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A New Type of Generalized Closed Sets via Hereditary Classes

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Abstract. A hereditary class $\mathcal{H}$ on a set X is a nonempty collection of subsets of X which is closed under taking subsets. In this paper, we introduce a new class of generalized closed sets in topological spaces called $\mathcal{H}$g-closed by using hereditary classes. We investigate the relationship of this type of closed sets with other types of closed sets of topological spaces and of generalized topological space. In addition, we obtain some characterizations of this class in T_1 topological spaces.

Keywords: Hereditary class, g-closed set, $\mathcal{H}$g-closed set.

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1. Introduction and preliminaries

Closed sets are fundamental objects in a topological space. In 1970, N. Levine [10] introduced and studied the generalized closed sets. By definition, a subset A of a topological space X is called generalized closed (briefly, g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open. This concept has been studied extensively in recent years by many topologists, for example see [1, 2, 3, 4]. Generalized closed sets are not only natural generalizations of closed sets, but they also suggest several new properties of topological spaces, such as weak separation axioms, submaximal spaces and extremally disconnected spaces.

For the sake of convenience, we begin with some basic concepts. For any $A \subseteq X$, the closure and the interior of A are denoted by $cl_\tau(A)$ (or $cl(A)$) and $int_\tau(A)$ (or $int(A)$), respectively. For every point $x \in X$, we denote by $\tau(x)$ the open neighborhood system at x, i.e., $\tau(x) := \{U \in \tau \mid x \in U\}$. Also, if we have several topology on a set X, we write τ-open (τ-closed) for openness (closedness) with respect to topology τ.

A subset A of a topological space X is called α-open (resp. semi-open, pre-open, semi-pre-open (or β-open)) if $A \subseteq intclint(A)$ (resp. $A \subseteq clintcl(A)$, $A \subseteq clint(A)$, $A \subseteq intcl(A)$, $A \subseteq clintcl(A)$). Moreover, A is said to be α-closed (resp. semi-closed, pre-closed, semi-pre-closed) if $X \setminus A$ is α-open (resp. semi-open, pre-open, semi-pre-open), or equivalently, if $clintcl(A) \subseteq A$ (resp. $intcl(A) \subseteq A$, $clint(A) \subseteq A$, $intclint(A) \subseteq A$). The α-closure (resp. semi-closure, pre-closure, semi-pre-closure) of $A \subseteq X$ is the smallest α-closed (resp. semi-closed, pre-closed, semi-pre-closed) set containing A. It is well-known that $\alpha cl(A) = A \cup clintcl(A)$ and $scl(A) = A \cup intcl(A)$, $pcl(A) = A \cup clint(A)$ and $spcl(A) = A \cup intclint(A)$. It is worth mentioning that the collection $\alpha O(X)$ of all α-open subsets of X is a topology on X [13] which is finer than the original one, and that a subset A of X is α-open if and only if A is semi-open and pre-open [15]. But the collections $PO(X)$ of all pre-open sets, $SO(X)$ of all semi-open sets and $SPO(X)$ of all semi-pre-open sets are generalized topologies on X in the sense of Császár. In [17], Talabeigi has presented a method for extracting the collection of pre-open sets from any desired topological space.

Definition 1.1. Let X be a topological space. A subset A of X is called:

(1) generalized closed (briefly, g-closed) [10] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open,
(2) semi-generalized closed (briefly, sg-closed) [5], if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open,

(3) generalized semi-closed (briefly, gs-closed) [3] if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is open,
(4) generalized α-closed (briefly, gα-closed) [12], if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open,
(5) α-generalized closed (briefly, αg-closed) [11], if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
(6) generalized semi-pre-closed (briefly, gsp-closed) [8] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
(7) regular generalized closed (briefly, r-g-closed) [14] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open.

In addition to above definition, a subset A of X is g-open [10] (sg-open [5]) if $X \setminus A$ is g-closed (sg-closed). Other classes of generalized open sets can be defined in a similar manner.

In this paper, we introduce a new class of generalized closed sets in topological spaces called $\mathcal{H}$g-closed by using hereditary classes. We investigate the relationship of this type of closed sets with other types of closed sets of topological spaces and of generalized topological spaces.

2. Hereditary spaces

In this section, we recall some basic concepts and results of hereditary classes and generalized topologies generated by them. An ideal $\mathcal{I}$ on a nonempty set X is a nonempty collection of subsets of X which is closed under taking subsets and finite unions; and a nonempty collection $\mathcal{H}$ of subsets of X is called a hereditary class if it is closed under taking subsets only. Thus every ideal is a hereditary class. Ideal is a fundamental concept in topological spaces and plays an important role in the study of topological spaces [9]. Similarly, hereditary classes are important in the study of generalized topological spaces [7, 17].

Example 2.1. Let X be a nonempty set and $A \subsetneq X$. Then the collections $\mathcal{H}_1 = \{\emptyset\}$, $\mathcal{H}_2 = P(X)$ the power set of X; and $\mathcal{H}_3 = \{B \subseteq X \mid B \subseteq A\}$ are ideals on X and hence are hereditary classes. Indeed, $\emptyset \in \mathcal{H}$ for any hereditary class $\mathcal{H}$, so $\mathcal{H}_1$ is the smallest ideal and hence the smallest hereditary class on X. Also, the collection $\mathcal{H}_p = \{\emptyset\} \bigcup \{\{x\} \mid x \in X\}$ is a hereditary class but not an ideal if X has at least two elements.

If (X, τ) is a space with a hereditary class $\mathcal{H}$ on X, then we say $(X, \tau, \mathcal{H})$ is a hereditary space. Also, we will consider the hereditary class $\mathcal{H}$ on a topological space (X, τ) which is not an ideal and hence $\mathcal{H} \neq P(X), \{\emptyset\}$, i.e., we assume the hereditary classes are nontrivial. Thus two important example of such classes are the collection $\mathcal{H}_i := \{A \subseteq X : int(A) = \emptyset\}$ and the collection $\mathcal{H}_{ici} := \{A \subseteq X : intclint(A) = \emptyset\}$.

Affected by [18], if for any hereditary class $\mathcal{H}$ on a space (X,τ), we set $dual(\mathcal{H}) = \{A \subseteq X : X \setminus A \notin \mathcal{H}\}$ then $dual(\mathcal{H})$ will also be a hereditary class on (X,τ) again. For example, if $\mathcal{H} = \mathcal{H}_i$, then $dual(\mathcal{H}_i) = \{A \subseteq X : clA \neq X\}$.

In the following, we need to recall some concepts and notations of generalized topological spaces which can be found in [6]. A subset μ of the power set $P(X)$ is called a generalized topology (briefly GT) on X and the pair (X,μ) is called a generalized topological space (briefly GTS) if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. A GTS (X,μ) is called strong if $X \in \mu$. A set $A \subseteq X$ is said to be μ-open if $A \in \mu$ and μ-closed if $X \setminus A \in \mu$.

A map $\gamma : P(X) \to P(X)$ is said to be monotone provided that $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$, where we write γA for $\gamma(A)$. Also, a map $\gamma : P(X) \to P(X)$ is said to be:

(1) idempotent if $\gamma^2 A = \gamma\gamma A = \gamma A$ for all $A \subseteq X$,
(2) restricting if $\gamma A \subseteq A$ for all $A \subseteq X$,
(3) enlarging if $A \subseteq \gamma A$ for all $A \subseteq X$.

Remark 2.2. [6] If μ is a GT on X, then the generalized interior operator $int_\mu : P(X) \to P(X)$ defined by $int_\mu A := \bigcup\{M \in \mu \mid M \subseteq A\}$ is monotone, idempotent and restricting; and the generalized closure operator $cl_\mu : P(X) \to P(X)$ defined by $cl_\mu A := \bigcap\{N \mid A \subseteq N, X \setminus N \in \mu\}$ is monotone, idempotent and enlarging. Moreover, int_μ and cl_μ are conjugate, i.e., $cl_\mu A = X \setminus (int_\mu(X \setminus A))$ for all $A \subseteq X$. Conversely, if $\gamma : P(X) \to P(X)$ is enlarging, monotone and idempotent, then the collection $\mu := \{A \mid \gamma(X \setminus A) = X \setminus A\}$ is a GT on X such that $cl_\mu A = \gamma A$ for all $A \subseteq X$.

Definition 2.3. [17] Let $(X,\tau,\mathcal{H})$ be a hereditary space and $A \subseteq X$. Then

$$A^*(\mathcal{H},\tau) := \{x \in X \mid U \cap A \notin \mathcal{H} \quad \text{for every } U \in \tau(x)\}$$

is called the local function of A with respect to τ and $\mathcal{H}$ and when no ambiguity is present we will simply write A^*.

Theorem 2.4. [17] *Let $(X,\tau,\mathcal{H})$ be a hereditary space. Then the operator $cl^* : P(X) \to P(X)$ defined by $cl^*(A) = A \cup A^*$ is a generalized closure operator, i.e., it is enlarging, monotone and idempotent. Hence the collection $\tau^*(\mathcal{H}) := \{A \subseteq X \mid cl^*(X \setminus A) = X \setminus A\}$ is a GT on X, called the GT induced by τ and $\mathcal{H}$.*

When no ambiguity is present we will simply write τ^*.

Theorem 2.5. [17] *Let $(X,\tau,\mathcal{H})$ be a hereditary space and A be a subset of X. Then the following statements hold:*

(1) $\tau \subseteq \tau^*$ *and* $\tau^{**} = \tau^*$.
(2) $A^{**} \subseteq (A \cup A^*)^* = A^* \subseteq clA$.
(3) $cl(A^*) = A^*$, *i.e., the set* A^* *is closed in* (X, τ) *and hence is also closed in the GTS* (X, τ^*).
(4) *If* $\mathcal{H} = \{\emptyset\}$, *then* $cl^*(A) = cl(A) = A^*$ *and* $\tau = \tau^*$.
(5) *The collection* $\beta := \{U \setminus H \mid U \in \tau, H \in \mathcal{H}\}$ *is a base for* τ^*.
(6) *For every* $H \in \mathcal{H}$, $H^* = \emptyset$ *and hence* H *is* τ^**-closed.*

The following example shows that in Theorem 2.4, the operator cl^* is not a Kuratowski closure operator.

Example 2.6. Let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{c\}, \{d\}, \{c, d\}\}$ be a topology and a hereditary class, respectively on $X = \{a, b, c, d\}$. Put $A = \{a\}$ and $B = \{b, c\}$. Then we have $A^* = \emptyset$, $B^* = \{b, c, d\}$ and $(A \cup B)^* = \{b, c, d\}$. So $cl^*(A) = \{a\}$ and $cl^*(B) = \{b, c, d\}$; and hence $cl^*(A) \cup cl^*(B) = \{b, c, d\}$, but $cl^*(A \cup B) = \{a, b, c, d\}$. Thus $cl^*(A) \cup cl^*(B) \neq cl^*(A \cup B)$.

Notice that by Definition 2.3, Theorem 2.4 and Theorem 2.5, we have the following result.

Corollary 2.7. *For any subset* A *of the hereditary space* $(X, \tau, \mathcal{H}_i)$, *we have* $A^*(\mathcal{H}_i) = clint(A)$, *and hence* $\tau^*(\mathcal{H}_i) = PO(X)$.

3. $\mathcal{H}$-generalized closed sets

In this section, we introduce a new class of generalized closed sets in topological spaces which is a generalization of g-*closed* sets.

Definition 3.1. Let $(X, \tau, \mathcal{H})$ be a hereditary space. Then a subset A of X is said to be g-*closed* with respect to $\mathcal{H}$ (briefly, $\mathcal{H}$g-closed) if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open in X. A subset A of X is said to be $\mathcal{H}$g-open if $X \setminus A$ is $\mathcal{H}$g-closed.

By Theorem 2.5, we have the following results.

Lemma 3.2. *Let* $(X, \tau, \mathcal{H})$ *be a hereditary space. Then the following statements hold:*

(1) *Every closed set in* X *is* $\mathcal{H}g$*-closed.*
(2) *For any subset* A *of* X, *we have* $(A^*)^* \subseteq A^*$, *so for any subset* A *of* X, *the set* A^* *is* $\mathcal{H}g$*-closed.*
(3) *For any subset* A *of* X, *the set* $cl^*(A)$ *is* $\mathcal{H}g$*-closed. Because if* $cl^*(A) \subseteq U$ *for some* $U \in \tau$, *then we have* $(cl^*(A))^* = (A \cup A^*)^* = A^* \subseteq cl^*(A) \subseteq U$.
(4) *Every member of* $\mathcal{H}$ *is* $\mathcal{H}g$*-closed .*
(5) *Every g-closed set is* $\mathcal{H}g$*-closed.*

The following example shows that the converse of part (5) in the above lemma is not true.

Example 3.3. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, \{b\}, \{b,c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}, \{c\}\}$. Then (X,τ) is a topological space and $\mathcal{H}$ is a hereditary class on X. Suppose $A = \{b\}$. Then it is easy to verify that A is not g-closed but is $\mathcal{H}$g-closed. In fact, $A \subseteq \{b,c\}$ but $cl(A) = X \nsubseteq \{b,c\}$, and $A^* = \emptyset$.

Also, two concepts $\mathcal{H}_i$g-closed and g-pre-closed coincide.

Using the special hereditary class $\mathcal{H}_i$ in Lemma 3.2 leads to the following corollary.

Corollary 3.4. *For any topological space (X,τ) we have:*

(1) *Every closed set of X is a g-pre-closed.*
(2) *Since for any subset A of X, we have $clint(clint(A)) = clint(A)$, it follows that for any subset A of X, the set $B = clint(A)$ is g-pre-closed, and hence every pre-closed set is g-pre-closed.*
(3) *Every subset of X with empty interior is g-pre-closed.*
(4) *Every g-closed set is g-pre-closed; that the converse is false as shown by the following example.*

Example 3.5. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$. It is easy to check that the set $\{b\}$ is g-closed but not g-pre-closed.

In what follows in this section, we derive certain characterizations and properties of $\mathcal{H}$g-closed sets.

Theorem 3.6. *Let $(X,\tau,\mathcal{H})$ be a hereditary space. For every $A \subseteq X$, the following are equivalent:*

(1) *A is $\mathcal{H}$g-closed,*
(2) *$cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is open,*
(3) *for all $x \in cl^*(A)$, $cl(\{x\}) \cap A \neq \emptyset$,*
(4) *$cl^*(A) \setminus A$ contains no nonempty closed set of (X,τ),*
(5) *$A^* \setminus A$ contains no nonempty closed set of (X,τ).*

Proof. (1) $\Rightarrow$ (2): Suppose A is $\mathcal{H}$g-closed and $A \subseteq U$ where U is open in (X,τ). Then $A^* \subseteq U$ so that $cl^*(A) = A \cup A^* \subseteq U$.

(2) $\Rightarrow$ (3): Suppose $x \in cl^*(A)$. If $cl(\{x\}) \cap A = \emptyset$, then $A \subseteq X \setminus cl(\{x\})$ and using (2), $cl^*(A) \subseteq X \setminus cl(\{x\})$, a contradiction since $x \in cl^*(A)$.

(3) $\Rightarrow$ (4): Suppose F is a closed set of (X,τ) contained in $cl^*(A) \setminus A$ and $x \in F$. Since $F \cap A = \emptyset$, we have $cl(\{x\}) \cap A = \emptyset$. Again, since $x \in cl^*(A)$, by (3) we have $cl(\{x\}) \cap A \neq \emptyset$, a contradiction. This proves (4).

(4) $\Rightarrow$ (5): It follows from the fact that $cl^*(A) \setminus A = A^* \setminus A$.

(5) $\Rightarrow$ (1): Suppose $A \subseteq U$ and U is open in (X, τ). Since A^* is closed in (X, τ) (by Theorem 2.5(3)) and $A^* \cap (X \setminus U) \subseteq A^* \setminus A$ holds, $A^* \cap (X \setminus U)$ is a closed set in (X, τ) contained in $A^* \setminus A$. Then by (5), $A^* \cap (X \setminus U) = \emptyset$ which gives $A^* \subseteq U$. Hence A is $\mathcal{H}$g-closed. □

Corollary 3.7. *Let $(X, \tau, \mathcal{H})$ be a hereditary T_1-space and $A \subseteq X$. Then A is $\mathcal{H}g$-closed if and only if A is τ^*-closed.*

Proof. Necessity: Follows from Theorem 3.6 (1) $\Rightarrow$ (3).
Sufficiency: Follows from part (2) of Lemma 3.2. □

Corollary 3.8. *Let $(X, \tau, \mathcal{H})$ be a hereditary space and A be a $\mathcal{H}g$-closed set. Then the following are equivalent:*

(1) *A is τ^*-closed,*
(2) *$cl^*(A) \setminus A$ is closed in (X, τ),*
(3) *$A^* \setminus A$ is closed in (X, τ).*

Proof. (1) $\Rightarrow$ (2): If A is τ^*-closed, then $cl^*(A) \setminus A = \emptyset$, so $cl^*(A) \setminus A$ is a closed set.

(2) $\Leftrightarrow$ (3): It is clear, since $cl^*(A) \setminus A = A^* \setminus A$.

(3) $\Rightarrow$ (1): If $A^* \setminus A$ is closed in (X, τ) and A is $\mathcal{H}$g-closed, then by Theorem 3.6, $A^* \setminus A = \emptyset$ and so A is τ^*-closed. □

Using the special hereditary class $\mathcal{H}_i$ in Theorem 3.6 gives the following remark.

Remark 3.9. The following statements hold:

(1) For every subset A of a topological space X; A is g-pre-closed if and only if $clint(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
(2) For every open subset A of X; A is g-pre-closed if and only if A is pre-closed.

Remark 3.10. We know that the concepts of $\mathcal{H}_i$g-closedness and g-pre-closedness are the same. More over if we consider the condition of being T_1 for the space, then the concepts of g-pre-closedness and pre-closedness will be the same (using Corollary 3.7).

Theorem 3.11. *Let $(X, \tau, \mathcal{H})$ be a hereditary space and A be a subset of X. Then A is $\mathcal{H}g$-open if and only if $F \subseteq int^*(A)$ whenever $F \subseteq A$ and F is closed.*

Proof. Assume A is $\mathcal{H}$g-open and $F \subseteq A$, where F is closed in (X, τ). Then $X \setminus A \subseteq X \setminus F$ so $(X \setminus A)^* \subseteq (X \setminus F)$. Hence $cl^*(X \setminus A) \subseteq (X \setminus F)$ and thus $F \subseteq int^*(A)$. Conversely, from $X \setminus A \subseteq U$, where U is open in (X, τ) we have $X \setminus U \subseteq int^*(A)$ which implies $cl^*(X \setminus A) \subseteq U$. Thus $X \setminus A$ is $\mathcal{H}$g-closed and hence A is $\mathcal{H}$g-open. □

Again, applying the special hereditary class $\mathcal{H}_i$ in the previous theorem gives the following result.

Corollary 3.12. *Let (X, τ) be a topological space. Then the following statements hold:*

(1) *for a subset A of X, A is g-pre-open if and only if $F \subseteq intcl(A)$ whenever $F \subseteq A$ and F is closed.*
(2) *For a closed subset A of X; A is g-pre-open if and only if A is pre-open.*

Definition 3.13. Let $(X, \tau, \mathcal{H})$ be a hereditary space. A subset A of X is said to be τ^*-dense in itself if $A \subseteq A^*$.

Substituting the specific hereditary class $\mathcal{H}_i$ in the above definition will lead to the definition of semi-open sets. Because being $\tau^*(\mathcal{H}_i)$-dense in itself is equivalent to $A \subseteq clint(A)$.

Lemma 3.14. *Let $(X, \tau, \mathcal{H})$ be a hereditary space and A be a subset of X. If A is τ^*-dense in itself, then $A^* = cl(A^*) = cl^*(A) = cl(A)$.*

Proof. Since A is τ^*-dense in itself, we get $A \subseteq A^*$. Thus $cl(A) \subseteq cl(A^*) = A^* \subseteq cl(A)$ (by Theorem 2.5), and thus we have $cl(A) = A^* = cl(A^*)$. Again, $cl^*(A) = A \cup A^* = A \cup cl(A) = cl(A)$. Consequently, $A^* = cl(A^*) = cl^*(A) = cl(A)$. □

As an obvious point, we know that the relation $clint(A) = clA$ is true for every open subset A of a space X, the following corollary shows the truth of this issue for semi-open sets as well.

Corollary 3.15. *If A is a semi-open set of a space X, then $clint(A) = clA$.*

Proof. By using $\mathcal{H} = \mathcal{H}_i$ in Lemma 3.14, the result is obtained. □

Theorem 3.16. *Let $(X, \tau, \mathcal{H})$ be a hereditary space and A be a τ^*-dense in itself subset of X. Then A is $\mathcal{H}g$-closed if and only if it is g-closed.*

Proof. Follows from Lemma 3.14. □

Corollary 3.17. *Let A be a semi-open set of a space X. Then A is g-pre-closed if and only if it is g-closed.*

Proof. By putting $\mathcal{H} = \mathcal{H}_i$ in Corollary 3.15 and Remark 3.10, the result is obtained. □

Theorem 3.18. *Let $(X, \tau, \mathcal{H})$ be a hereditary space. The following statements are equivalent:*

(1) *Every subset of X is $\mathcal{H}g$-closed.*

(2) *Every open subset of* X *is* τ^**-closed.*

Proof. (1) $\Rightarrow$ (2): Suppose A is open set of X. Then by assumption, A is $\mathcal{H}$g-closed so that $A^* \subseteq A$. Thus, A is τ^*-closed.

(2) $\Rightarrow$ (1): Let $A \subseteq X$ and U be open set of X such that $A \subseteq U$. Then by hypothesis, $U^* \subseteq U$. Again, $A \subseteq U$ implies that $A^* \subseteq U^* \subseteq U$, which implies that A is $\mathcal{H}$g-closed. □

Corollary 3.19. *For any space* X *the following statements are equivalent:*

(1) *Every subset of* X *is g-pre-closed.*
(2) *Every open subset of* X *is pre-closed.*

Proof. Put $\mathcal{H} = \mathcal{H}_i$ in Theorem 3.18. □

Theorem 3.20. *Let* $(X, \tau, \mathcal{H})$ *be a hereditary space. The following statements are equivalent:*

(1) A *is* $\mathcal{H}g$*-closed.*
(2) $A \cup (X \setminus A^*)$ *is* $\mathcal{H}g$*-closed.*
(3) $A^* \setminus A$ *is* $\mathcal{H}g$*-open.*

Proof. (1) $\Rightarrow$ (2): Suppose $A \cup (X \setminus A^*) \subseteq U$, where U is open in X. Then $X \setminus U \subseteq X \setminus (A \cup (X \setminus A^*)) = A^* \setminus A$. Since A is $\mathcal{H}$g-closed, by Theorem 3.6 we have $X \setminus U = \emptyset$, i.e., $X = U$. Since X is the only open set containing $A \cup (X \setminus A^*)$, $A \cup (X \setminus A^*)$ is $\mathcal{H}$g-closed.

(2) $\Rightarrow$ (1): Suppose $F \subseteq A^* \setminus A$ where F is closed in (X, τ). Then $A \cup (X \setminus A^*) \subseteq X \setminus F$ and so by (2), $[A \cup (X \setminus A^*)]^* \subseteq X \setminus F$. So $A^* \cup (X \setminus A^*)^* \subseteq X \setminus F$. Thus $F \subseteq X \setminus A^*$. Again, since $F \subseteq A^*$ we have $F = \emptyset$. Hence by Theorem 3.6, A is $\mathcal{H}$g-closed.

(2) $\Leftrightarrow$ (3): Follows from the fact that $X \setminus (A^* \setminus A) = A \cup (X \setminus A^*)$. □

Corollary 3.21. *Let* X *be a space and* A *be a subset of* X*. Then* A *is g-pre-closed if and only if the set* $clint(A) \setminus A$ *is g-pre-open.*

Proof. Put $\mathcal{H} = \mathcal{H}_i$ in Theorem 3.20. □

Theorem 3.22. *Let* $(X, \tau, \mathcal{H})$ *be a hereditary space and* A*,* B *be subsets of* X *such that* $A \subseteq B \subseteq cl^*(A)$*. If* A *is* $\mathcal{H}g$*-closed, then* B *is* $\mathcal{H}g$*-closed. Especially;* τ^**-closure of every* $\mathcal{H}g$*-closed set is* $\mathcal{H}g$*-closed.*

Proof. Suppose $B \subseteq U$, where U is open in X. Since A is $\mathcal{H}$g-closed, $A^* \subseteq U$, so $cl^*(A) \subseteq U$. Now, $A \subseteq B \subseteq cl^*(A)$ implies $cl^*(A) = cl^*(B)$. Thus $cl^*(B) \subseteq U$ and hence B is $\mathcal{H}$g-closed. □

Putting $\mathcal{H} = \mathcal{H}_i$ in Theorem 3.22 leads us to the following results.

Corollary 3.23. *Let* X *be a space. The following statements hold:*

(1) *If* $A \subseteq B \subseteq clint(A)$ *and* A *is g-pre-closed, then* B *is g-pre-closed.*
(2) *pre-closure of every g-pre-closed set is g-pre-closed.*

Theorem 3.24. *Let* $(X, \tau, \mathcal{H})$ *be a hereditary space and Let* A, B *be subsets of* X *such that* $A \subseteq B \subseteq A^*$*. If* A *is* $\mathcal{H}$*g-closed, then* A *and* B *are g-closed.*

Proof. Since $A \subseteq B \subseteq A^*$, we obtain $A \subseteq B \subseteq cl^*(A)$, and hence by Theorem 3.22, B is $\mathcal{H}$g-closed. Again, from $A \subseteq B \subseteq A^*$, By Theorem 2.5, we have $A^* \subseteq B^* \subseteq (A^*)^* \subseteq A^*$, which shows that $A^* = B^*$. Thus, A and B are τ^*-dense in themselves and hence by Theorem 3.16, A and B are g-closed. □

By putting $\mathcal{H} = \mathcal{H}_i$ in Theorem 3.24, we have:

Corollary 3.25. *Let* X *be a space and* A, B *be subsets of* X *such that* $A \subseteq B \subseteq clint(A)$*. If* A *is g-pre-closed, then* A *and* B *are g-closed.*

The following result gives a precise form of a $\mathcal{H}$g-closed set.

Theorem 3.26. *Let* $(X, \tau, \mathcal{H})$ *be a hereditary space and* A *be a subset of* X*. Then* A *is* $\mathcal{H}$*g-closed if and only if* $A = F \setminus N$ *where* F *is* τ^**-closed and* N *is a set not containing any nonempty closed set.*

Proof. Assume A is $\mathcal{H}$g-closed. Then by Theorem 3.6, $A^* \setminus A = N$ (say) contains no nonempty closed set. If $F = cl^*(A)$, then F is a τ^*-closed set such that $F \setminus N = (A \cup A^*) \setminus (A^* \setminus A) = A$.

Conversely, suppose $A = F \setminus N$, where F is τ^*-closed and N contains no nonempty closed set. Suppose $A \subseteq U$ where U is open in X. Then $F \setminus N \subseteq U$ which implies that $F \cap (X \setminus U) \subseteq N$. Now, $A \subseteq F$ and $F^* \subseteq F$ imply that $A^* \cap (X \setminus U) \subseteq F^* \cap (X \setminus U) \subseteq F \cap (X \setminus U) \subseteq N$. Since $A^* \cap (X \setminus U)$ is closed, we have by hypothesis that $A^* \cap (X \setminus U) = \emptyset$ and hence $A^* \subseteq U$, proving A to be $\mathcal{H}$g-closed. □

The selection of the hereditary class $\mathcal{H}_i$ in the above theorem gives a result, in which an exact form of the g-pre-closed set is specified.

Corollary 3.27. *Let* X *be a space and* A *be a subset of* X*. Then* A *is g-pre-closed if and only if* $A = F \setminus N$ *where* F *is pre-closed and* N *is a set not containing any nonempty closed set.*

Recall that an ideal $\mathcal{I}$ on a space (X, τ) is called τ-codense if $\tau \cap \mathcal{I} = \{\emptyset\}$. It has been shown that $\mathcal{I}$ is τ-codense if and only if $X = X^*$ if and only if $I \in \mathcal{I}$ implies $int_\tau(I) = \emptyset$ [9]. Similarly, we call a hereditary class $\mathcal{H}$ on a space (X, τ), τ-codense if $\tau \cap \mathcal{H} = \{\emptyset\}$ and hence we need the following definition.

Definition 3.28. If $(X, \tau, \mathcal{H})$ is a hereditary space such that $\mathcal{H}$ is τ-codense. Then we say that $(X, \tau, \mathcal{H})$ is a codense-hereditary space.

Theorem 3.29. *Let $(X, \tau, \mathcal{H})$ be a codense-hereditary space. Then for any $U \in \tau$; $U \subseteq U^*$ and $U^* = clU$.*

Proof. Refer to theorems 2.10 and 2.12 of [17]. □

The hereditary space $(X, \tau, \mathcal{H}_i)$ is an example of a codense-hereditary space. Because if $A \in \tau \cap \mathcal{H}_i$, then $A = intA$ and $intA = \emptyset$. Hence $A = \emptyset$ and therefore $\tau \cap \mathcal{H}_i = \{\emptyset\}$. So all hereditary spaces $(X, \tau, \mathcal{H}_i)$ are codense-hereditary spaces and we will show that maximal codense-hereditary spaces are among these spaces.

Theorem 3.30. *The following statements are equivalent:*

(1) $\tau \cap \mathcal{H} = \{\emptyset\}$,
(2) $\mathcal{H} \subseteq \mathcal{H}_i$.

Proof. (1) $\Rightarrow$ (2): First suppose that $\tau \cap \mathcal{H} = \{\emptyset\}$. If $A \in \mathcal{H}$, then $intA \in \mathcal{H}$ and also $intA \in \tau$. Therefore, by assumption $intA = \emptyset$ and so, $A \in \mathcal{H}_i$.

(2) $\Rightarrow$ (1): Let $A \in \tau \cap \mathcal{H}$ such that $A \neq \emptyset$. Then $A \in \mathcal{H}$ and $\emptyset \neq A = intA$. Thus, $A \notin \mathcal{H}_i$. This is impossible. □

Theorem 3.31. *For any codense-hereditary space $(X, \tau, \mathcal{H})$ we have $\tau^* \subseteq PO(X)$.*

Proof. Obviously if $\mathcal{H}_1 \subseteq \mathcal{H}_2$, then $\tau^*(\mathcal{H}_1) \subseteq \tau^*(\mathcal{H}_2)$. Attention to Theorem 3.30 completes the proof. □

The equivalent conditions presented in the following theorem brings to mind the conditions related to the property of normality.

Theorem 3.32. *In any codense-hereditary space $(X, \tau, \mathcal{H})$, the following statements are equivalent:*

(1) *For each pair of disjoint closed sets F and K, there exist disjoint $\mathcal{H}g$-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.*
(2) *For each closed set F and any open set V containing F, there is a $\mathcal{H}g$-open set U such that $F \subseteq U \subseteq cl^*(U) \subseteq V$.*

Proof. (1) $\Rightarrow$ (2): Assume F is a closed set and V is an open set of X such that $F \subseteq V$. Then F and $X \setminus V$ are disjoint closed sets and so by (1), there exist disjoint $\mathcal{H}$g-open sets U and W such that $F \subseteq U$ and $X \setminus V \subseteq W$. Since W is $\mathcal{H}$g-open and $X \setminus V \subseteq W$ where $(X \setminus V)$ is closed, we have by Theorem 2.11, $X \setminus V \subseteq int^*(W)$, and so $X \setminus int^*(W) \subseteq V$. Again, $U \cap W = \emptyset \Rightarrow U \cap int^*(W) = \emptyset$, and so $cl^*(U) \subseteq X \setminus int^*(W) \subseteq V$. Thus $F \subseteq U \subseteq cl^*(U) \subseteq V$, where U is a $\mathcal{H}$g-open set.

(2) $\Rightarrow$ (1): Suppose F and K be two disjoint closed sets of X. Putting $V = X \setminus K$. Now, F is a closed set and V is an open set of X such that $F \subseteq V$. So by (2) there exist a $\mathcal{H}$g-open set U such that $F \subseteq U \subseteq cl^*(U) \subseteq V$. Putting $W = X \setminus cl^*(U)$, we have $F \subseteq U$ and $K = X \setminus V \subseteq W$, where U and W are disjoint $\mathcal{H}$g-open sets. □

Corollary 3.33. *The following statements are equivalent:*

(1) *For each pair of disjoint closed sets F and K, there exist disjoint g-pre-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.*
(2) *For each closed set F and any open set V containing F, there is a g-pre-open set U such that $F \subseteq U \subseteq pcl(U) \subseteq V$.*

Proof. By putting $\mathcal{H} = \mathcal{H}_i$ in Theorem 3.32, the result holds. □

The equivalent conditions presented in the following theorem brings to mind the conditions related to the property of regularity.

Theorem 3.34. *Let $(X, \tau, \mathcal{H})$ be a codense-hereditary T_1-space. Then the following are equivalent:*

(1) *For each closed set F and each $x \notin F$, there exist disjoint $\mathcal{H}$g-open sets U and V such that $x \in V$ and $F \subseteq U$.*
(2) *For each open set V of X and each point $x \in V$, there exists a $\mathcal{H}$g-open set U such that $x \in U \subseteq cl^*(U) \subseteq V$.*

Proof. Since X is a T_1-space, it is sufficient to put $K = \{x\}$ in part (1) of Theorem 3.32 and also to put $F = \{x\}$ in part (2) of the same theorem. □

Corollary 3.35. *Let X be a T_1-space. Then the following are equivalent:*

(1) *For each closed set F and each $x \notin F$, there exist disjoint g-pre-open sets U and V such that $x \in V$ and $F \subseteq U$.*
(2) *For each open set V of X and each point $x \in V$, there exists a g-pre-open set U such that $x \in U \subseteq cl^*(U) \subseteq V$.*

Proof. By putting $\mathcal{H} = \mathcal{H}_i$ in Theorem 3.34, the result is obtained. □

It is appropriate to mention here that since every open set is a $\mathcal{H}$g-open set, therefore, being T_3 (resp. normality) of a space requires establishing Theorem 3.34 (resp. Theorem 3.32).

As a final point, for further research, we suggest interested researchers to read the article [16], entitled "An approach to change the topology of a topological space with the help of its closed sets in the presence of grills". In the mentioned paper, the author has changed the initial topology of any topological space to another topology by focusing on its closed sets. Therefore, interested researchers can follow that approach by focusing on the generalized closed sets considered in the present paper.

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