\}.\end{aligned}$$

and

$$\begin{aligned}\nu_A(\lambda_1 x_1 + ... + \lambda_n x_n) &\leq \nu_{\lambda_1 A_1 + ... + \lambda_n A_n}(\lambda_1 x_1 + ... + \lambda_n x_n)\\ &\leq max\{\nu_{\lambda_1 A_1}(\lambda_1 x_1), ..., \nu_{\lambda_n A_n}(\lambda_n x_n)\}\\ &\leq max\{\nu_{A_1}(x_1), ..., \nu_{A_n}(x_n)\}.\end{aligned}$$

$(ii) \Rightarrow (i)$ By rearing the order if necessary, we may assume that $\lambda_1 \neq 0$ for $i = 1, ..., k$ and $\lambda_i = 0$ for $k < i \leq n$. Let $x_1, ..., x_k$ be elements of X.

$$\mu_A(\lambda_1 x_1 + ... + \lambda_k x_k) \geq min\{\mu_{A_1}(x_1), ..., \mu_{A_k}(x_k), \mu_{A_{k+1}}(y_1), ..., \mu_{A_n}(y_{n-k})\}.$$

Since $\mu_{0A_j}(0) = sup_{y\in X}\mu_{A_j}(y)$, we get

$$\mu_A(\lambda_1 x_1 + ... + \lambda_k x_k) \geq min\{\mu_{A_1}(x_1), ..., \mu_{A_k}(x_k), \mu_{0A_{\alpha+1}}(0), ..., \mu_{0A_n}(0)\}.$$

Now

$$\mu_{\lambda_1 A_1 + \lambda_n A_n}(z) = sup_{x_1 + ... + x_k = z}[min\ \mu_{\lambda_1 A_1}(x_1), ..., \mu_{\lambda_n A_n}(x_n)]$$
$$= sup_{x_1 + ... + x_k = z}[min\ \mu_{\lambda_1 A_1}(x_1), ..., \mu_{\lambda_k A_k}(x_k), \mu_{0A_{k+1}}(0), ..., \mu_{0A_n}(0)]$$
$$= sup_{x_1 + ... + x_k = z}[min\ \mu_{A_1}((1/\lambda_1)x_1), ..., \mu_{A_k}((1/\lambda_k)x_k), \mu_{0A_{k+1}}(0), ..., \mu_{0A_n}(0)\}]$$
$$\leq sup_{x_1 + ... + x_k = z}\ \mu_A(\lambda_1(1/\lambda_1)x_1 + ... + \lambda_k(1/\lambda_k)x_k) = \mu_A(z).$$

Also

$$\nu_A(\lambda_1 x_1 + ... + \lambda_k x_k) \leq max\{\nu_{A_1}(x_1), ..., \nu_{A_k}(x_k), \nu_{A_{k+1}}(y_1), ..., \nu_{A_n}(y_{n-k})\}.$$

Since $\nu_{0A_j}(0) = inf_{y\in X}\nu_{A_j}(y)$, we get

$\nu_A(\lambda_1 x_1 + ... + \lambda_k x_k) \leq max\{\nu_{A_1}(x_1), ..., \nu_{A_k}(x_k), \nu_{0A_{\alpha+1}}(0), ..., \nu_{0A_n}(0)\}$.

Now

$$\nu_{\lambda_1 A_1 + \lambda_n A_n}(z) = sup_{x_1+...+x_k=z}[min\ \nu_{\lambda_1 A_1}(x_1), ..., \nu_{\lambda_n A_n}(x_n)]$$
$$= sup_{x_1+...+x_k=z}[min\ \nu_{\lambda_1 A_1}(x_1), ..., \nu_{\lambda_k A_k}(x_k), \nu_{0A_{k+1}}(0), ..., \mu_{0A_n}(0)]$$
$$= inf_{x_1+...+x_k=z}[max\ \nu_{A_1}((1/\lambda_1)x_1), ..., \nu_{A_k}((1/\lambda_k)x_k), \nu_{0A_{k+1}}(0), ..., \nu_{0A_n}(0)\}]$$
$$\geq inf_{x_1+...+x_k=z}\ \nu_A(\lambda_1(1/\lambda_1)x_1 + ... + \lambda_k(1/\lambda_k)x_k) = \nu_A(z).$$

□

Lemma 4.12. Let $A = (\mu_A, \nu_A)$ is an ILF set of X. Then the following are equivalent:
(i)A is an ILF subspace of X;
(ii)For all scalars $k, m \in F$, we have $kA + mA \subset A$;
(iii)For all scalars $k, m \in F$ and all $x,\ y \in X$, we have: $\mu_A(kx + my) \leq min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(kx + my) \geq max\{\mu_A(x), \mu_A(y)\}$.

Proof. Clearly, we have $1 \to 2$ holds. Also (2) and (3) are equivalent by Proposition 4.11.
$(ii) \to (i)$

$$\mu_A + \mu_A = 1\mu_A + 1\mu_A \subset \mu_A,$$

and

$$\nu_A + \nu_A = 1\nu_A + 1\nu_A \supset \nu_A;$$

also

$$k\mu_A = k\mu_A + 0\mu_A \subset \mu_A,$$

and

$$k\nu_A = k\nu_A + 0\nu_A \supset \nu_A,$$

□

Proposition 4.13. Let f be a linear map from X_1 into X_2. If $A = (\mu_A, \nu_A)$ is an ILF subspace of X_1, then $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ is an ILF subspace of X_2. Similary, $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ is an ILF subspace of X_2 whenever $B = (\mu_B, \nu_B)$ is an ILF subspace of X_2.

Proof. For k, m scalars we have

$$k\mu_{f(A)} + m\mu_{fA} = f(k\mu_A + m\mu_A) \subset \mu_{fA},$$

and

$$k\nu_{fA} + m\nu_{fA} = f(k\nu_A + m\nu_A) \supset \nu_{fA},$$

which shows f(A) is an ILF subspace of F. Also,

$$\mu_{f^{-1}(B)}(kx + my) = \mu_B(f(kx + my)) = \mu_B(kf(x) + mf(y))$$

$$\geq min\{\mu_B f(x), \mu_B f(y)\}$$
$$= min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\},$$

and

$$\nu_{f^{-1}(B)}(kx{+}my) = \nu_B(f(kx{+}my)) = \nu_B(f(kx{+}my)) = \nu_B(kf(x){+}mf(y))$$
$$\leq min\{\nu_B f(x), \nu_B f(y)\}$$
$$= min\{\nu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(y)\}$$

. Hence $f^{-1}(B)$ is an ILF subspace by Lemma 4.12. □

Theorem 4.14. Let $f\ : X_1 \longrightarrow X_2$ be a surjective algebra homomorphism. If $A = (\mu_A, \nu_A)$ is an ILF subalgebra of X_1, then $\tilde{f}(A)$ is also an ILF subalgebra of X_2.

Proof. Since $f(0) = 0$, $\mu_A(0) = 1$ and $\nu_A(0) = 0$, it is clear that $\tilde{f}(A)(0) = 1$ and $\tilde{f}(\nu_A)(0) = 0$. By proposition 4.13 $\tilde{f}(A)$ is an ILF subspace of X_2.

Let $x,\ y \in X_2$. it is enough to show that $\tilde{f}(A)(x.y) \geq \tilde{f}(A)(x) \wedge \tilde{f}(A)(y)$. If $x.y \in f(X_1)$, assume that$\tilde{f}(\mu_A)(x.y) < \tilde{f}(\mu_A)(x) \wedge \tilde{f}(\mu_A)(y)$. Then $\tilde{f}(\mu_A)(x.y) < \tilde{f}(\mu_A)(x)$ or $\tilde{f}(\mu_A)(x.y) < \tilde{f}(\mu_A)(y)$.

We can choose a number $t \in [0, 1]$ such that $\tilde{f}(\mu)(x.y) < t < \tilde{f}(\mu_A)(x)$ and $\tilde{f}(\nu_A)(x.y) < t < \tilde{f}(\nu_A)(x)$. There exist an $a \in f^{-1}(x) \subseteq X_1$, $b \in f^{-1}(y) \subseteq X_1$ such that $\mu_A(a) > t$ and $\mu_A(b) > t$.

Since $f(a.b) = f(a).f(b) = x.y$, we have $f^{-1}(x.y) \neq \oslash$, and $\tilde{f}(\mu_A)(x.y) = sup\{\mu_A(z)\ \ : z \in f^{-1}(x.y)\} \geq \mu_A(a.b) \geq \mu_A(a) > t > \tilde{f}(\mu_A)(x.y)$. That is a contradiction. Similary, if $\nu_{\tilde{f}(A)}(x.y) \geq \nu_{\tilde{f}(A)}(x) \wedge \nu_{\tilde{f}(A)}(y)$, then we get a contradiction. Hence, $\tilde{f}(A)$ is an ILF subalgebra of X_2. □

Theorem 4.15. Let $f\ : X_1 \longrightarrow X_2$ be a surjective algebra homomorphism. If $A = (\mu_A, \nu_A)$ is an ILF ideal of X_1, then $\tilde{f}(A)$ is also an ILF ideal of X_2.

Proof. Since $f(0) = 0$, $\mu_A(0) = 1$ and $\nu_A(0) = 0$, it is clear that $\tilde{f}(A)(0) = 1$ and $\tilde{f}(\nu_A)(0) = 1$. By proposition 4.13 $\tilde{f}(A)$ is an ILF subspace. Let $x,\ y \in X_2$. it is enough to show that $\tilde{f}(A)(x.y) \geq \tilde{f}(A)(x) \vee \tilde{f}(A)(y)$. Assume that$\tilde{f}(\mu_A)(x.y) < \tilde{f}(\mu_A)(x) \vee \tilde{f}(\mu_A)(y)$. Then $\tilde{f}(\mu_A)(x.y) < \tilde{f}(\mu_A)(x)$ or $\tilde{f}(\mu_A)(x.y) < \tilde{f}(\mu_A)(y)$ and $\tilde{f}(\nu_A)(x.y) > \tilde{f}(\nu_A)(x)$ or $\tilde{f}(\nu_A)(x.y) > \tilde{f}(\nu_A)(y)$. Without loss of generality, we can choose a number $t \in [0, 1]$ such that $\tilde{f}(\mu_A)(x.y) < t < \tilde{f}(\mu_A)(x)$ and $\tilde{f}(\nu_A)(x.y) > t > \tilde{f}(\nu_A)(x)$. There exist an $a \in f^{-1}(x) \subseteq X_1$ such that $\mu_A(a) > t$ and $\nu_A(a) < t$. Since f is surjective, there exist $b \in X_1$ such that $A(b) = y$. Since $f(a.b) = f(a).f(b) = x.y$, we have $f^{-1}(x.y) \neq \oslash$, and $\tilde{f}(\mu_A)(x.y) = sup\{\mu_A(z)\ \ : z \in f^{-1}(x.y)\} \geq \mu_A(a.b) \geq \mu_A(a) > t >$

$\tilde{f}(\mu_A)(x.y)$. That is a contradiction. Similary we can prove the other case. Hence, $\tilde{f}(A)$ is an ILF ideal in X_2. □

Theorem 4.16. (i) Let $A = (\mu_A, \nu_A)$ be any ILF of a ring R and let $t = A(0)$. Then the ILF subset A^* of R/A_t, defined by $A^*(x+A_t) = A(x)$ for all $x \in R$, is an ILF ideal of R/A_t.
(ii) If $A = (\mu_A, \nu_A)$ is an ILF ideal of R and θ is an ILF ideal of R/A such that $\theta(x + A) = \theta(A)$ only when $x \in A$, then there exists an ILF ideal $A = (\mu_A, \nu_A)$ of R such that $A_t = A$, where $t = A(0)$ and $\theta = A^*$.
Proof. (i) Since A is an ILF ideal of R, A_i is an ILF ideal of R, too. Now A^* is well defined because $x + \mu_A = y + \mu_A$ where $x, y \in R \to x - y \in \mu_t$ $\to \mu_A(x - y) = \mu_A(0) \to \mu_A(x) = \mu_A(y) \to \mu_A^*(x + \mu_t) = \mu_A^*(y + \mu_t)$.
Next, we show that μ_A^* is an ILF ideal of R. To this end, for any $x, y \in R$, we have $\mu_A^*((x + \mu_t) - (y + \mu_t)) = \mu_A^*((x - y) + \mu_t) = \mu_A(x - y) \geq min(\mu_A(x), \mu_A(y)) = min(\mu_A^*(x + \mu_t), \mu_A^*(x + \mu_t))$, and $\mu_A((x + \mu_t)(y + \mu_t)) = \mu_A^*(xy + \mu_t) = \mu_A(xy)$ Also for every $x, y \in R$, $x + \nu_A = y + \nu_A \to x - y \in \nu_t \to \nu_A(x - y) = \nu_A(0) \to \nu_A(x) = \nu_A(y) \to \nu_A^*(x + \nu_t) = \nu_A^*(y + \nu_t)$. Next, we show that ν_A^* is an ILF ideal of R. To this end, for any $x, y \in R$, we have $\nu_A^*((x+\nu_t)-(y+\nu_t)) = \nu_A^*((x-y)+\nu_t) = \nu_A(x-y)$ $\leq max(\nu_A(x), \nu_A(y)) = max(\nu_A^*(x+\nu_t), \nu_A^*(x+\nu_t))$ and $\nu_A((x+\nu_t)(y+\nu_t)) = \nu_A^*(xy + \nu_t) = \nu_A(xy)$.
(ii) Define an ILF ideal A of R by $A(x) = \theta(x + A)$ for all $x \in R$. A routine computation shows that A is an ILF ideals of R. Also, $A_t = A$, since $x \in \mu_t \leftrightarrow \mu_A(x) = \mu_A(0) \leftrightarrow \theta(x + A) = \theta(A), x \in A$. Finally $\mu_A^* = \theta$, because $\mu_A^*(x + A) = \mu_A^*(x + \mu_t) = \mu_A(x) = \theta(x + A)$ and since $x \in \nu_t \leftrightarrow \nu_A(sx) = \nu_A(0) \leftrightarrow \theta(x + A) = \theta(A), x \in A$. Finally $\nu_A^* = \theta$, because $\nu_A^*(x + A) = \nu_A^*(x + \nu_t) = \nu_A(x) = \theta(x + A)$. So there exists an ILF ideal $A = (\mu_A, \nu_A)$ of R such that $A_t = A$, where $t = A(0)$; and $\theta = A^*$. □

Theorem 4.17. Let $f : X_1 \longrightarrow X_2$ be an algebra homomorphism, then (i) If $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are ILF subalgebras of X_1, then $\tilde{f}(A \oplus B) = \tilde{f}(A) \oplus \tilde{f}(B)$.
(ii) If $\{A_i : i \in I\}$ be a set of ILF subalgebras of X_1 , then $\tilde{f}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \tilde{f}(A_i)$.
(iii) If $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are ILF subalgebras of X_1, then $\tilde{f}(A \otimes B) = \tilde{f}(A) \otimes \tilde{f}(B)$.
Proof. (i) and (ii) can be proved by the same method as the proof of Theorem 4.16. It is sufficient to prove (iii). Let $x \in X_2$. We prove that $\mu_{\tilde{f}(A\otimes B)}(x) = (\mu_{\tilde{f}(A)} \otimes \mu_{\tilde{f}(B)})(x)$ and $(\nu_{\tilde{f}(A\otimes B)})(x) = (\nu_{\tilde{f}(A)})(x) \otimes (\nu_{\tilde{f}(B)})(x)$.
If $x = y.z \in \tilde{f}(X_1)$, we have $y \notin \tilde{f}(X_1)$ or $z \notin \tilde{f}(X_1)$. By the proof of theorem 4.16 we get $\tilde{f}(\mu_A \oplus \mu_B)(x) = 0$ and $(\tilde{f}(\mu_A) \otimes \tilde{f}(\mu_B))(x) = \tilde{f}(\mu_A)(x) \otimes \tilde{f}(\mu_B)(x) = sup\{\tilde{f}(\mu_A)(y) \wedge \tilde{f}(\mu_B)(z) \ : x = y\} = 0$. and

$\tilde{f}(\nu_A \oplus \mu_B)(x) = 1$ and $(\tilde{f}(\nu_A) \otimes \tilde{f}(\nu_B))(x) = \tilde{f}(\nu_A)(x) \otimes \tilde{f}(\nu_B)(x) = inf\{\tilde{f}(\nu_A)(y) \vee \tilde{f}(\nu_B)(z) \ : x = y.\} = 1$. Let $x = y.z \notin f(X_1)$ or $\tilde{f}(\mu_A \otimes \mu_B)(x) < \tilde{f}(\mu_A)(x) \otimes \tilde{f}(\mu_B)(x)$ or $\tilde{f}(\nu_A \otimes \nu_B)(x) > \tilde{f}(\nu_A)(x) \otimes \tilde{f}(\nu_B)(x)$. We can choose an element $t \in L$ such that $\tilde{f}(\mu_A \otimes \mu_B)(x) < t < \tilde{f}(\mu_A)(x) \otimes (\mu_B)(x)$ or $\tilde{f}(\nu_A \otimes \nu_B)(x) > t > \tilde{f}(\nu_A)(x) \otimes (\nu_B)(x)$, respectively. Since $\tilde{f}(\mu_A)(x) \otimes \tilde{f}(\mu_B)(x) = sup\{\tilde{f}(\mu_A)(y) \wedge \tilde{f}(\mu_B)(z) \ : x = y.z\}$ and $\tilde{f}(\nu_A)(x) \otimes \tilde{f}(\nu_B)(x) = inf\{\tilde{f}(\nu_A)(y) \vee \tilde{f}(\nu_B)(z) \ : x = y.z\}$, there exist $y, z \in X_2$ such that $x = y.z$ with $\tilde{f}(\mu_A)(y) > t$, $\tilde{f}(\mu_B)(z) > t$ or $\tilde{f}(\nu_A)(y) < t$, $\tilde{f}(\nu_B)(z) < t$. Since $x \in f(X_1)$, there exist an $x_1 \in X_1$ such that $f(x_1) = x$ and $x_1 = y_1.z_1$ for $y_1 \in f^{-1}(y)$, $z_1 \in f^{-1}(z)$ with $\mu_A(y_1) > t$ and $\mu_B(z_1) > t$ and $\nu_A(y_1) < t$, $\nu_B(z_1) < t$. Since $f(y_1.z_1) = f(y_1).f(z_1) = y.z = x$, we have

$$\begin{aligned} \tilde{f}(\mu_A \otimes \mu_B)(x) &= sup\{(\mu_A \otimes \mu_B)(x_1) | f(x_1 = x\} \\ &= sup\{\mu_A(a) \wedge \mu_B(b) | f(ab) = x\} \\ &\geq \mu_A(y_1) \wedge \mu_B(z_1) > t. \end{aligned}$$

or

$$\begin{aligned} \tilde{f}(\nu_A \otimes \nu_B)(x) &= inf\{(\nu_A \otimes \nu_B)(x_1) | f(x_1) = x\} \\ &= inf\{\nu_A(a) \vee \nu_B : f(x_1) = x\} \\ &\leq \nu_A(y_1) \vee \nu_B(z_1) < t, \end{aligned}$$

respectively. This is a contradiction. Similarly, for the case $\tilde{f}(A \otimes B)(x) > (\tilde{f}(A \otimes \tilde{f}(B))(x)$ or $\tilde{f}(\nu_A \otimes \nu_B)(x) < (\tilde{f}(\nu_A \otimes \tilde{f}(\nu_B))(x)$, we get a contradiction. Hence $\tilde{f}(A \otimes B) = \tilde{f}(A) \otimes \tilde{f}(B)$. □

Theorem 4.18. Let $f : X_1 \longrightarrow X_2$ be a surjective algebra homomorphism, then (i) if $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are ILF ideals of X_1 then $\tilde{f}(A \oplus B) = \tilde{f}(A) \oplus \tilde{f}(B)$.
(ii) if $\{A_i : i \in I\}$ is a set of ILF ideals of X_1, then $\tilde{f}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \tilde{f}(A_i)$.
(iii) if $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are ILF ideals of X_1, then $\tilde{f}(A \otimes B) = \tilde{f}(A) \otimes \tilde{f}(B)$.

Proof. (i) and (ii) can be proved by the same method as proof of Theorem 4.16. It is sufficient to prove (iii). Let $x \in X_2$. We prove that $\tilde{f}(\mu_A \otimes \mu_B) = (\tilde{f}(\mu_A) \otimes \tilde{f}(\mu_B))(x)$ and $\tilde{f}(\nu_A \otimes \nu_B) = (\tilde{f}(\nu_A) \otimes \tilde{f}(\nu_B))(x)$. Assume that $\tilde{f}(\mu_A \otimes \mu_B)(x) < (\tilde{f}(\mu_A) \otimes \tilde{f}(\mu_B))(x)$ and $\tilde{f}(\nu_A \otimes \nu_B)(x) > (\tilde{f}(\nu_A) \otimes \tilde{f}(\nu_B))(x)$. We can choose an element $t \in L$ such that $\tilde{f}(\mu_A)(x) \otimes \tilde{f}(\mu_B)(x) = sup\{\tilde{f}(\mu_A)(y) \wedge \tilde{f}(\mu_B)(z) : x = y.z\}$ and $t \in L$ such that $\tilde{f}(\nu_A)(x) \otimes \tilde{f}(\nu_B)(x) = inf\{\tilde{f}(\nu_A)(y) \vee \tilde{f}(\nu_B)(z) : x = y.z\}$, there exist $y, z \in X_2$, such that $x = y.z$ with $\tilde{f}(\mu_A)(y) > t$ and $\tilde{f}(\mu_B)(z) > t$.

$\tilde{f}(\nu_A)(y) < t$ and $\tilde{f}(\nu_B)(z) < t$. Since f is surjective, there exists a $x_1 \in X_1$, such that $f(x_1) = x$ and $x_1 = y_1.z_1$ for $y_1 \in f^{-1}(y), z_1 \in f^{-1}(z)$ with $\mu_A(y_1) > t$ and $\mu_B(z_2) > t$ and $\nu_A(y_1) > t$ and $\mu_B(z_2) > t$. Since $f(y_1.z_1) = f(y_1).f(z_1) = y.z = x$, we have $\tilde{f}(\mu_A \otimes \mu_B)(x) = sup\{(\mu_A \otimes \mu_B)(x_1) : f(x_1)\} = sup\{sup\{\mu_A(a) \wedge \mu_B(b) : f(x_1) = f(a.b) = x\}\} = sup\{\mu_A(a) \wedge \mu_B(b) : f(a.b) = x\}\} \geq \mu_A(y_1) \wedge \mu_B(z_2) > t$, and $\tilde{f}(\nu_A \otimes \nu_B)(x) = inf\{(\nu_A \otimes \nu_B)(x_1) : f(x_1)\} = inf\{inf\{\nu_A(a) \vee \nu_B(b) : f(x_1) = f(a.b) = x\}\} = inf\{\nu_A(a) \vee \nu_B(b) : f(a.b) = x\}\} \leq \nu_A(y_1) \vee \nu_B(z_2) < t$. This is a contradiction. Similarly, for the case $\tilde{f}(A \otimes B)(x) > (\tilde{f}(A) \otimes \tilde{(B)})(x)$, we get a contradiction. Hence $\tilde{f}(A) \otimes \tilde{f}(B)$.

□

5. CONCLUSION

In this paper, the concept of ILFSs of Novikov algebras is introduced. ILFSs include deeper aspects of uncertainty and vagueness where traditional fuzzy sets may not fully succeed, by using membership and non-membership degrees. Furthermore, the notions of ILF ideals and ILF subalgebras are explored alongside essential and fundamental theorems and lemmas. Also key properties of ILF algebras, including operations such as intersection, sum, and product are investigated. Additionally, significant conditions, such as when an ILF subspace is an ILF ideal are discussed. Finally, homomorphisms on ILF ideals is introduced and illustrative examples are also added.

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