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a-Roberts orthogonality in C^*-algebras and its characterization via a-numerical ranges

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Abstract. Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$ be a positive and invertible element. Set

$$\mathcal{S}_a(\mathcal{A}) = \{\frac{f}{f(a)} \, : \, f \in \mathcal{S}(\mathcal{A}),\, f(a) \neq 0\},$$

where $\mathcal{S}(\mathcal{A})$ is the set of all states on $\mathcal{A}$.

In this paper, by using the concept of algebraic a-Davies-Wielandt shell of elements of $\mathcal{A}$, we obtain a characterization of Roberts orthogonality with respect to the norm:

$$\|x\|_a = \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \sqrt{\varphi(x^*ax)} \quad (x \in \mathcal{A}),$$

in C^*-algebra $\mathcal{A}$, so called, a-Roberts orthogonality. More precisely, for any a-isometry $x \in \mathcal{A}$, we prove that x is a-Roberts orthogonal to $1_{\mathcal{A}}$ if and only if algebraic a-numerical range of x is symmetric with respect to the origin.

Keywords: C^*-algebra, state space of C^*-algebras, a-Birkhoff–James orthogonality, a-Roberts orthogonality, algebraic a-numerical range.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$. We denote by $\mathcal{A}'$ the topological dual space of $\mathcal{A}$. The adjoint of any element $x \in \mathcal{A}$ is denoted by x^*. An element a of $\mathcal{A}$ is called positive (written by $a \geq 0$), if a is selfadjoint whose spectrum $\sigma(a)$ is contained in $[0, \infty)$. A linear functional f on $\mathcal{A}$ is called positive if $f(a) \geq 0$ for every positive element $a \in \mathcal{A}$. The set of all states of $\mathcal{A}$, that is, the set of all positive linear functionals of $\mathcal{A}$ of norm 1, is denoted by $\mathcal{S}(\mathcal{A})$; cf. [12].

Let a be a positive and invertible element of $\mathcal{A}$. A generalization of state space of $\mathcal{A}$ has been introduced in [8] as follows:

$$\mathcal{S}_a(\mathcal{A}) := \{\varphi \in \mathcal{A}' : \varphi \geq 0,\, \varphi(a) = 1\} = \{\frac{f}{f(a)} : f \in \mathcal{S}(\mathcal{A}),\, f(a) \neq 0\}.$$

Observe that if $a = 1_{\mathcal{A}}$, then $\mathcal{S}_a(\mathcal{A}) = \mathcal{S}(\mathcal{A})$. It has been proved in [8] that $\mathcal{S}_a(\mathcal{A})$ is a nonempty convex and w^*-compact subset of $\mathcal{A}'$. For any element $x \in \mathcal{A}$, let

$$\|\cdot\|_a : \mathcal{A} \to [0, \infty), \quad \|x\|_a := \sup\{\sqrt{\varphi(x^*ax)} : \varphi \in \mathcal{S}_a(\mathcal{A})\}.$$

It was shown in [8] that $\|\cdot\|_a$ is a sub-multiplicative norm on $\mathcal{A}$. Consequently, $\|\cdot\|_{1_{\mathcal{A}}}$ agrees with the C^*-norm $\|\cdot\|$ of $\mathcal{A}$. The algebraic a-numerical range of any element $x \in \mathcal{A}$ is defined by

$$V_a(x) = \{\varphi(ax) : \varphi \in \mathcal{S}_a(\mathcal{A})\}.$$

Observe that $V_{1_{\mathcal{A}}}(x) = V(x) = \{f(x) : f \in \mathcal{S}(\mathcal{A})\}$ which is known as algebraic numerical range of x. It has been proved in [8, Theorem 4.7] that $V_a(x)$ is a nonempty convex subset of complex numbers for all $x \in \mathcal{A}$. For more information about algebraic a-numerical ranges and its fundamental properties the reader is referred to [1].

One of the most well-known concept in study of the geometry of normed linear spaces, and also operator spaces is the notion of orthogonality. In 1934, Roberts introduced the first orthogonality in normed linear spaces [13]. Let $(X, \|\cdot\|)$ be a normed linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, whose dimension is at least 2. A vector $x \in X$ is said to be orthogonal in the sense of Roberts to a vector $y \in X$, denoted by $x \perp_R y$, if

$$\|x - \lambda y\| = \|x + \lambda y\| \qquad (\forall \lambda \in \mathbb{K}).$$

Later, in 1935 Birkhoff introduced one of the most important orthogonality type [7]. This notion of orthogonality was developed by James in [11]. A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ in the sense of Birkhoff–James, written as $x \perp_{BJ} y$, if

$$\|x + \lambda y\| \geq \|x\| \qquad (\forall \lambda \in \mathbb{K}).$$

Birkhoff-James orthogonality of bounded linear operators defined on Hilbert spaces studied by Bhatia and Šemrl in [6]. Some characterizations of Birkhoff-James orthogonality in C^*-algebra $\mathcal{A}$ and in a more general setting Hilbert C^*-modules over $\mathcal{A}$ in terms of the elements of state space $\mathcal{S}(\mathcal{A})$ have been obtained in [2, 3, 5, 14]. Also, a characterization of Roberts orthogonality in terms of the algebraic Davis-Wielandt shell of elements of $\mathcal{A}$ is obtained in [4]; see also [9].

In this paper, we consider a generalization of the notion of algebraic Davies-Wielandt shell of $x \in \mathcal{A}$ [4], namely algebraic a-Davies-Wielandt shell of x, which can be defined naturally as follows:

$$DV_a(x) = \{(\varphi(ax), \varphi(x^*ax)) \,:\, \varphi \in \mathcal{S}_a(\mathcal{A})\}.$$

Recently, the notion of Birkhoff-James orthogonality with respect to $\|\cdot\|_a$ in unital C^*-algebra $\mathcal{A}$, so called a-Birkhoff-James orthogonality, has been investigated in [10]. A characterization of a-Birkhoff-James orthogonality in terms of the elements of $\mathcal{S}_a(\mathcal{A})$ has been obtained in [10](see Theorem 2.5). Using this characterization and the concept of algebraic a-Davies-Wielandt shells, we obtain a characterization of Roberts orthogonality in unital C^*-algebra $\mathcal{A}$ with respect to $\|\cdot\|_a$. In fact, we prove that an a-isometry $x \in \mathcal{A}$ is a-Roberts orthogonal to $1_{\mathcal{A}}$ if and only if algebraic a-numerical range of x is symmetric with respect to the origin. Our results cover and extend some known results of [4] .

2. a-Roberts orthogonality in unital C^*-algebras

Throughout this paper, we assume that $\mathcal{A}$ is a unital C^*-algebra with unit $1_{\mathcal{A}}$ and $a \in \mathcal{A}$ is positive and invertible. We start this section with introducing the notions of a-Birkhoff-James orthogonality and a-Roberts orthogonality in $\mathcal{A}$.

We say that an element $x \in \mathcal{A}$ is Birkhoff-James orthogonal with respect to $\|\cdot\|_a$ (a-Birkhoff-James orthogonal) to an element $y \in \mathcal{A}$, in short $x \perp_{BJ}^a y$, if

$$\|x + \lambda y\|_a \geq \|x\|_a \quad (\forall \lambda \in \mathbb{C}).$$

Let us introduce the concept of a-Roberts orthogonality in C^*-algebras.

Definition 2.1. We say that an element $x \in \mathcal{A}$ is Roberts orthogonal with respect to $\|\cdot\|_a$ (a-Roberts orthogonal) to an element $y \in \mathcal{A}$, in short $x \perp_R^a y$, if

$$\|x - \lambda y\|_a = \|x + \lambda y\|_a \quad (\forall \lambda \in \mathbb{C}).$$

It is easy to check that a-Roberts orthogonality implies a-Birkhoff-James orthogonality. Indeed, if $x \perp_R^a y$, then for all $\lambda \in \mathbb{C}$, we have

$$2\|x\|_a = \|(x + \lambda y) + (x - \lambda y)\|_a \leq \|x + \lambda y\|_a + \|x - \lambda y\|_a = 2\|x + \lambda y\|_a.$$

An element $x^\sharp \in \mathcal{A}$ is called an a-adjoint of $x \in \mathcal{A}$ if $ax^\sharp = x^*a$. The set of all a-adjointable elements of $\mathcal{A}$ is denoted by $\mathcal{A}_a$. An element $x \in \mathcal{A}$ is said to be a-selfadjoint if ax is hermitian; i.e., $ax = x^*a$. It has been proved in [8] that if

$x \in \mathcal{A}$ is a-selfadjoint, then $V_a(x) \subseteq \mathbb{R}$. Moreover, in [8, Corollary 4.9] was shown that if $x \in \mathcal{A}_a$ and $x^\sharp$ is an a-adjoint of it, then

$$\|x\|_a^2 = \|xx^\sharp\|_a = \|x^\sharp x\|_a = \|x^\sharp\|_a^2.$$

We now discuss the main properties of a-Roberts orthogonality. The following Proposition follows by the definition of a-Roberts orthogonality; and we omit the proof.

Proposition 2.2. *If $x, y \in \mathcal{A}$, then the following statements hold:*

i) $\alpha x \perp_R^a \beta y$ for any $\alpha, \beta \in \mathbb{C}$; i.e., a-Roberts orthogonality is homogenous;
ii) a-Roberts orthogonality is non-degenerated; i.e., if $x \perp_R^a x$, then $x = 0$.
iii) If $x \perp_R^a y$, then $x^\sharp \perp_R^a y^\sharp$;
iv) For any two nonzero elements $x, y \in \mathcal{A}$, if $x \perp_R^a y$, then x and y are linearly independent.

Proposition 2.3. *For any $x, y \in \mathcal{A}$, if $x^*ay = 0$, then $x \perp_R^a y$.*

Proof. For each $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\|x + \lambda y\|_a^2 &= \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \varphi((x + \lambda y)^* a (x + \lambda y)) \\
&= \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \big(\varphi(x^*ax) + 2\mathrm{Re}\varphi(x^*ay) + |\lambda|^2 \varphi(y^*ay)\big) \\
&= \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \big(\varphi(x^*ax) - 2\mathrm{Re}\varphi(x^*ay) + |\lambda|^2 \varphi(y^*ay)\big) \\
&= \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \varphi((x - \lambda y)^* a (x - \lambda y)) = \|x - \lambda y\|_a^2.
\end{aligned}$$

Hence $\|x + \lambda y\|_a = \|x - \lambda y\|_a$ for all $\lambda \in \mathbb{C}$, and therefore $x \perp_R^a y$. □

In [10] have been described that if $\mathcal{A}$ is a commutative and unital C^*-algebra and $a \in \mathcal{A}$ is positive and invertible, then a-Roberts orthogonality and the Roberts orthogonality are equivalent. But it is not true in noncommutative C^*-algebra even when a is invertible. The following example illustrate that a-Roberts orthogonality and Roberts orthogonality may not be equivalent in noncommutative C^*-algebras.

Example 2.4. Let $\mathbb{M}_2(\mathbb{C})$ be the C^*-algebra of all 2×2 complex matrices, and let Tr be the usual trace functional on $\mathbb{M}_2(\mathbb{C})$. According to the Example 2.2 of [8], we have

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C})) = \{\varphi_h \,:\, h \in \mathbb{M}_2(\mathbb{C})^+ \quad \text{and} \quad \mathrm{Tr}(ha) = 1\},$$

where

$$\varphi_h(x) := \mathrm{Tr}(hx), \qquad (x \in \mathbb{M}_2(\mathbb{C})).$$

Now, let $a = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then with some simple matrix computations, we conclude that

$$\mathcal{S}_a(\mathbb{M}_2(\mathbb{C})) = \{\varphi_h \,:\, h \in \mathcal{L}_a\},$$

where

$$\mathcal{L}_a := \{h = \begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})^+ \,:\, h_{12} \in \mathbb{C},\, h_{11}, h_{22} \geq 0 \text{ and } 2h_{11} + h_{22} = 1\}.$$

Let $x = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. It can easily be seen that

$$\|x + \lambda y\|^2 = \left|1 - \lambda\right|^2 + \left|1 + \lambda\right|^2 = \|x - \lambda y\|^2$$

for all $\lambda \in \mathbb{C}$. So $x \perp_R y$. On the other hand, for all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\|x + \lambda y\|_a^2 &= \sup_{h \in \mathcal{L}_a} \varphi_h((x + \lambda y)^* a (x + \lambda y)) = \sup_{h \in \mathcal{L}_a} \mathrm{Tr}(h(x + \lambda y)^* a (x + \lambda y)) \\
&= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}\big(\begin{bmatrix} h_{11} & h_{12} \\ \overline{h}_{12} & h_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2|1 + \lambda|^2 + |1 - \lambda|^2 \end{bmatrix} \big) \\
&= \sup_{2h_{11} + h_{22} = 1, h_{11}, h_{22} \geq 0} (2|1 + \lambda|^2 + |1 - \lambda|^2) h_{22} \\
&= 2|1 + \lambda|^2 + |1 - \lambda|^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|x - \lambda y\|_a^2 &= \sup_{h \in \mathcal{L}_a} \mathrm{Tr}(h(x + \lambda y)^* a (x + \lambda y)) \\
&= \sup_{2h_{11} + h_{22} = 1, h_{11}, h_{22} \geq 0} (2|1 - \lambda|^2 + |1 + \lambda|^2) h_{22} \\
&= 2|1 - \lambda|^2 + |1 + \lambda|^2.
\end{aligned}$$

Let $\lambda = 1$. Then $\|x + \lambda y\|_a^2 = 8 \neq 4 = \|x - \lambda y\|_a^2$ which implies that $x \not\perp_R^a y$.

Now, let $x' = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ and $y' = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. By Proposition 2.3, since $(x')^* a y' = 0$, we conclude that $x' \perp_a^R y'$. But for every $\lambda \in \mathbb{C}$, we have

$$\|x' + \lambda y'\|^2 = |\frac{1}{2} + \lambda|^2 + |1 - \lambda|^2$$

and

$$\|x' - \lambda y'\|^2 = |\frac{1}{2} - \lambda|^2 + |1 + \lambda|^2.$$

So, $\|x' - \lambda y'\| \neq \|x' + \lambda y'\|$, for $\lambda = 1$. Therefore $x' \not\perp_R y'$.

The following characterization of a-Birkhoff-James orthogonality in a unital C^*-algebra $\mathcal{A}$ based on the elements of its generalized state space $\mathcal{S}_a(\mathcal{A})$ has been presented in [10].

Theorem 2.5. [10, Theorem 2.6] *Let $\mathcal{A}$ be a unital C^*-algebra, $x, y \in \mathcal{A}$ and let a be positive and invertible element of $\mathcal{A}$. Then the following statements are equivalent:*

(i) $x \perp_{BJ}^a y$.

(ii) *There is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax) = \|x\|_a^2$ and $\varphi(y^*ax) = 0$ ($\varphi(x^*ay) = 0$).*

Now, we present our first main result in the next Proposition. Note that Theorem 2.5 has a key role for proving this Proposition.

Proposition 2.6. *Let $x \in \mathcal{A}$ be a-selfadjoint. If $x \perp_R^a 1_{\mathcal{A}}$, then $V_a(x)$ is symmetric with respect to the origin.*

Proof. Assume that $x \in \mathcal{A}$ is a-selfadjoint and $x \perp_R^a 1_{\mathcal{A}}$. Since $\perp_R^a \subseteq \perp_{BJ}^a$, we conclude from Theorem 2.5 that $(0, \|x\|_a^2) \in DV_a(x)$. So there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(x^*ax) = \|x\|_a^2$ and $\varphi(ax) = 0$. It follows that $0 \in V_a(x)$. Note that $V_a(x)$ is compact, and so is a bounded subset of $\mathbb{R}$. Hence, there are $\alpha \le 0 \le \beta$ such that $V_a(x) = [\alpha, \beta]$, since $V_a(x)$ is a convex subset of $\mathbb{R}$. Without loss of generality, we may assume that $-\alpha \le \beta$. Now, from the definition of $V_a(x)$, there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $\varphi(ax) = \beta$. On the other hand, for every $n \in \mathbb{N}$, there exists $\varphi_n \in \mathcal{S}_a(\mathcal{A})$ such that

$$\|x + n1_{\mathcal{A}}\|_a^2 = \varphi_n((x + n1_{\mathcal{A}})^* a(x + n1_{\mathcal{A}})) = \varphi_n(x^*ax) + 2n\varphi_n(ax) + n^2. \quad (2.1)$$

Also, there exists $\psi_n \in \mathcal{S}_a(\mathcal{A})$ such that

$$\|x - n1_{\mathcal{A}}\|_a^2 = \psi_n((x - n1_{\mathcal{A}})^* a(x - n1_{\mathcal{A}})). \quad (2.2)$$

But for each $\varphi \in \mathcal{S}_a(\mathcal{A})$, we have

$$\begin{aligned}\varphi((x + n1_{\mathcal{A}})^* a(x + n1_{\mathcal{A}})) &\le \sup_{\varphi \in \mathcal{S}_a(\mathcal{A})} \varphi((x + n1_{\mathcal{A}})^* a(x + n1_{\mathcal{A}})) \\ &= \|x + n1_{\mathcal{A}}\|_a^2 = \varphi_n((x + n1_{\mathcal{A}})^* a(x + n1_{\mathcal{A}})).\end{aligned}$$

Hence for each $n \in \mathbb{N}$,

$$\varphi(x^*ax) + 2n\varphi(ax) + n^2 \le \varphi_n(x^*ax) + 2n\varphi_n(ax) + n^2,$$

and so

$$2n(\varphi(ax) - \varphi_n(ax)) \le \varphi_n(x^*ax) - \varphi(x^*ax) \qquad (\forall n \in \mathbb{N}).$$

On the other hand, for every $n \in \mathbb{N}$, $\varphi_n \in \mathcal{S}_a(\mathcal{A})$. Then $\varphi_n(ax) \in V_a(x) = [\alpha, \beta]$. So, $\varphi_n(ax) \le \beta$ for all $n \in \mathbb{N}$. Hence, $0 \le \beta - \varphi_n(ax)$, and therefore

$$0 \le \varphi(ax) - \varphi_n(ax) \qquad (\forall n \in \mathbb{N}).$$

Now, let $\varepsilon > 0$ for which $\varphi(ax) - \varphi_n(ax) > \varepsilon$ for all $n \in \mathbb{N}$. Then

$$n \le \frac{\varphi_n(x^*ax) - \varphi(x^*ax)}{2(\varphi(ax) - \varphi_n(ax))} < \frac{\varphi_n(x^*ax) - \varphi(x^*ax)}{2\varepsilon} \le \frac{\varphi_n(x^*ax)}{2\varepsilon} \le \frac{\|x\|_a^2}{2\varepsilon},$$

which is a contradiction, since the set of natural numbers is not bounded above. Therefore for each $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $\varphi(ax) - \varphi_{n_\varepsilon}(ax) \leq \varepsilon$. Hence, if we take $\varepsilon = \frac{1}{k}$ $(k \in \mathbb{N})$, then there is $n_k \in \mathbb{N}$ such that $\varphi(ax) - \varphi_{n_k}(ax) \leq \frac{1}{k} \longrightarrow 0$ as $k \to \infty$. So $\varphi_{n_k}(ax) \to \varphi(ax)$. Further, (3.2), (2.2) and $x \perp_R^a 1_\mathcal{A}$ imply that

$$\begin{aligned}\varphi_n((x + n1_\mathcal{A})^* a(x + n1_\mathcal{A})) &= \|x + n1_\mathcal{A}\|_a^2 = \|x - n1_\mathcal{A}\|_a^2 \\ &= \psi_n((x + n1_\mathcal{A})^* a(x + n1_\mathcal{A})) \quad (\forall n \in \mathbb{N}).\end{aligned}$$

Thus $\varphi_n(x^*ax) + 2n\varphi_n(ax) + n^2 = \psi_n(x^*ax) + 2n\psi_n(ax) + n^2$ for all $n \in \mathbb{N}$, and so

$$\varphi_n(ax) + \psi_n(ax) = \frac{\psi_n(x^*ax) - \varphi_n(x^*ax)}{2n} \quad (\forall n \in \mathbb{N}).$$

Since φ_n and ψ_n are bounded for all $n \in \mathbb{N}$, we conclud that

$$\frac{\psi_n(x^*ax) - \varphi_n(x^*ax)}{2n} \overset{n\to\infty}{\longrightarrow} 0,$$

that implies $\varphi_n(ax) + \psi_n(ax) \to 0$. Therefore

$$\lim_{k\to\infty} \psi_{n_k}(ax) = -\lim_{k\to\infty} \varphi_{n_k}(ax) = -\varphi(ax) = -\beta$$

Moreover, for each $k \in \mathbb{N}$, we have

$$\psi_{n_k}(ax) \in V_a(x) = [\alpha, \beta].$$

It follows that $-\beta \in [\alpha, \beta]$, and so $\alpha = -\beta$ which implies that $V_a(x) = [\alpha, \beta] = [-\beta, \beta]$. □

Remark 2.7. By using the similar technique used in the proof of Proposition 2.1 of [4], we can show that Proposition 2.6 holds for every $x \in \mathcal{A}$ for which $x \perp_R^a 1_\mathcal{A}$ as well. In the last result of this paper (Theorem 3.4), we obtain some special classes of elements $x \in \mathcal{A}$ for which the symmetry of $V_a(x)$ with respect to the origin is a sufficient condition for the a-Roberts orthogonality to $1_\mathcal{A}$.

3. Characterization of a-Roberts orthogonality in C^*-algebras

Assume that $x \in \mathcal{A}$ is arbitrary. The concept of algebraic Davis-Wielandt shell of $x \in \mathcal{A}$ was introduced in [4] as follows:

$$DV(x) = \{(\varphi(x), \varphi(x^*x)) \,:\, \varphi \in \mathcal{S}(\mathcal{A})\}.$$

It is well-known that the $DV(x)$ is a compact convex subspace of $\mathbb{C} \times \mathbb{R}$; (see [4]). Let $a \in \mathcal{A}$ be positive and invertible. We introduce a generalized notion for $DV(x)$, namely algebraic a-Davies-Wielandt shell of x as follows:

$$DV_a(x) = \{(\varphi(ax), \varphi(x^*ax)) \,:\, \varphi \in \mathcal{S}_a(\mathcal{A})\}.$$

To achieve the desired result, we need the following lemma.

Lemma 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$ be positive and invertible and $x \in \mathcal{A}$. If there is $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $(-\varphi(ax), \varphi(x^*ax)) \notin DV_a(x)$, then there exists $\lambda \in \mathbb{C}$ such that either*

$$\varphi(x^*ax) - 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) > \psi(x^*ax) + 2\mathrm{Re}(\overline{\lambda}\psi(ax)), \qquad \forall \psi \in \mathcal{S}_a(\mathcal{A}),$$

or

$$\varphi(x^*ax) - 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) < \psi(x^*ax) + 2\mathrm{Re}(\overline{\lambda}\psi(ax)), \qquad \forall \psi \in \mathcal{S}_a(\mathcal{A}).$$

Proof. For every $y \in \mathcal{A}$ and every $\psi \in \mathcal{S}_a(\mathcal{A})$, we have

$$DV_a(y) = \{(\psi(\mathrm{Re}\, y), \psi(\mathrm{Im}\, y), \psi(y^*ay)) \,:\, \psi \in \mathcal{S}_a(\mathcal{A})\}.$$

From the fact that $(-\varphi(ax), \varphi(x^*ax)) \notin DV_a(x)$, we conclude that $\big(\varphi(\mathrm{Re}(ax)), \varphi(\mathrm{Im}(ax)), \psi(x^*ax)\big) \notin DV_a(-x)$. Since $DV_a(-x)$ is closed and convex set in $\mathbb{R}^3$, separation theorem follows that there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\varphi(x^*ax) - \beta\varphi(\mathrm{Re}(ax)) - \gamma\varphi(\mathrm{Im}(ax)) \\ &\qquad > \alpha\psi(x^*ax) + \beta\psi(\mathrm{Re}(ax)) + \gamma\psi(\mathrm{Im}(ax)) \, (\forall \psi \in \mathcal{S}_a(\mathcal{A}))\end{aligned}$$

or

$$\begin{aligned}&\alpha\varphi(x^*ax) - \beta\varphi(\mathrm{Re}(ax)) - \gamma\varphi(\mathrm{Im}(ax)) \\ &\qquad < \alpha\psi(x^*ax) + \beta\psi(\mathrm{Re}(ax)) + \gamma\psi(\mathrm{Im}(ax)) \, (\forall \psi \in \mathcal{S}_a(\mathcal{A})).\end{aligned}$$

Now, it is enough to take $\lambda := \dfrac{\beta + i\gamma}{2\alpha}$ for $\alpha \neq 0$. □

Let $x \in \mathcal{A}$. The upper boundary of $DV(x)$ is the set

$$DV_{ub}(x) = \{(\mu, r) \in DV(x) \,:\, r = \max \mathcal{L}_\mu(x)\},$$

where $\mathcal{L}_\mu(x) = \{\varphi(x^*x) \,:\, \varphi \in \mathcal{S}(\mathcal{A}), \varphi(x) = \mu\}$. We need to consider the upper boundary of $DV_a(x)$ as in the following definition.

Definition 3.2. Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be positive and invertible and $x \in \mathcal{A}$. The a-upper boundary of $DV_a(x)$ is the set

$$DV^a_{ub}(x) = \{(\mu, r) \in DV_a(x) \,:\, r = \max \mathcal{L}^a_\mu(x)\},$$

where $\mathcal{L}^a_\mu(x) = \{\varphi(x^*ax) \,:\, \varphi \in \mathcal{S}_a(\mathcal{A}), \varphi(ax) = \mu\}$.

First, note that $\mathcal{L}^a_\mu(x)$ is a compact subset of $\mathbb{R}$, since $\mathcal{S}_a(\mathcal{A})$ is w^*-compact. Now, let $(\mu, r) \in DV_a(x)$. Then

$$\begin{aligned}DV_a(-x) =& \{\varphi(-ax), \varphi((-x)^*a(-x)) \,:\, \varphi \in \mathcal{S}_a(\mathcal{A})\} \\ =& \{\varphi(-ax), \varphi(x^*ax) \,:\, \varphi \in \mathcal{S}_a(\mathcal{A})\},\end{aligned}$$

and so $(-\mu, r) \in DV_a(-x)$. Also, we have $\mathcal{L}^a_{-\mu}(-x) = \mathcal{L}^a_\mu(x)$. Thus

$$(\mu, r) \in DV^a_{ub}(x) \Leftrightarrow (-\mu, r) \in DV^a_{ub}(-x). \tag{3.1}$$

The next result gives us the relation between a-Roberts orthogonality and a-upper boundary of $DV_a(x)$. The proof method employed in this theorem follows Arambašić et. al [4] approach.

Theorem 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$ be positive and invertible and $x \in \mathcal{A}$. The following statements are equivalent:*

(i) $x \perp_R^a 1_{\mathcal{A}}$,
(ii) $DV_{ub}^a(x) = DV_{ub}^a(-x)$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $x \perp_R^a 1_{\mathcal{A}}$. We just prove $DV_{ub}^a(x) \subseteq DV_{ub}^a(-x)$, since a-Roberts orthogonality is homogenous, and so we can substitute x with $-x$. First, we prove that

$$DV_{ub}^a(x) \subseteq DV_a(-x). \tag{3.2}$$

Suppose that $(\mu, r) \in DV_{ub}^a(x)$. Then there exists $\varphi \in \mathcal{S}_a(\mathcal{A})$ such that $(\mu, r) = (\varphi(ax), \varphi(x^*ax))$. To the contrary, we assume that $(-\mu, r) = (-\varphi(ax), \varphi(x^*ax)) \notin DV_a(x)$. By perviuos lemma, there exists $\lambda \in \mathbb{C}$ such that either

$$\varphi(x^*ax) - 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) > \psi(x^*ax) + 2\mathrm{Re}(\overline{\lambda}\psi(ax)) \ (\forall \psi \in \mathcal{S}_a(\mathcal{A})) \tag{3.3}$$

or

$$\varphi(x^*ax) - 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) < \psi(x^*ax) + 2\mathrm{Re}(\overline{\lambda}\psi(ax)) \ (\forall \psi \in \mathcal{S}_a(\mathcal{A})). \tag{3.4}$$

Let (3.3) holds. Then

$$\begin{aligned}&\psi((x + \lambda 1_{\mathcal{A}})^* a(x + \lambda 1_{\mathcal{A}})) = \psi(x^*ax) + 2\mathrm{Re}(\overline{\lambda}\psi(ax)) + |\lambda|^2 \\ &< \varphi(x^*ax) - 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) + |\lambda|^2 = \varphi((x - \lambda 1_{\mathcal{A}})^* a(x - \lambda 1_{\mathcal{A}})) \\ &= \|x - \lambda 1_{\mathcal{A}}\|_a^2.\end{aligned}$$

Taking supremum of all $\psi \in \mathcal{S}_a(\mathcal{A})$ give us

$$\|x + \lambda 1_{\mathcal{A}}\|_a^2 = \sup_{\psi \in \mathcal{S}_a(\mathcal{A}} \psi((x + \lambda 1_{\mathcal{A}})^* a(x + \lambda 1_{\mathcal{A}})) < \|x - \lambda 1_{\mathcal{A}}\|_a^2 \tag{3.5}$$

which is a contradiction to $x \perp_R^a 1_{\mathcal{A}}$.

Now, let (3.4) holds. Since $x \perp_R^a 1_{\mathcal{A}}$, Proposition 2.6 implies that $V_a(x)$ is symmetric with respect to the origin. Then there exists $\psi \in \mathcal{S}_a(\mathcal{A})$ such that $\psi(ax) = -\varphi(ax) = -\mu$. By (3.4), we get

$$\varphi(x^*ax) - 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) < \psi(x^*ax) + 2\mathrm{Re}(\overline{\lambda}\psi(ax)),$$

and so

$$\varphi(x^*ax) < \psi(x^*ax) \tag{3.6}$$

Also, we have $(-\psi(ax), \psi(x^*ax)) \in DV_a(x)$. If it is not true, by Lemma 3.1 there is $\alpha \in \mathbb{C}$ such that either

$$\psi(x^*ax) - 2\mathrm{Re}(\overline{\alpha}\psi(ax)) > \rho(x^*ax) + 2\mathrm{Re}(\overline{\alpha}\rho(ax)) \ (\forall \rho \in \mathcal{S}_a(\mathcal{A})) \tag{3.7}$$

or

$$\psi(x^*ax) - 2\mathrm{Re}(\overline{\alpha}\psi(ax)) < \rho(x^*ax) + 2\mathrm{Re}(\overline{\alpha}\rho(ax)) \ (\forall \rho \in \mathcal{S}_a(\mathcal{A})). \tag{3.8}$$

If (3.7) holds, similar way of proving (3.5) yeilds that $\|x - \alpha 1_{\mathcal{A}}\|_a > \|x + \alpha 1_{\mathcal{A}}\|_a$ that is contradiction with hypothesis $x \perp_R^a 1_{\mathcal{A}}$, and if (3.8) holds, with taking $\rho = \varphi$, we get $\psi(x^*ax) < \varphi(x^*ax)$ which is deduce (3.6) not hold.

Consequently there exists $\rho \in \mathcal{S}_a(\mathcal{A})$ such that $\rho(ax) = \psi(ax) = \mu$ and $\rho(x^*ax) = -\psi(x^*ax)$. From (3.6), we obtain that $\rho(x^*ax) = \psi(x^*ax) > \varphi(x^*ax) = r$ that is contradiction with the fact that $(\mu, r) \in DV_{ub}^a(x)$. Then (3.5) does not hold, and hence $(-\mu, r) = (-\varphi(ax), \rho(x^*ax)) \in DV_a(x)$. It means $(\mu, r) \in DV_a(-x)$. This show $DV_{ub}^a(x) \subseteq DV_a(-x)$, since $(\mu, r) \in DV_{ub}^a(x)$ is an arbitrary.

Now, we shall show that $DV_{ub}^a(x) \subseteq DV_{ub}^a(-x)$. Assume $(\mu, r) \in DV_a(x)$. By using (3.2), we get $(\mu, r) \in DV_a(-x)$, and so $(-\mu, r) \in DV_a(x)$. Let $\psi \in \mathcal{S}_a(\mathcal{A})$ such that $\psi(ax) = -\mu$ and $\psi(x^*ax) = r$. We want to show $(-\mu, r) \in DV_{ub}^a(x)$. To do this, we prove that $\rho(x^*ax) \leq r$ for all $\rho \in \mathcal{S}_a(\mathcal{A})$ such that $\rho(ax) = -\mu$. Suppose that it does not hold. It means that there exists $\rho \in \mathcal{S}_a(\mathcal{A})$ such that $\rho(ax) = -\mu$ and

$$\rho(x^*ax) > r = \psi(x^*ax). \tag{3.9}$$

Assume that $(-\mu, r) \in DV_a(x)$. Then $(\rho(ax), \rho(x^*ax)) \in DV_{ub}^a(x)$ and so $(\rho(ax), \rho(x^*ax)) \in DV_a(-x)$ by (3.2). Hence there exists $\pi \in \mathcal{S}_a(\mathcal{A})$ such that $\pi(ax) = -\rho(ax) = \mu$ and $\pi(x^*ax) = \rho(x^*ax)$. From $(\mu, r) \in DV_{ub}^a(x)$ and $\pi(ax) = \mu$ we conclude that $\pi(x^*ax) \leq r$ and so $\rho(x^*ax) = \pi(x^*ax) \leq r$ that is contradiction with (3.9). Hence for all $\rho \in \mathcal{S}_a(\mathcal{A})$ such that $\rho(ax) = -\mu$, we have $(-\mu, r) \in DV_{ub}^a(x)$ and so $(\mu, r) \in DV_{ub}^a(-x)$. Since $(\mu, r) \in DV_{ub}^a(x)$ is arbitrary, we conclude that $DV_{ub}^a(x) \subseteq DV_{ub}^a(-x)$.

$(ii) \Rightarrow (i)$ Assume that $DV_{ub}^a(x) = DV_{ub}^a(-x)$ for every $x \in \mathcal{A}$. For each $\lambda \in \mathbb{C}$, by using (3.1), we have

$$\begin{aligned}
\|x + \lambda 1_{\mathcal{A}}\|_a^2 &= \sup\{\big(\varphi(x + \lambda 1_{\mathcal{A}})^* a(x + \lambda 1_{\mathcal{A}})\big) \ : \ \varphi \in \mathcal{S}_a(\mathcal{A})\} \\
&= \sup\{\varphi(x^*ax) + 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) + |\lambda|^2 \ , \ \varphi \in \mathcal{S}_a(\mathcal{A})\} \\
&= \sup\{r + 2\mathrm{Re}(\overline{\lambda}\mu) + |\lambda|^2 \ : \ (\mu, r) \in DV_{ub}^a(x)\} \\
&= \sup\{r + 2\mathrm{Re}(\overline{\lambda}\mu) + |\lambda|^2 \ : \ (-\mu, r) \in DV_{ub}^a(-x)\} \\
&= \sup\{r + 2\mathrm{Re}(\overline{\lambda}\mu) + |\lambda|^2 \ : \ (-\mu, r) \in DV_{ub}^a(x)\} \\
&= \sup\{r + 2\mathrm{Re}(\overline{\lambda}(-\mu)) + |\lambda|^2 \ : \ (\mu, r) \in DV_{ub}^a(x)\} \\
&= \sup\{\varphi(x^*ax) - 2\mathrm{Re}(\overline{\lambda}\varphi(ax)) + |\lambda|^2 \ : \ (\mu, r) \in DV_{ub}^a(x)\} \\
&= \sup\{\big(\varphi(x - \lambda 1_{\mathcal{A}})^* a(x - \lambda 1_{\mathcal{A}})\big) \ : \ \varphi \in \mathcal{S}_a(\mathcal{A})\} = \|x - \lambda 1_{\mathcal{A}}\|_a^2.
\end{aligned}$$

Therefore $x \perp_R^a 1_{\mathcal{A}}$. □

Finally, we want to explain when the algebraic a-numerical range is symmetric. To this, we recall the concept of a-isometry in unital C^*-algebras. An element

$x \in \mathcal{A}$ is called an a-isometry if $x^\sharp x = 1_{\mathcal{A}}$, which equivalent to $x^*ax = a$, where a is positive and invertible element.

Theorem 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$ be positive and invertible and $x \in \mathcal{A}$. If $x \in \mathcal{A}$ is a-isometry, then the following statements are equivalent:*
(i) $x \perp_R^a 1_{\mathcal{A}}$.
(ii) $V_a(x)$ is symmetric with respect to the origin.

Proof. Note that x is an a-isometry. So $\varphi(x^*ax) = \varphi(ax^\sharp x) = \varphi(a) = 1_{\mathcal{A}}$. Then

$$\begin{aligned} DV_{ub}^a(x) &= \{(\varphi(ax), \varphi(x^*ax)) \in DV_a(x) \ : \ \varphi \in \mathcal{S}_a(\mathcal{A})\} \\ &= \{(\varphi(ax), \varphi(a)) \in DV_a(x) \ : \ \varphi \in \mathcal{S}_a(\mathcal{A})\} \\ &= \{(\varphi(ax), 1_{\mathcal{A}}) \in DV_a(x) \ : \ \varphi \in \mathcal{S}_a(\mathcal{A})\} = V_a(x) \times 1_{\mathcal{A}}. \end{aligned}$$

Therefore $DV_{ub}^a(x) = DV_{ub}^a(-x)$ if and only if $V_a(x) = V_a(-x)$. Pervious Theorem yields that $DV_{ub}^a(x) = DV_{ub}^a(-x)$ if and only if $x \perp_R^a 1_{\mathcal{A}}$ and this fact complete the proof. □

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