Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
https://doi.org/10.22080/cjms.2025.28681.1745
Caspian J Math Sci. **14**(1)(2025), 137-142 (Research Article)

Constructing a new class of H_v-semigroups

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Abstract. Let X be a non-empty set and H_X be the set of all mappings from X to $P^*(X)$, when $P^*(X)$ is the family of all non-empty subsets of X. In this paper, we define the hyperoperation $\circledcirc$ on H_X such that $(H_X, \circledcirc)$ is an H_v-semigroup. Then we prove that the fundamental relation β^* on H_X is the trivial relation.

Keywords: hyperoperation, H_v-semigroup, fundamental relation.

2000 Mathematics subject classification: 20N20.

1. Introduction

Algebraic hyperstructures are a generalization of classical algebraic structures. In a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a non-empty set. More exactly, let H be a non-empty set. Then a map $\circ : H \times H \to P^*(H)$ is called a hyperoperation when $P^*(H)$ is the family of non-empty subsets of H and the couple $(H, \circ)$ is called a hypergroupoid. The hyper product of two subsets A and B of

[1]Corresponding author: dheidari82@gmail.com
Received: 22 February 2025
Revised: 19 April 2025
Accepted: 19 May 2025
How to Cite: Heidari, Dariush. Constructing a new class of H_v-semigroups, Casp.J. Math. Sci.,**14**(1)(2025), 137-142.

H defines as follows

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

Also, the notion $x \circ A$ is used for $\{x\} \circ A$ for every $x \in H$.

We deal with the large class of hyperstructures called H_v-semigroups were introduced by Vougiouklis [10] as a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule and so on). After the introduction of the notion of Hv-structures, several authors studied different aspects of Hv-structures. For instance, Vougiouklis [11], Spartalis [8, 9], Davvaz [5]. The hyperstructure $(H, \circ)$ is called an H_v-semigroup if the weak associativity property holds i. e. $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$ for every $x, y, z \in H$. Also, an H_v-semigroup $(H, \circ)$ is called an H_v-group if satisfies the reproductive property: $x \circ H = H \circ x = H$ for every $x \in H$.

In [8] Spartalis introduced a wide class of H_v-semigroups, obtained from a semigroup, using families of non-empty sets. These constructions, called $S{-}H_v$-semigroups, are more general than the well known complete semihypergroups. Also, Corsini and Vougiouklis in 1989 introduced the uniting elements method as follows: Take the partition in a group G for which put in the same class, all pairs of elements that causes the non-validity of a not valid property d. The quotient by this partition G/d is an Hv-structure [3].

The main tool to study hyperstructures is the fundamental relation β^* that is defined as the smallest equivalence relation so that the quotient be the corresponding classical structures. Let $(H, \circ)$ be an H_v-semigroup and $\mathcal{U}$ be the set of all finite hyperproducts of elements of H. Then the relation β defines as $x\beta y$ if and only if $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$ and the fundamental relation β^* is the transitive closure of β. One can see [1, 2, 4] for more details.

Also, the hyperstructure $(H, \circ)$ is called a semihypergroup if the associative property hold, i. e. $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in H$.

For the basic concepts and terminology of semihypergroup, the reader is refereed to the fundamental book [5].

As we know, transformation semigroups are also of utmost importance for semigroup theory, as every semigroup is isomorphic to a transformation semigroup. The main motivation behind this paper is to construct an H_v-semigroup that has a role similar to transformation semigroup.

In Section 2, we analyse the structure of the a generalization of the full transformation semigroupa, say H_X, prove some properties of their ideals and give an example. Also, we define an order $\leq$ such that H_X becomes an ordered H_v-semigroup. Finally, we prove that the fundamental relation β^* on H_X is the trivial relation.

2. Generalized transformations

In this section we generalize the concept of a transformation on a non-empty set and define a new hyperoperation on the set of all generalized transformations on a non-empty set and construct an H_v-semigroup that is a generalization of the full transformations semigroup.

Definition 2.1. Let X be a non-empty set and $P^*(X)$ denote the set of all non-empty subset of X. Then, every function from X to $P^*(X)$ is called an H_v-transformation on X and the set of all H_v-transformations on X is denoted by H_X.

Obviously, every function on a non-empty set X is an H_v-transformation. We define a hyperoperation $\circledcirc$ on H_X that is weak associative and so the hyperstructure $(H_X, \circledcirc)$ is an H_v-semigroup. Thus the full transformation semigroup is a subsemigroup of H_X.

Definition 2.2. Let X be a non-empty set and $H_X = \{f : X \to P^*(X)\}$. For every $f \in H_X$ let Λ_f be the set of all choice functions on the family $\{f(x)\}_{x\in X}$. Then we define the hyperoperation $\circledcirc$ on H_X as follows:

$$f \circledcirc g = \{(f, \lambda_g) | \lambda_g \in \Lambda_g\}, \forall f, g \in H_X$$

where, for every $x \in X$, $(f, \lambda_g)(x) = f(\lambda_g(g(x)))$.

Remark 2.3. Since, the hyperoperation $\circledcirc$ is closed thus $(H_X, \circledcirc)$ is a hypergroupoid.

From now on, we focus on the case that X is a finite set. We consider $X = \{1, 2, ..., n\}$ and use H_n in place of H_X. Thus $(H_n, \circledcirc)$ is a hypergroupoid with $(2^n - 1)^n$ elements.

Also, we denote every $f \in H_X$ by $[F_1, F_2, \cdots, F_n]$ where $f(k) = F_k, k = 1, 2, \cdots, n$. For example consider the identity and universal mappings on X that are denoted by id_X and u_X, respectively, where for every $k \in \{1, 2, \cdots, n\}$, $id_X(k) = \{k\}$ and $u_X(k) = X$. Then $id_X = [\{1\}, \{2\}, \cdots, \{n\}]$ and $u_X = [X, X, \cdots, X]$.

By use of this notation we can say that every element of H_n is an n-tuple $[F_1, F_2, \cdots, F_n]$, $[F_k]$ for short, such that every F_k's is a non-empty subset of $\{1, 2, \cdots, n\}$. Moreover, the hyperoperation $\circledcirc$ is simplify as below:

$$[F_k] \circledcirc [G_k] = \{[A_k] | A_k = F_{\lambda(G_k)}, \lambda : \{G_k\}_{k=1}^n \to \cup_{k=1}^n G_k, \lambda(G_k) \in G_k\}.$$

Theorem 2.4. *Let n be a natural number. Then $(H_n, \circledcirc)$ is an H_v-semigroup with the right identity element id_X.*

Proof. For every $f = [F_k], g = [G_k], r = [R_k] \in H_n$, let $m = [M_k]$ such that for every $k \in \{1, 2, \cdots, n\}$, $M_k = F_{\min(G_{\min(R_k)})}$. Then $m \in$

$f \circledcirc (g \circledcirc r) \cap (f \circledcirc g) \circledcirc r$ thus $(H_n, \circledcirc)$ satisfies the weak associativity property.

$\square$

Example 2.5. Let $n = 2$. Then $X = \{1, 2\}$,

$$H_2 = \{[1,1],[1,2],[1,X],[2,1],[2,2],[2,X],[X,1],[X,2],[X,X]\}$$

and we have the following Cayley table, where h_i is the i'th element of H_2 for $1 \leq i \leq 9$

$\circledcirc$	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1
h_2	h_1	h_2	$\{h_1,h_2\}$	h_4	h_5	$\{h_4,h_5\}$	$\{h_1,h_4\}$	$\{h_2,h_5\}$	$\{h_1,h_2,h_4,h_5\}$
h_3	h_1	h_3	$\{h_1,h_3\}$	h_7	h_9	$\{h_7,h_9\}$	$\{h_1,h_7\}$	$\{h_3,h_9\}$	$\{h_1,h_3,h_7,h_9\}$
h_4	h_5	h_4	$\{h_4,h_5\}$	h_2	h_1	$\{h_1,h_2\}$	$\{h_2,h_5\}$	$\{h_1,h_4\}$	$\{h_1,h_2,h_4,h_5\}$
h_5	h_5	h_5	h_5	h_5	h_5	h_5	h_5	h_5	h_5
h_6	h_5	h_6	$\{h_5,h_6\}$	h_8	h_9	$\{h_8,h_9\}$	$\{h_5,h_8\}$	$\{h_6,h_9\}$	$\{h_5,h_6,h_8,h_9\}$
h_7	h_9	h_7	$\{h_7,h_9\}$	h_3	h_1	$\{h_1,h_3\}$	$\{h_3,h_9\}$	$\{h_1,h_7\}$	$\{h_1,h_3,h_7,h_9\}$
h_8	h_9	h_8	$\{h_8,h_9\}$	h_6	h_5	$\{h_5,h_6\}$	$\{h_6,h_9\}$	$\{h_5,h_8\}$	$\{h_5,h_6,h_8,h_9\}$
h_9	h_9	h_9	h_9	h_9	h_9	h_9	h_9	h_9	h_9

Let $f = [F_k] \in H_n$ and $<< f >>= f \circledcirc u_X$. Then $<< f >>$ is the subset of H_n that contains all $r = [R_k]$ such that for every $k \in \{1, 2, \cdots, n\}$ we have $R_k \in \left\{F_j | j \in \{1, 2, \cdots, n\}\right\}$. In the following we prove some properties.

Lemma 2.6. *For every $f \in H_n$ the following assertions hold.*

(1) *If σ is a permutation on $\{1, 2, \cdots, n\}$ and $f_\sigma = [F_{\sigma(k)}]$, then $<< f >>=<< f_\sigma >>$.*
(2) *If $F = \{F_1\} \cup \{F_2\} \cup \cdots \cup \{F_n\}$, then $<< f >>$ has $|F|^n$ elements.*
(3) *If f is one-to-one, then $<< f >>$ has n^n elements.*
(4) *If f is a constant mapping, then $<< f >>= \{f\}$.*
(5) *$<< f >>$ is a right ideal of H_n.*

Proof. (1), (2), (3) and (4) are straightforward.

5) Suppose that $g \in << f >>$ and $r \in H_n$ then, for every $t \in g \circledcirc r$, there exist choice functions λ and μ such that for every $1 \leq k \leq n$

$$T_k = G_{\lambda(R_k))} = F_{\mu(G_\lambda(G_K))}$$

so $t \in << f >>$, thus, $<< f >>$ is a right ideal of H_n.

$\square$

Let F_1,F_2,$\cdots$ and F_n be distinct subsets of X, then we have the following chain of right ideals:

$$\begin{aligned} << F_1, F_1, \cdots, F_1 >> &\subseteq << F_1, F_2, F_1, F_1, \cdots, F_1 >> \\ &\subseteq << F_1, F_2, F_3, F_1, F_1 \cdots, F_1 >> \\ &\subseteq \cdots \subseteq << F_1, F_2, \cdots, F_n >> \end{aligned}$$

In [7], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an ordered semihypergroup $(S, \circ, \leq$ $)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is compatible with the hyperoperation, meaning that for any $x, y, z \in S$, we have $x \leq y \Longleftrightarrow z \circ x \leq z \circ y$ and $x \circ z \leq y \circ z$. Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Lemma 2.7. *$(H_n, \circledcirc, \leq)$ is a right ordered H_v-semigroup, where for every $f, g \in H_n$ the order $\leq$ defines as follows*

$$f \leq g \Leftrightarrow F_k \subseteq G_k, \forall 1 \leq k \leq n.$$

Proof. Let $f, g \in H_n$ such that $f \leq g$. Then for every $r \in H_n$ and $t \in f \circledcirc r$ there exists a choice function λ such that for every $1 \leq k \leq n$ we have

$$T_k = F_{\lambda(R_k)} \subseteq G_{\lambda(R_k)} \in g \circledcirc r$$

so, $t \leq [G_{\lambda(R_k)}]$, thus $(H_n, \circledcirc, \leq)$ is a right ordered H_v-semigroup. □

Lemma 2.8. *The H_v-semigroup H_n has n^n minimal and only one maximal element.*

Proof. An element $[F_k]$ is minimal if and only if for every $1 \leq k \leq n$, F_k has no empty subset so $|F_k| = 1$ thus we have n^n minimal element. Also, u is only maximal element. □

Theorem 2.9. *The fundamental relation β is not transitive and β^* is trivial relation on H_n for every natural number n.*

Proof. If $n = 1$, then the result hold trivially. Let $n > 1$ and $f = [\{1\}, \{1\}, \cdots, \{1, n\}]$. Then there is no hyper product contains i and f, so $(i, f) \notin \beta$.

Also, for every f and g in H_n we have

$$f\beta[F_1, F_1, \cdots, F_1]\beta[G_1, F_1, F_1, \cdots, F_1]\beta[G_1, G_1, \cdots, G_1]\beta g$$

Thus $(f, g) \in \beta^*$ so β^* is the trivial relation on H_n. □

3. Conclusion

We introduced the structure of the a generalization of the full transformation semigroup more precisely let X be a non-empty set and $P^*(X)$ denote the set of all non-empty subset of X. Then, every function from X to $P^*(X)$ is called an H_v-transformation on X and the set of all H_v-transformations on X is denoted by H_X. Morever, we proved that H_X is an H_v-semigroup such that contains the full transformation semigroup. However the following question remains for the future work; "Is every H_v-semigroup can be embedded in H_X?"

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