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# Spectral approximation of bounded linear operators on ultrametric Banach spaces

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Abstract. In this paper, we introduce and study the concepts of stable convergence and square-norm convergence of bounded linear operators on ultrametric Banach spaces. We establish a results on convergence and error estimates of operators and we prove many results related to the stable convergence, the square-convergence and the completely compact convergence on ultrametric Banach spaces. Finally, we give several examples about them.

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## 1. Introduction

In the classical setting, Anselone [4] introduced and studied the collectively compact convergence of bounded linear operators and he applied it to numerical integration approximations of integral operators. There

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are many other concepts of the convergence as the norm convergence, the pointwise convergence and the $\nu$-convergence in [3].

In ultrametric operator theory, the authors [1] extended and studied the concepts of the collectively compact convergence and the $\nu$-convergence of bounded linear operators on ultrametric Banach spaces.

Throughout this paper, $E$ and $F$ are ultrametric Banach spaces over an ultrametric complete valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|$, $\mathcal{L}(E,F)$ is the set of all bounded linear operators from $E$ into $F$ and $\mathbb{Q}_p$ denotes the field of $p$-adic numbers. For more details, see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21].

Remember that a free Banach space $E$ is an ultrametric Banach space for which there is a family $(e_i)_{i\in\mathbb{N}}$ in $E\backslash\{0\}$ such that any $u \in E$ is written as the form $u = \sum\limits_{i\in\mathbb{N}} x_i e_i$, $x_i \in \mathbb{K}$ and $\|u\| = \sup\limits_{i\in\mathbb{N}} |x_i|\|e_i\|$. The family $(e_i)_{i\in\mathbb{N}}$ is called an orthogonal basis.

In this paper, the appproximation of the solutions of the problem of the form

$$Ax = y, \tag{1.1}$$

where $A \in \mathcal{L}(E,F)$ is studied. Consider

$$A_n x_n = y_n, \tag{1.2}$$

where $A_n \in \mathcal{L}(E,F)$. We will suppose that (1.1) is well-posed i.e., $A$ is bijective and $A^{-1} : F \to E$.

## 2. Preliminaries

We begin with some preliminaries.

**Definition 2.1** ([1]). Let $E$ be an ultrametric Banach space over $\mathbb{K}$, let $A, (A_n) \in \mathcal{L}(E)$. Then

(i) The sequence $(A_n)_n$ is said to be norm convergent to $A$ denoted by $A_n \to A$ if $\lim\limits_{n\to\infty} \|A_n - A\| = 0$;

(ii) The sequence $(A_n)_n$ is said to be pointwise convergent to $A$, denoted by $A_n \xrightarrow{p} A$, if for all $x \in E$, $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$.

**Theorem 2.2** ([21]). *Let $E$ be an ultrametric Banach space over $\mathbb{K}$ and let $F$ be an ultrametric normed space over $\mathbb{K}$. If $S$ is a subset of $\mathcal{L}(E,F)$ with for each $x \in E$, the set $\{Ax : A \in S\}$ is bounded in $F$, hence $S$ is a bounded set in $\mathcal{L}(E,F)$.*

**Corollary 2.3** ([21]). *Let $E$ be an ultrametric Banach space over $\mathbb{K}$ and let $F$ be an ultrametric normed space over $\mathbb{K}$. If $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}(E,F)$ such that for each $x \in E, Ax = \lim_{n\to\infty} A_n x$ exists, then $A \in \mathcal{L}(E,F)$.*

**Definition 2.4** ([1]). Let $E$ be an ultrametric Banach space over a locally compact field $\mathbb{K}$ and let $A, A_n \in \mathcal{L}(E)$. A sequence $(A_n)_{n\in\mathbb{N}}$ is said to be convergent to $A$ in the collectively compact convergence, denoted by $A_n \overset{c.c}{\to} A$, if $A_n \overset{p}{\to} A$ and for some $N \in \mathbb{N}$,

$$\bigcup_{n\geq N} \{(A_n - A)x \ : x \in E,\ \|x\| \leq 1\}$$

has compact closure of $E$.

**Lemma 2.5** ([7]). *Let $E$ be an ultrametric Banach space over $\mathbb{K}$. Let $A \in \mathcal{L}(E)$ with $\|A\| < 1$, hence $(I - A)^{-1}$ exists and $\|(I - A)^{-1}\| \leq 1$.*

## 3. Main results

The following lemma holds.

**Lemma 3.1.** *Let $E$ be an ultrametric Banach space over $\mathbb{K}$. Let $A \in \mathcal{L}(E)$ with $\|A^k\| < 1$ for some $k \in \mathbb{N}$, hence $(I - A)^{-1}$ exists and*

$$\|(I - A)^{-1}\| \leq \frac{\|\sum_{i=0}^{k-1} A^i\|}{1 - \|A^k\|}.$$

*Proof.* Since $I - A^k = (\sum_{i=0}^{k-1} A^i)(I - A)$, using Lemma 2.5, one can get the result. $\square$

We continue by the following results.

**Definition 3.2.** Let $E$ and $F$ be two ultrametric Banach spaces over $\mathbb{K}$, let $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}(E, F)$, $(A_n)_{n\in\mathbb{N}}$ is said to be stable with index $N \in \mathbb{N}$ if

(i) $(\|A_n\|)_{n\in\mathbb{N}}$ is bounded;
(ii) For all $n \geq N$, $A_n$ is invertible;
(iii) $\{\|A_n^{-1}\| : n \geq N\}$ is bounded.

**Proposition 3.3.** *Let $E$ and $F$ be two ultrametric Banach spaces over $\mathbb{K}$, let $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}(E, F)$. Assume that there is $N \in \mathbb{N}$ with for each $n \geq N$, $A_n$ is invertible and $A_n \overset{p}{\to} A$. We have:*
*(i) If $\{\|A_n^{-1}\| : n \geq N\}$ is bounded, hence $A$ is injective;*
*(ii) If $(A_n^{-1}y)_n$ (with $y \in F$) converges, hence $y \in R(A)$;*
*In particular, if $(A_n)_{n\in\mathbb{N}}$ is stable and for each $x \in E$, $A_n x \to Ax$ as $n \to \infty$ and $(A_n^{-1}y)_n$ converges for any $y \in F$, hence (1.1) is well-posed.*

*Proof.*
(i) Let $x \in E$ with $Ax = 0$, we shall prove that $x = 0$. Since $\{\|A_n^{-1}\| : n \geq N\}$ is bounded i.e., there is $C > 0$ with $n \geq N$, $\|A_n^{-1}\| \leq C$. Then

$$\|x\| = \|A_n^{-1}A_n x\| = \|A_n^{-1}(A_n x - Ax)\| \leq C\|(A_n x - Ax)\| \to 0,$$

as $n \to \infty$, thus $A$ is injective.

(ii) Since for each $x \in X, A_n x \to Ax$ as $n \to \infty$. Hence, there is $M > 0$ with for all $n \in \mathbb{N}, \|A_n\| \leq M$. Set $x = \lim_{n\to\infty} A_n^{-1} y$, hence

$$\|y - A_n x\| = \|A_n(A_n^{-1} y - x)\| \leq M\|A_n^{-1} y - x\| \to 0$$

as $n \to \infty$, then

$$\|y - Ax\| = \|y - A_n x + A_n x - Ax\| \leq \max\{\|y - A_n x\|, \|A_n x - Ax\|\} \to 0$$

as $n \to \infty$, thus $y \in R(A)$. □

We may see that if $(x_n)_n$ is a solution of (1.2), hence for each $x \in E$,

$$A_n(x - x_n) = A_n x - y_n. \tag{3.1}$$

**Theorem 3.4.** *Let $E$ and $F$ be two ultrametric Banach spaces over $\mathbb{K}$. Suppose that $(A_n)_n$ is stable with index $N \in \mathbb{N}$ in $\mathcal{L}(E, F)$. Let $(x_n)_n$ be the unique solution of (1.2) for $n \geq N$. Hence for each $x \in E$,*

$$M_1\|A_n x - y_n\| \leq \|x - x_n\| \leq M_2\|A_n x - y_n\|, \tag{3.2}$$

*where $M_1, M_2 > 0$ such that for each $n \geq N$,*

$$\|A_n\| \leq M_1^{-1} \text{ and } \|A_n^{-1}\| \leq M_2.$$

*In addition, if $x \in E, y \in F$ such that $\beta_n = \|(A_n - A)x + y - y_n\| \to 0$ as $n \to \infty$, then*

$$x_n \to x \text{ if, and only if, } Ax = y,$$

*and in this case $\beta_n = \|A_n x - y_n\|$.*

*Proof.* Let $(x_n)_n$ be the unique solution of (1.2) for $n \geq N$. From (3.1), we have

$$\|x_n - x\| = \|A_n^{-1}(A_n(x - x_n))\| \leq \|A_n^{-1}\|\|A_n x - y_n\|, \text{ for any } n \geq N,$$

and

$$\|A_n x - y_n\| = \|A_n(x - x_n)\| \leq \|A_n\|\|x - x_n\|, \text{ for all } n \geq N,$$

thus

$$M_1\|A_n x - y_n\| \leq \|x - x_n\| \leq M_2\|A_n x - y_n\|,$$

where $M_1, M_2 > 0$ such that for each $n \geq N$,

$$\|A_n\| \leq M_1^{-1} \text{ and } \|A_n^{-1}\| \leq M_2.$$

From (3.2), we get $x_n$ converges to $x$ as $n \to \infty$, if and only if $A_n x - y_n \to 0$ as $n \to \infty$. Also, we have $\beta_n \to 0$ if, and only if, $A_n x - y_n \to Ax - y$. By hypothesis, $\beta_n \to 0$ and two last equivalences, we get $x_n \to x$ if, and only if, $Ax = y$ and in this case $\beta_n = \|A_n x - y_n\|$. □

By Theorem 3.4, we have.

**Theorem 3.5.** *Let $E$ and $F$ be two ultrametric Banach spaces over $\mathbb{K}$. Assume that $A^{-1}$ exists and $(A_n)_n$ is stable with index $N \in \mathbb{N}$ in $\mathcal{L}(E,F)$. Let $x$ be the unique solution of* (1.1) *and $(x_n)_n$ be the unique solution of* (1.2) *for $n \geq N$. Assume also that*

$$y_n \to y, \text{ and } \|(A_n x - Ax)\| \to 0 \text{ as} \to \infty.$$

*Hence $\beta_n = \|A_n x - y_n\| \to 0$ as $n \to \infty$ and*

$$M_1\beta_n \leq \|x - x_n\| \leq M_2\beta_n, \text{ for all } n \geq N, \tag{3.3}$$

*where $M_1, M_2 > 0$ such that for each $n \geq N$,*

$$\|A_n\| \leq M_1^{-1} \text{ and } \|A_n^{-1}\| \leq M_2.$$

*In addition, $x_n \to x$ as $n \to \infty$.*

*Remark* 3.6. Let $E$ and $F$ be two ultrametric Banach spaces over $\mathbb{K}$ and let $A \in \mathcal{L}(E,F)$. If $B = A - \lambda I$ where $\lambda \in \mathbb{K}$ and $(A_n)_n$ converges to $A$ in $\mathcal{L}(E,F)$. Setting $B_n = A_n - \lambda I$, then

$$\|B - B_n\| = \|A - A_n\| \to 0, \text{ as } n \to \infty.$$

**Theorem 3.7.** *Let $E$ and $F$ be two ultrametric Banach spaces over $\mathbb{K}$ and let $A \in \mathcal{L}(E,F)$. Assume that $A$ is bijective and $(A_n)_n \in \mathcal{L}(E,F)$ converges in norm to $A$. Then $(A_n)_n$ is stable.*

*Proof.* Set $B_n = A_n - A$, hence $B_n A^{-1} \to 0$ as $n \to \infty$. Thus there is $N \in \mathbb{N}$ with $n \geq N$, $\|B_n A^{-1}\| \leq \frac{1}{2}$. By Lemma 2.5, $(I - B_n A^{-1})^{-1}$ exists and $\|(I - B_n A^{-1})^{-1}\| \leq 1$. Hence for each $n \geq N$, we have

$$\|A_n^{-1}\| = \|A^{-1}(I - B_n A^{-1})^{-1}\| \leq \|A^{-1}\|\|(I - B_n A^{-1})^{-1}\| \leq \|A^{-1}\|.$$

Consequently, $(A_n)_n$ is stable. □

**Definition 3.8.** Let $E$ and $F$ be two ultrametric Banach spaces over $\mathbb{K}$ and let $A, A_n \in \mathcal{L}(E,F)$. A sequence $(A_n)_n$ converges in square-norm to $A$ if $\lim\limits_{n\to\infty} \|(A_n - A)^2\| = 0$.

**Lemma 3.9.** *Let $E$ be an ultrametric Banach space over $\mathbb{K}$. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$. If $(A_n)_n$ converges in norm to $A$, then $(A_n)_n$ converges in square-norm to $A$.*

*Proof.* It suffices to apply that for all $B \in \mathcal{L}(E)$, $\|B^2\| \leq \|B\|^2$. □

*Remark* 3.10. The converse of Lemma 3.9 is not true in general.

We illustrate that by the following countre-example.

**Example 3.11.** Let $\mathbb{K} = \mathbb{Q}_p$. If for all $n \in \mathbb{N}$,

$$A_n = \begin{pmatrix} 1 & p^{-1} \\ p^n & 2 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

We have for all $n \in \mathbb{N}$,

$$(A_n - A)^2 = \begin{pmatrix} p^{n-1} & 0 \\ 0 & p^{n-1} \end{pmatrix},$$

then $\lim_{n\to\infty} \|(A_n - A)^2\| = 0$ and $\lim_{n\to\infty} \|A_n - A\| = p \neq 0$. Consequently, $(A_n)_n$ converges in square-norm to $A$ but $(A_n)_n$ is not convergent in norm to $A$.

**Theorem 3.12.** *Let $E$ be an ultrametric Banach space over $\mathbb{K}$. Let $A, (A_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$. Assume that $A$ is bijective,* $\lim_{n\to\infty} \|((A_n - A)A^{-1})^2\| = 0$ *and $(\|A_n\|)_n$ is bounded. Hence $(A_n)_{n\in\mathbb{N}}$ is stable.*

*Proof.* Since $(\|A_n\|)_n$ is bounded, hence for any $n \in \mathbb{N}$, $\|(A_n - A)A^{-1}\| \leq M$. Assume that $A$ is bijective and $\lim_{n\to\infty} \|((A_n - A)A^{-1})^2\| = 0$, then there is $N \in \mathbb{N}$ with for each $n \geq N$, $\|((A_n - A)A^{-1})^2\| \leq \frac{1}{2}$. By Lemma 3.1, $(A_n)_{n\in\mathbb{N}}$ is bijective and

$$\|A_n^{-1}\| \leq \frac{\|A^{-1}\|(1 + \|(A_n - A)A^{-1}\|)}{1 - \|((A_n - A)A^{-1})^2\|} \leq 2(1+M)\|A^{-1}\|, \text{ for all } n \geq N.$$

□

Consider the following condtions:

$(C_1)$ $\|(A - A_n)A\| \to 0$;
$(C_2)$ $\|(A - A_n)A_n\| \to 0$;
$(C_3)$ $\|(A - A_n)^2\| \to 0$.

Note that for each $A, (A_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$,

$$(A - A_n)A = (A - A_n)A_n + (A - A_n)^2. \tag{3.4}$$

**Proposition 3.13.** *Let $E$ be an ultrametric Banach space over $\mathbb{K}$. Let $A, (A_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$. Any two of $(C_1)$-$(C_3)$ imply the third.*

*Proof.* It suffices to use (3.4) and hypothesis. □

**Proposition 3.14.** *Let $E$ be an ultrametric Banach space of countable type over $\mathbb{K}$. Assume that $(P_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$ is a sequence of projections with $(\|P_n\|)_{n\in\mathbb{N}}$ is bounded and $\|(I - P_n)A\| \to 0$ as $n \to \infty$. Let*

$$A_n \in \{P_nA, AP_n, P_nAP_n\}.$$

*Then $(A_n)_{n\in\mathbb{N}}$ satisfies $(C_1)$-$(C_3)$.*

*Proof.* Obvious. □

**Proposition 3.15.** *Let $E$ be an ultrametric Banach space over a locally compact field $\mathbb{K}$. Let $(A_n)_{n\in\mathbb{N}}, A \in \mathcal{L}(E)$ such that $A_n \stackrel{p}{\to} A$ and $S \subset E$ with $cl(S)$ is compact. Hence*

$$\lim_{n\to\infty} \sup_{x\in S} \|A_n x - Ax\| = 0.$$

*In particular, if $B \in \mathcal{L}(X)$ is compact, hence*

$$\lim_{n\to\infty} \|(A_n - A)B\| = 0.$$

*Proof.* From Theorem 2.2, there is $C > 0$ such that for each $n \in \mathbb{N}, \|A_n\| \le C$. Let $\varepsilon > 0$ be given. Since $cl(S)$ is compact, there exist $x_1, \cdots, x_k$ in $S$ such that

$$S \subseteq \bigcup_{i=1}^{k} \{x \in E : \|x - x_i\| < \varepsilon\}.$$

For $i \in \{1, \cdots, k\}$, let $N_i \in \mathbb{N}$ with for any $n > N_i$, $\|A_n x_i - Ax_i\| < \varepsilon$. Let $x \in S$ and let $j \in \{1, \cdots, k\}$ with $\|x_j - x\| < \varepsilon$. Hence for all $n \ge N$ where $N = \max\{N_i : i = 1, \cdots, k\}$, we get

$$\begin{aligned}
\|A_n x - Ax\| &= \|A_n x - A_n x_i + A_n x_i - Ax_i + Ax_i - Ax\| \\
&\le \max\{\|A_n x - A_n x_i\|, \|A_n x_i - Ax_i\|, \|Ax_i - Ax\|\} \\
&\le \max\{\|A_n\|\|x - x_i\|, \|A_n x_i - Ax_i\|, \|A\|\|x_i - x\|\} \\
&\le \max\{C, 1, \|A\|\}\varepsilon.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \sup_{x\in S} \|A_n x - Ax\| = 0.$$

From $S = \{Bx : \|x\| \le 1\}$, we get the particular case. □

**Theorem 3.16.** *Let $E$ be an ultrametric Banach space over a locally compact field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}(X)$ such that $A_n \stackrel{cc}{\to} A$, then $(C_3)$ is satisfied. Furthermore if $A$ is compact, then $(C_1)$-$(C_2)$ are also satisfied.*

*Proof.* From $A_n \stackrel{cc}{\to} A$, we get for any $x \in X, A_n x \to Ax$ and there is $N \in \mathbb{N}$ with $n \ge N$, the collection

$$S = \bigcup_{n\ge N} \{(A_n - A)x : x \in E, \|x\| \le 1\},$$

has a compact closure. Then by Proposition 3.15,

$$\|(A_n - A)^2\| \le \sup_{x\in S} \|(A_n - A)x\| \to 0 \quad \text{as} \quad n \to 0.$$

Since $A$ is compact, by Proposition 3.15, then $\lim_{n\to\infty} \|(A_n - A)A\| = 0$ and $\lim_{n\to\infty} \|(A_n - A)A_n\| = 0$. $\square$

Put

$(C_0)$ $\|[(A_n - A)(\lambda I - A)^{-1}]^2\| \to 0$ for all $\lambda \in \rho(A)$. We get

**Theorem 3.17.** *Let $E$ be an ultrametric Banach space over $\mathbb{K}$. Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$ with $\|(A_n)\|_{n\in\mathbb{N}}$ is bounbed. Hence any two of the conditions $(C_1)$-$(C_3)$ imply $(C_0)$.*

*Proof.* By Proposition 3.13, it is enough to show that conditions $(C_1)$ and $(C_3)$ imply $(C_0)$. Then for all $0 \neq \lambda \in \rho(A)$, we get

$$(\lambda I - A)^{-1} = \frac{1}{\lambda}(I + A(\lambda I - A)^{-1}).$$

Setting $R(\lambda, A) = (\lambda I - A)^{-1}$ and $S(\lambda, A) = I + A(\lambda I - A)^{-1}$, we get

$$\begin{aligned} \lambda^2[(A - A_n)R(\lambda, A)]^2 &= [(A - A_n)S(\lambda, A)]^2 \\ &= [(A - A_n)^2 + (A - A_n)AR(\lambda, A)(A - A_n)]S(\lambda, A). \end{aligned}$$

Since the conditions $(C_1)$ and $(C_3)$ hold and $(\|(A_n)\|)_{n\in\mathbb{N}}$ is bounbed, hence $(C_0)$ is satisfied. $\square$

**Definition 3.18.** Let $A, (A_n)_n \in \mathcal{L}(E)$, then $\lambda - A_n$ is a square-approximation of $A$ if for any $\lambda \in \rho(A), \lim_{n\to\infty} \|[(A_n - A)(\lambda I - A)^{-1}]^2\| = 0$ and is denoted by $\lambda I - A_n \overset{sq}{\to} \lambda I - A$.

From Theorem 3.17, we get.

**Theorem 3.19.** *Let $E$ be an ultrametric Banach space over a locally compact field $\mathbb{K}$ such that $\|E\| \subseteq |\mathbb{K}|$. Let $(A_n)_{n\in\mathbb{N}}, A \in \mathcal{L}(E)$ with $A_n \overset{p}{\to} A$ and $A$ is compact and $\lambda \notin \sigma(A)$. If $\|(A - A_n)^2\| \to 0$ as $n \to \infty$, then $\lambda I - A_n$ is a square-approximation of $\lambda I - A$.*

*Proof.* Since $A_n \overset{p}{\to} A$. By Theorem 2.2, $(\|A_n\|)_n$ is bounded and by the condition $A$ is compact and Theorem 3.15, we get $\lim_{n\to\infty} \|(A - A_n)A\| = 0$. From $\|(A - A_n)^2\| \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} \|(A - A_n)A\| = 0$ and Theorem 3.17, we get $\lambda I - A_n$ is a square-approximation of $A$ $\square$

**Theorem 3.20.** *Let $E$ be an ultrametric Banach space over $\mathbb{K}$, let $(A_n)_n, A \in \mathcal{L}(X)$. If any two of the conditions $(C_1)$-$(C_3)$ are satisfied, hence there is $N \in \mathbb{N}$, we have*

$$\textit{for each } n \geq N,\ \sigma(A_n) \subset \sigma(A).$$

*Proof.* Let $\lambda \in \rho(A)$, by Theorem 3.13, we suppose the conditions $(C_1)$ and $(C_3)$, i.e., $\lim_{n\to\infty} \|(A_n - A)A_n\| = 0$ and $\lim_{n\to\infty} \|(A_n - A)^2\| = 0$. Put for $n \in \mathbb{N}$,

$$C_n = \lambda I - (A - A_n),\ D_n = \lambda I + (\lambda I - A)^{-1}(A - A_n)A_n.$$

One can see that

$$C_n(\lambda I - A_n) = (\lambda I - A)D_n. \tag{3.5}$$

Let $0 < M < 1$ and $N \in \mathbb{N}$ such that

$$\|(A_n - A)^2\| \leq (M|\lambda|)^2$$

and

$$\|(A_n - A)A_n\| \leq \frac{M|\lambda|}{\|(\lambda I - A)^{-1}\|}$$

for each $n \geq N$. By Lemmas 2.5 and 3.1, $C_n$ and $D_n$ are invertible for any $n \geq N$. Using (3.5), we get for all $n \geq N$, $(\lambda I - A_n)^{-1} = B_n^{-1}(\lambda I - A)^{-1}A_n$, thus $\lambda \in \rho(A_n)$ for all $n \geq N$. ☐

**Theorem 3.21.** *Let $E$ and $F$ be two free Banach spaces over $\mathbb{K}$. Assume that $(A_n)_n$ is stable with index $N \in \mathbb{N}$ in $\mathcal{L}(E, F)$. Let $(x_n)_n$ be the unique solution of (1.2) for $n \geq N$. Hence for each $x \in E$,*

$$\|x_n - x\| \leq \max\{\|x - P_n x\|, M\|A_nP_nx - y_n\|, \tag{3.6}$$

*where $M > 0$ with for each $n \geq N$,*

$$\|A_n^{-1}\| \leq M.$$

*In addition, if $x \in E, y \in F$ such that $\alpha_n = \|(A_nP_n - A)x + y - y_n\| \to 0$ and $P_nx \to x$ as $n \to \infty$, hence*

$$x_n \to x \textit{ if and only if } Ax = y,$$

*and in this case $\alpha_n = \|A_nP_nx - y_n\|$.*

*Proof.* Let $x \in E$, we get

$$x_n - x = x - P_nx + A_n^{-1}(A_nP_nx - y_n). \tag{3.7}$$

Then

$$\begin{aligned} \|x_n - x\| &= \|x - P_nx + A_n^{-1}(A_nP_nx - y_n)\| \\ &\leq \max\{\|x - P_nx\|, M\|A_nP_nx - y_n\|\}. \end{aligned}$$

If $x \in E, y \in F$ such that $\alpha_n = \|(A_nP_n - A)x + y - y_n\| \to 0$ and $P_nx \to x$ as $n \to \infty$, then

$$A_nP_n - y_n \to Ax - y, \text{ as } n \to \infty.$$

Since $(\|A_n\|)_n$ and $(\|A_n^{-1}\|)_{n\geq N}$ are bounded and from (3.7), we get $x_n \to x$ if, and only if, $Ax = y$ and in this case $\beta_n = \|A_nP_nx - y_n\|$. ☐

Similarly to the proof of Theorem 3.21, we get.

**Theorem 3.22.** *Let $E$ and $F$ be two free Banach spaces over $\mathbb{K}$. Assume that $A^{-1}$ exists and $(A_n)_n$ is stable with index $N \in \mathbb{N}$ in $\mathcal{L}(E,F)$. Let $x$ be the unique solution of* (1.1) *and $(x_n)_n$ be the unique solution of* (1.2) *for $n \geq N$. Assume also that*

$$y_n \to y, \; P_n x \to x, \quad \textit{and} \quad \|(A_n P_n x - Ax)\| \to 0 \; as \to \infty.$$

*Then*

$$\|x_n - x\| \leq \max\{\|x - P_n x\|, M\|A_n P_n x - y_n\|, \tag{3.8}$$

*where $M > 0$ with for each $n \geq N$,*

$$\|A_n^{-1}\| \leq M.$$

*In addition, $x_n \to x$ as $n \to \infty$.*

**Example 3.23.** Let $E$ and $F$ be two free Banach spaces over $\mathbb{K}$. Assume that $(P_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$ and $(Q_n)_{n\in\mathbb{N}} \in \mathcal{L}(E)$ are projections with for any $x \in E$ and $y \in R(A)$, $P_n x \to x$ and $Q_n y \to yx$ as $n \to \infty$. Then

$$\|AP_n x - Ax\| \leq \|A\|\|AP_n x - Ax\| \to 0 \quad \text{as} \quad n \to 0,$$

and

$$\|Q_n Ax - Ax\| \to 0 \quad \text{as} \quad n \to 0,$$

for all $x \in E$. Thus $(AP_n)_n$ and $(Q_n A)_n$ are pointwise approximations of $A$. Suppose that $(\|Q_n\|)_n$ is bounded, hence

$$\|Q_n AP_n - Ax\| \quad \leq \quad \max\{\|Q_n\|\|(AP_n x - Ax)\|, \|Q_n Ax - Ax\|\} \to 0,$$

as $n \to \infty$ for each $x \in E$.

**Example 3.24.** Let $E = c_0(\mathbb{Q}_p)$ and $(e_n)_{n\geq 1}$ be basis of $E$. Let $(P_n)_n$ be defined on $c_0(\mathbb{Q}_p)$ by

$$P_n x = \sum_{i=1}^{n} < x, e_i > e_i.$$

Then

(i) For all $n \in \mathbb{N}$, $(P_n)_n$ is orthogonal projection.
(ii) For each $x \in E, \|P_n x - x\| \to 0$ as $n \to \infty$.

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