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On τ-Lindelöf spaces

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Abstract. A topological space X is said to be *τ-Lindelöf* if every open cover of X contains a subcover of cardinality $\leq \tau$. This paper aims to study those spaces in which all points have a τ-Lindelöf neighborhood, calling them *locally τ-Lindelöf*. This work extends the known results from Lindelöf and locally Lindelöf spaces to τ-Lindelöf and locally τ-Lindelöf spaces.

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1. Introduction

Compactness is one of the most fundamental concepts in topology that is a generalization of the "closed and bounded" property in the Euclidean space $\mathbb{R}$. A topological space is said to be *compact* if every open cover has a finite subcover. The definition of compactness adopted here was given by Alexandroff and Urysohn in 1923 [3]. They deeply analyzed the

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concept of compactness in [4]. Not all important spaces are compact-even the real line is not. As a generalization of compactness we consider local compactness: a Hausdorff space X is called *locally compact* if every point $x \in X$ has a compact neighborhood. The Euclidean space $\mathbb{R}^n$ and any infinite discrete space are locally compact. However, the space $\mathbb{Q}$ is not locally compact. The notion of a locally compact space was introduced, independently, by Alexandroff in [1, 2] and Tietze in [11].

In 1903, Lindelöf proved that any family of open subsets of $\mathbb{R}^n$ contains a countable subfamily with the same union, see [9]. Motivated by this result, Alexandroff and Urysohn introduced the notion of a Lindelöf space in [4] as follows: a space X is called *Lindelöf* if every open cover of X has a countable subcover. Obviously, every compact space is Lindelöf. However, the converse is not true.

More than a century has passed since the invention of the notion of Lindelofness, and researchers in topology and analysis are still interested in this concept and its generalizations. In this paper, among the generalizations and types of Lindelofness, we are interested in the concept *τ-Lindelöfness* defined by Vesko Valov [12]. A space X is called *τ-Lindelöf* if every open cover of X contains a subcover of cardinality $\leq \tau$. We continue this approach and investigate spaces in which all points have a τ-Lindelöf neighborhood. We aim to extend the known results about Lindelöf and locally Lindelöf spaces.

Throughout this paper, all considered spaces are assumed to be **Hausdorff**. We usually denote by X a topological space and by $\tau(X)$ the topology on X. The "Axiom of Choice" is assumed throughout this paper. If X is a set, $|X|$ denotes the cardinality of X. For any infinite cardinal number τ, τ^+ stands for the *successor cardinal* of τ, i.e., the least cardinal greater than τ. Assuming the axiom of choice, it is known that each cardinal has a successor cardinal. Our notation and terminology are mainly as in [6] and [13].

2. τ-Lindelöf spaces

We cite the following definition from [12].

Definition 2.1. A space X is called *τ-Lindelöf* if every open cover of X contains a subcover of cardinality $\leq \tau$.

Notice immediately that Lindelöf spaces (e.g., compact spaces) are precisely $\aleph_0$-Lindelöf. In general, every infinite discrete space of cardinality τ is τ-Lindelöf. Hereafter, we will restrict our attention to only $\tau > \aleph_0$ without loss of generality.

Lemma 2.2. *The following statements hold.*

(1) *Every closed subspace of a τ-Lindelöf space is τ-Lindelöf.*

(2) *The continuous image of a τ-Lindelöf space is τ-Lindelöf.*

Proof. (1) Let X be a τ-Lindelöf space and $F \subseteq X$ a closed subspace. Then for any open cover $\{U_\alpha\}_{\alpha \in A}$ of F, we have the open cover $\{U_\alpha\} \cup \{X \setminus F\}$ of X. Since X is τ-Lindelöf, $\{U_\alpha\} \cup \{X \setminus F\}$ contains a subcover of cardinality $\leq \tau$, say $\{U_i\}$. If necessary, we remove $X \setminus F$, and we have the desired subcover of F. Thus, F is also τ-Lindelöf.
(2) Let $f : X \to f(X)$ be a continuous map and let $\mathcal{A}$ be any covering of $f(X)$. Then $\left\{f^{-1}(A) | A \in \mathcal{A}\right\}$ is a collection of sets covering X; these sets are open in X by the continuity of f. Since X is τ-Lindelöf, $\left\{f^{-1}(A) | A \in \mathcal{A}\right\}$ contains a subcover of cardinality $\leq \tau$, say $\left\{f^{-1}(A_i)\right\}$. Then, $\{A_i\}$ forms a subcover of the original cover $\mathcal{A}$ with cardinality $\leq \tau$. Thus, $f(X)$ is τ-Lindelöf, as desired. □

Lemma 2.3. *Every subspace of X is τ-Lindelöf if and only if every open subspace of X is τ-Lindelöf.*

Proof. If every subspace of X is τ-Lindelöf, there is nothing to prove. Now, let S be an arbitrary subset of X and let $\{O_i\}_{i \in I}$ be an open cover of S. The family $\{O_i\}_{i \in I}$ is an open cover of the open set $\bigcup \{O_i\}_{i \in I}$. By hypothesis, $\bigcup \{O_i\}_{i \in I}$ contains a subcover of cardinality $\leq \tau$, say $\left\{O_{i_j}\right\}_{j \in J}$. This subcover is also a cover of the set S, as desired. □

Lemma 2.4. *Let $\{C_i\}_{i \in I}$ be a family of τ-Lindelöf subspaces of a topological space X, where the index set I has cardinality $\lambda < \tau$. Then $\bigcup_{i \in I} \{C_i\}$ is τ-Lindelöf.*

Proof. Consider a collection $\{C_i\}_{i \in I}$, where $|I| = \lambda < \tau$, of τ-Lindelöf subspaces of X. Let $C = \bigcup C_i$ and $\mathcal{A}$ be an arbitrary cover for C. For each i, the cover $\mathcal{A}$ of C_i contains a subcover of cardinality $\leq \tau$. Assuming the axiom of choice, we have $\lambda \times \tau = \max\{\lambda, \tau\} = \tau$. Then, the cover $\mathcal{A}$ of C contains a subcover of cardinality $\leq \tau$. Therefore, C is τ-Lindelöf. □

Let us recall that a family Ω of subsets of X has the *finite intersection property* if the intersection of any finite subcollection of Ω is non-empty. For more details, see [13, Definition 17.3]. We shall be interested in a counterpart of this definition.

Definition 2.5. We say that a family $\mathcal{F}$ of non-empty subsets of X has the α *intersection property*, if the intersection of any subcollection of $\mathcal{F}$ with cardinality α is non-empty.

With the above definition, we have the following useful characterization of the τ-Lindelöf property.

Lemma 2.6. *A space X is τ-Lindelöf if and only if every family of closed non-empty subsets of X which has the τ intersection property has a non-empty intersection.*

Proof. Let $\mathcal{F}$ be a family of closed subsets of X with the τ intersection property. Suppose that $\bigcap_{F\in\mathcal{F}} F=\emptyset$. Then $\mathcal{U}=\{X\setminus F : F\in\mathcal{F}\}$ is an open cover for X. Indeed,

$$\bigcup_{U\in\mathcal{U}} U=\bigcup_{F\in\mathcal{F}} X\setminus F = X\setminus\bigcap_{F\in\mathcal{F}} F = X\setminus\emptyset = X.$$

Define $\mathcal{F}'=\{X\setminus U : U\in\mathcal{U}'\}\subseteq\mathcal{F}$. Since X is τ-Lindelöf, $\bigcup_{U\in\mathcal{U}} U$ contains a subcover of cardinality $\leq\tau$, say $\bigcup_{U\in\mathcal{U}'} U$. However, this means that

$$\emptyset = X\setminus\bigcup_{U\in\mathcal{U}'} U=\bigcap_{U\in\mathcal{U}'} X\setminus U=\bigcap_{F\in\mathcal{F}'} F.$$

This contradicts the τ intersection property. Hence $\bigcap_{F\in\mathcal{F}} F\neq\emptyset$.

Conversely, let $\mathcal{U}$ be an open cover of X. Then $\mathcal{F}=\{X\setminus U : U\in\mathcal{U}\}$ is a family of closed subsets of X. Suppose that X is not τ-Lindelöf, so for all τ subsets of $\mathcal{U}'\subseteq\mathcal{U}$, there exists $x\in X$ with $x\in X\setminus\bigcup_{U\in\mathcal{U}'} U$. Then $x\in X\setminus\bigcup_{U\in\mathcal{U}'} U=\bigcap_{U\in\mathcal{U}'}(X\setminus U)=\bigcap_{F\in\mathcal{F}'} F$. So, $\mathcal{F}$ has the τ intersection property. Thus, we have

$$\emptyset\neq\bigcap_{F\in\mathcal{F}} F=\bigcap_{U\in\mathcal{U}} X\setminus U = X\setminus\bigcup_{U\in\mathcal{U}} U=\emptyset.$$

That is a contradiction. Hence X is τ-Lindelöf. □

Before proceeding, we need to recall a definition. Following [8], the intersection of any family with cardinality less than λ of open subsets of a topological space X is called a G_λ-*set* (or λ-*open*). The complement of a λ-open set is said to be F_λ-*set* (or λ-*closed*). The topological space X is said to be a P_λ-*space* whenever each G_λ-set in X is open. Obviously, $P_{\aleph_1}$-spaces are precisely P-spaces. Recall that a space X is said to be a P-*space*, if every G_δ is open. Conditions that are equivalent to a space being classified as a P-space are presented in [7, 4J and 14.29].

Lemma 2.7. *The following statements are equivalent:*

(1) *X is a P_λ-space.*
(2) *For every family $\{A_\alpha\}_{\alpha\in I}$ of subsets of X, with $|I|<\lambda$, we have $(\bigcap_{\alpha\in I} A_\alpha)^\circ=\bigcap_{\alpha\in I} A_\alpha^\circ$.*
(3) *For every family $\{A_\alpha\}_{\alpha\in I}$ of subsets of X, with $|I|<\lambda$, we have $\overline{\bigcup_{\alpha\in I} A_\alpha}=\bigcup_{\alpha\in I}\overline{A}_\alpha$.*
(4) *For every family $\{A_\alpha\}_{\alpha\in I}$ of closed subsets of X, with $|I|<\lambda$, $\bigcup_{\alpha\in I} A_\alpha$ is closed.*

Proof. Most of the other implications being obvious or easy, we shall only prove (1) $\Rightarrow$ (2). Assume that $\{A_\alpha\}_{\alpha\in I}$ is a family of subsets of X, with $|I| < \lambda$. Clearly, $(\bigcap_{\alpha\in I} A_\alpha)^\circ \subseteq \bigcap_{\alpha\in I} A_\alpha^\circ$. On the other hand, by hypothesis, $\bigcap_{\alpha\in I} A_\alpha^\circ$ is open and so $\bigcap_{\alpha\in I} A_\alpha^\circ \subseteq (\bigcap_{\alpha\in I} A_\alpha)^\circ$. $\square$

Lemma 2.8. *Every τ-Lindelöf subspace of a P_{τ^+}-space is closed.*

Proof. Let X be a P_{τ^+}-space and K be a τ-Lindelöf subspace of X. Fix $x \in X \setminus K$. Since X is Hausdorff, for each $y \in K$ there are disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Obviously, $\{V_y : y \in K\}$ is an open cover of K. Since K is τ-Lindelöf, $\{V_y : y \in K\}$ contains a subcover of cardinality $\leq \tau$, say $\{V_i : i \in I\}$. Define $U := \bigcap_{i\in I} U_y$. Since X is a P_{τ^+}-space, U is an open neighborhood of x disjoint from K. Since x was an arbitrary point of $X \setminus K$, K must be closed. $\square$

We need the following lemma.

Lemma 2.9. *Let Y be a P_{τ^+}-space. If $C \subseteq Y$ is τ-Lindelöf and disjoint from x, then there exist disjoint open neighborhoods V of x and V' of C.*

Proof. Since $x \notin C$, we can find disjoint open neighborhoods V_y of x and V'_y of y for each $y \in C$. The collection $\left\{V'_y\right\}_{y\in C}$ is a cover of C. By τ-Lindelöfness, $\left\{V'_y\right\}_{y\in C}$ contains a subcover of cardinality $\leq \tau$, say $\left\{V'_{y_i}\right\}_{i\in I}$. Define $V := \bigcap_{i\in I} V'_{y_i}$ and $V' := \bigcup_{i\in I} V'_{y_i}$. Since Y is a P_{τ^+}-space, V, being an intersection of open neighborhoods, is open. Clearly, $V \cap V' = \emptyset$, $x \in V$ and $C \subseteq V'$, as desired. $\square$

We conclude this section with the following fact.

Corollary 2.10. *If X is a τ-Lindelöf P_{τ^+}-space, then X is a normal space.*

Proof. Let A and B be two disjoint closed subsets of X. Lemma (2.2)(1) implies that A and B are τ-Lindelöf. Thus, by Lemma (2.9), the result follows. $\square$

3. Locally τ-Lindelöf spaces

A space X is called *locally compact* if each point of X has a compact neighborhood. We refer the reader to [13, 18, §6] and [6, §3] for more details.

There are three definitions of *locally Lindelöf* in the literature:

(1) Every point has a Lindelöf neighborhood.
(2) Every point has a neighborhood whose closure is Lindelöf.
(3) Every neighborhood of a point contains a neighborhood whose closure is Lindelöf.

Obviously, (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Furthermore, (1) $\Rightarrow$ (3) if X is a P-space.

The following notion is motivated by the above definitions.

Definition 3.1. We say a space X is *locally τ-Lindelöf* if every point $x \in X$ has a τ-Lindelöf neighborhood.

We will now examine a significant concept that establishes the connection between locally τ-Lindelöf spaces and τ-Lindelöf spaces. Before proceeding, let us recall that a space Y is said to be an *extension* of a space X if Y contains X as a dense subspace. An extension Y of X is called a *one-point extension* of X if $Y \setminus X$ is a singleton. Alexandroff proved that any locally compact non-compact space X has a one-point compact extension, called the *one point compactification* of X. In order to achieve our aims, we make the following definition.

Definition 3.2. Let Y be τ-Lindelöf. If $X \hookrightarrow Y$ is an embedding such that X is dense in Y, then we say Y is a *τ-Lindelöfication of X*. If $Y \setminus X$ is a singleton, then we say Y is *the one-point τ-Lindelöfication of X*.

Theorem 3.3. *Let X be a P_{τ^+}-space. The following statements are equivalent:*

(1) *X is a locally τ-Lindelöf space.*
(2) *There exists a one-point τ-Lindelöfication Y of X.*

Moreover, if such a τ-Lindelöfication Y exists, then it is unique up to homeomorphism.

Proof. (2) $\Rightarrow$ (1) Suppose $Y = X \cup \{\infty\}$ is a one-point τ-Lindelöfication. The subspace $X \subseteq Y$ is Hausdorff, and if $x \in X$ choose disjoint neighborhoods $x \in U$ and $\infty \in V$. Let $C = Y \setminus V$. By Lemma 2.2(1), C is τ-Lindelöf. And

$$x \in U \subseteq C = Y \setminus V \subseteq Y \setminus \{\infty\} = X.$$

Therefore, X is locally τ-Lindelöf at x.

(1) $\Rightarrow$ (2) Suppose X is locally τ-Lindelöf. Define $Y := X \cup \{\infty\}$, where the element $\infty \notin X$ is a distinct symbol. Define a topology on Y by

$$\tau = \{U : U \subseteq X \text{ open}\} \cup \{Y \setminus C : C \subseteq X \ \tau\text{-Lindelöf}\}.$$

The sets of the form $\{U : U \subseteq X \text{ and open}\}$ are precisely those that are already open in X. The sets $\{Y \setminus C : C \subseteq X \ \tau\text{-Lindelöf}\}$ are those containing ∞ whose complements are τ-Lindelöf subsets of X.

(1) We first confirm that this indeed defines a topology. Clearly, $\emptyset$ and X are in $\tau(X)$. Let $\{C_i\}_{i \in I} \subseteq X$ be a collection of τ-Lindelöf subspaces. It is easy to see that $\bigcap_{i \in I} C_i$ is τ-Lindelöf. Lemma 2.4 yields that $\bigcup_{k=i_1}^{i_n} C_k$ is τ-Lindelöf for each $n \in \mathbb{N}$.

Now assume that $U \subseteq X$ is open and $C \subseteq X$ is τ-Lindelöf. By Lemma 2.8, we have C is closed, and so $Y \setminus C$ is open. Thus,

$$U \cap (Y \setminus C) = U \cap (X \setminus C) \subseteq X$$

is an element of $\tau(X)$. On the other hand, $C \cap (X \setminus U)$ is closed and so it is τ-Lindelöf, by Lemma 2.2(1). Hence,

$$U \cup (Y \setminus C) = Y \setminus (C \cap (X \setminus U))$$

is an element of $\tau(X)$.
Moreover, since X is open in Y, the subspace topology on X induced by $\tau(X)$ is the original topology on X.

(2) (Y is Hausdorff.) If two points $x, y \in Y$ are in X, then they can be separated by the corresponding open neighborhoods that arise from the Hausdorff $X \subseteq Y$. To separate $x \in X$ and ∞, local τ-Lindelöfness implies there exists a τ-Lindelöf C containing an open neighborhood U of x. Hence, U and $Y \setminus C$ separate x and ∞.
(3) (Y is τ-Lindelöf.) Let $\{U_i\}_{i \in I}$ be an open cover. ∞ lies in some $U_0 = Y \setminus C$, for C τ-Lindelöf. Now $\{U_i \cap C\}_{i \in I}$ is an open cover of C. By τ-Lindelöfness, C contains a subcover of cardinality $\leq \tau$, say $\bigcup_J (U_j \cap C)$. Thus, $C \subseteq \bigcup_J U_j$, and $Y = U_0 \cup \bigcup_J U_j$.
(4) (Y is unique up to homeomorphism.) Assume that there is another $Y' = X \cup \{p\}$ that is τ-Lindelöf such that subspace topology on X agrees with our original topology on X. We will show that the map $Y \to Y'$ defined by the identity on X and $\infty \to p$ is a homeomorphism, so that the only difference between Y and Y' is the naming of the added point.
   - Since Y' is Hausdorff, $\{p\}$ is closed. Thus, $X \subseteq Y'$ is open. This yields that the subspace topology on X consists exactly of open subsets of Y' which contain X. Hence, The collection $\{U\}$ of the sets $\{U : U \subseteq X \text{ open}\}$ open sets in Y' are exactly the open sets of X.
   - If $V \subseteq Y'$ is open and $p \in V$, then $C = Y' \setminus V$ is closed in Y'. Hence, C is τ-Lindelöf by Lemma 2.2(1). But in fact $C \subseteq Y' \setminus \{p\} = X$, so $V = Y' \setminus C$, where $C \subseteq X$, is τ-Lindelöf. Conversely, if $C \subseteq X$ is τ-Lindelöf, then it is closed in Y' by Lemma 2.8. This yields that $Y' \setminus C$ is open in Y'.

□

Proposition 3.4. *A P_{τ^+}-space X is locally τ-Lindelöf if and only if it is homeomorphic to an open subset of a τ-Lindelöf space Y.*

Proof. ($\Rightarrow$) It follows from the fact that X is open in the τ-Lindelöf space $Y = X \cup \{\infty\}$, see Theorem 3.3.
($\Leftarrow$) Define $Y_\infty = Y \setminus X$. Since Y_∞ is closed in Y, it is τ-Lindelöf by Lemma 2.2(1). Using Lemma 2.9, we can find disjoint open sets U and $Y_\infty \subseteq V$ in Y, where $x \in U$. Then $K = Y \setminus V$ is the desired τ-Lindelöf neighborhood of x in X. □

Proposition 3.5. *Let X be a P_{τ^+}-space. The following statements are equivalent:*

1. *X is locally τ-Lindelöf.*
2. *For all $x \in X$ and neighborhoods U of x, there exists a neighborhood V of x such that $\overline{V} \subseteq U$ and $\overline{V}$ is τ-Lindelöf.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and U be a neighborhood of x. Let Y be the one-point τ-Lindelöfication of X. By Theorem 3.3, Y is τ-Lindelöf. Define $C := Y \setminus U$. Since C is closed in Y, it is τ-Lindelöf by Lemma 2.2(1). Lemma 2.9 implies that there exists disjoint neighborhoods V around x and V' around $Y \setminus U$. Thus we have

$$x \in V \subseteq \overline{V} \subseteq Y \setminus V' \subseteq (Y \setminus C) = U.$$

(2) $\Rightarrow$ (1) Assume (2). Take $U = X$, at each points there is a τ-Lindelöf $\overline{V}$ containing x that contains the open neighborhood V of x. □

Proposition 3.5 yields the following corollary.

Corollary 3.6. *Let X be a locally τ-Lindelöf P_{τ^+}-space, K be a τ-Lindelöf set in X, and U be an open subset, with $K \subseteq U$. Then there exists an open set V such that:*

1. *$\overline{V}$ is a τ-Lindelöf space;*
2. *$K \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. By Proposition 3.5, for every $x \in K$, we find an open set $V(x)$ such that $\overline{V(x)}$ is τ-Lindelöf and $x \in V(x) \subseteq \overline{V(x)} \subseteq U$. We have $K \subseteq \bigcup_{x\in K} V(x)$. By τ-Lindelöfness, $\bigcup_{x\in K} V(x)$ contains a subcover of cardinality $\leq \tau$, say $\bigcup_{i\in I} V(x_i)$. Notice that if we take $V = \bigcup_{i\in I} V(x_i)$, then we deduce that $K \subseteq V \subseteq \overline{V} \subseteq \bigcup_{i\in I} \overline{V(x_i)} \subseteq U$, as desired. □

Lemma 3.7. *Any open or closed subset of a locally τ-Lindelöf P_{τ^+}-space is locally τ-Lindelöf.*

Proof. Let X be a locally τ-Lindelöf space. If $Y \subseteq X$ is open, then any point in Y has a neighborhood whose closure is τ-Lindelöf and contained in Y by Proposition 3.5. Hence, Y is locally τ-Lindelöf. Now suppose $Z \subseteq X$ is closed. Any $x \in Z$ has a τ-Lindelöf neighborhood in X, say U. We note that $\overline{U \cap Z} = \overline{U} \cap Z$ is a closed subset of the τ-Lindelöf set U. From Lemma 2.2(1), we deduce that $\overline{U} \cap Z$ is τ-Lindelöf. Hence, $\overline{U} \cap Z$ is a τ-Lindelöf neighborhood of x in Z. Thus, Z is locally τ-Lindelöf. □

For invariance properties, we have the following lemma.

Lemma 3.8. (1) *A locally τ-Lindelöf subset A of a P_{τ^+}-space Y is of the form $V \cap F$, where V is open and F is closed in Y.*

(2) *A subspace of a locally τ-Lindelöf P_{τ^+}-space is locally τ-Lindelöf if and only if it is of the form $V \cap F$, where V is open and F is closed.*

Proof. (1) Assume that A is locally τ-Lindelöf. Each $a \in A$ has a neighborhood $V(a)$ in Y such that $\overline{V(a)} \cap A$ is τ-Lindelöf and is closed in Y by Lemma 2.8. Define $V := \bigcup\{V(a)|a \in A\}$. Obviously, V is open in Y and contains A. Moreover, the formula $V(a) \cap A = V(a) \cap (\overline{V(a)} \cap A)$ shows that each $V(a) \cap A$ is closed in $V(a)$. This implies that A is closed in V. Thus, $A = V \cap F$, where F is a closed set in Y, as desired.
(2) By (1), it suffices to show that if $A = V \cap F$, then A is locally τ-Lindelöf. Take $a \in A$. By Proposition 3.5, we can find a τ-Lindelöf open U in Y satisfying $a \in U \subseteq \overline{U} \subseteq V$. Consider the neighborhood $U \cap A$ of a in A. The closure of this neighborhood in A is $\overline{U} \cap A = \overline{U} \cap (V \cap F) = \overline{U} \cap F$, which is a set closed in $\overline{U}$, as desired. □

The following is an easy consequence of Lemma 3.8.

Corollary 3.9. *A dense subspace D of a locally τ-Lindelöf P_{τ^+}-space X is locally τ-Lindelöf if and only if it is open in X.*

Corollary 3.10. *Let X be a locally τ-Lindelöf P_{τ^+}-space. A subset $A \subseteq X$ is open if and only if its intersection with each τ-Lindelöf $C \subseteq X$ is open in C.*

Proof. We only prove the converse. Assume $A \cap C$ is open in C for each τ-Lindelöf C. Take $a \in A$. By Proposition 3.5, we can find a τ-Lindelöf neighborhood $V(a)$. Since $A \cap \overline{V(a)}$ is open in $\overline{V(a)}$, we infer that $A \cap V(a)$ is open in $V(a)$. Hence, $A \cap V(a)$ is open in X. Thus, each $a \in A$ has a neighborhood in A. This implies that A is open in X, as desired. □

We now look at the products of locally τ-Lindelöf spaces.

Corollary 3.11. *If X and Y are locally τ-Lindelöf P_{τ^+}-spaces, then so is $X \times Y$.*

Proof. Let $x \in X$, $y \in Y$. By Proposition 3.5, there are open sets $U \subseteq X$, $V \subseteq Y$ with $\overline{U}$ and $\overline{V}$ τ-Lindelöf. Hence, $U \times V$ is an open neighbourhood of (x, y) in $X \times Y$ with τ-Lindelöf closure $\overline{U} \times \overline{V}$, and we are done. □

One of the most important results in analysis and topology is Urysohn's Lemma which states that any two disjoint closed sets in a normal space are completely separated, see [7, 3.13]. A version of Urysohn's Lemma for locally compact Hausdorff spaces is stated in [10, 2.12]. We now state the following.

Theorem 3.12. (Urysohn's Lemma for locally τ-Lindelöf spaces) *Let X be a locally τ-Lindelöf P_{τ^+}-space, and let $K, F \subseteq X$ be two disjoint sets, where K is τ-Lindelöf and F is closed. Then there exists a continuous function $f : X \to [0,1]$ such that $f\big|_K = 1$ and $f\big|_F = 0$.*

Proof. With the help of Corollary 3.6 for the pair $K \subseteq X \setminus F$, we can find an open set E, where E is τ-Lindelöf, such that $K \subseteq E \subseteq \overline{E} \subseteq X \setminus F$. Again, Corollary 3.6 yields that for the pair $K \subseteq E$, we can find another open set G with G τ-Lindelöf, such that $K \subseteq G \subseteq \overline{G} \subseteq E$. $\overline{E}$ (equipped with the induced topology) is a τ-Lindelöf space, hence it is normal by Lemma 2.10. Using Urysohn's Lemma, there is a continuous function $g : \overline{E} \to [0,1]$ such that $g\big|_K = 1$ and $g\big|_{\overline{E}\setminus G} = 0$. Define the function $f : X \to [0,1]$ by

$$f(x) = \begin{cases} g(x) & x \in \overline{E} \\ 0 & x \in X \setminus \overline{E}. \end{cases}$$

Obviously, $f\big|_E = g\big|_E$, and we have $f\big|_E$ is continuous. Take the open set $A = X \setminus \overline{G}$. Hence, it is clear that $f\big|_A = 0$. Therefore, we have two open sets E and A, where $A \cup E = X$. Note that both $f\big|_A$ and $f\big|_E$ are continuous. Thus, we deduce that f is continuous. The other two properties $f\big|_K = 1$ and $f\big|_F = 0$ are clear. □

The intersection of two dense sets may not be dense, but if the sets are also open, the intersection is dense. A space X is called a *Baire space* if the intersection of each countable family of dense sets in X is dense. It is known that every locally compact space is a Baire space, see [13, 25.4] for example. We now make the following.

Theorem 3.13. (Baire's Theorem for locally τ-Lindelöf spaces) *If $\{V_i | i \in I\}$ is a τ^+ collection of open dense subsets in a locally τ-Lindelöf P_τ^+-space X, then $\bigcap V_i$ is also dense.*

Proof. Let $W_{i_0} \subseteq X$ be a non-empty open set. Since V_{i_1} is dense, $V_{i_1} \cap W_{i_0}$ is non-empty open. By Proposition 3.5, there is a non-empty open set W_{i_1}, where $\overline{W_{i_1}}$ is τ-Lindelöf and $\overline{W_{i_1}} \subseteq V_{i_1} \cap W_{i_0}$. Inductively, we get open sets W_{i_n} such that $\overline{W_{i_n}} \subseteq V_{i_n} \cap W_{i_{n-1}}$ for all $n \geq 1$. $\overline{W_{i_n}}$ is a decreasing sequence of τ-Lindelöf sets, so has τ intersection property. Lemma 2.6 yields $K = \bigcap \overline{W_i}$ is non-empty τ-Lindelöf. Since $K \subseteq W_{i_0}$ and $K \subseteq V_i$ for all $i \in I$, we infer that $W_{i_0} \cap (\bigcap V_i)$ is non-empty. Thus $\bigcap V_i$ intersects every non-empty open set and so is dense. □

The second half of this section is devoted to the study of k_τ-spaces. First, let us recall that a space X is said to be a *k-space* if it has the weak topology determined by the family of its compact subspaces. The following is a counterpart of this definition.

Definition 3.14. We say a space X is a *k_τ-space* if it has the weak topology determined by the family of its τ-Lindelöf subspaces.

Obviously, every locally τ-Lindelöf space is a k_τ-space. To describe the relation of locally τ-Lindelöf spaces and k_τ-spaces, we need recall the following definition and theorem from [5].

Definition 3.15. ([5, Definition 8.4, §VI]) Let $\{Y_\alpha | \alpha \in \mathcal{A}\}$ be a family of spaces. For each α, let Y'_α be the space $\{\alpha\} \times Y_\alpha$, so that $Y'_\alpha \cong Y_\alpha$ and the family $\{Y'_\alpha | \alpha \in \mathcal{A}\}$ is pairwise disjoint. The *free union* of the given family $\{Y_\alpha | \alpha \in \mathcal{A}\}$ is the set $\bigcup_\alpha Y'_\alpha$ with the weak topology determined by the space Y'_α; this space is denoted by $\sum_\alpha Y'_\alpha$.

Theorem 3.16. ([5, Theorem 8.5, §VI]) *Let (X, τ) be a space with the weak topology determined by the covering $\{A_\alpha | \alpha \in \mathcal{A}\}$. Let $A = \sum_\alpha A'_\alpha$ be the free union of $\{A_\alpha | \alpha \in \mathcal{A}\}$. For each α, let $h_\alpha : A'_\alpha \to A_\alpha \subseteq X$ be the homeomorphism $(\alpha, a) \to a$. Define $h : \sum_\alpha A'_\alpha \to X$ by $h = h_\alpha$. Then $h\big|_{A'_\alpha}$ is continuous and $A/Ker(h) \cong X$.*

We end this paper with a result describing the relationship between locally τ-Lindelöf spaces and k_τ-spaces.

Theorem 3.17. *A space X is a k_τ-space if and only if it is a quotient space of a locally τ-Lindelöf space.*

Proof. ($\Rightarrow$) Assume that X is a k_τ-space. By Theorem 3.16, X is a quotient space of the free union of its τ-Lindelöf subspaces. On the other hand, it is clear that the free union of τ-Lindelöf spaces is locally τ-Lindelöf, as desired.
($\Leftarrow$) Let $f : Y \to X$ be the identification map, where Y is locally τ-Lindelöf. Let $U \subseteq X$ be such that $U \cap C$ is open in C for each τ-Lindelöf C. We claim that U is open in X. For each relatively τ-Lindelöf open set $V \subseteq Y$, we have $U \cap f(\overline{V})$ is open in the τ-Lindelöf $f(\overline{V})$, that is, $U \cap f(\overline{V}) = f(\overline{V}) \cap G$, where G is open in X. Since $f^{-1}(U) \cap f^{-1}f(\overline{V}) = f^{-1}f(\overline{V}) \cap f^{-1}(G)$, it follows $f^{-1}(U) \cap V$ is open in Y. Since there is a covering $Y := \bigcup_\alpha V_\alpha$ consisting of relatively τ-Lindelöf open sets, the formula $f^{-1}(U) = \bigcup_\alpha f^{-1}(U) \cap V_\alpha$ shows that $f^{-1}(U)$ is open in Y, and so U is open in X. □

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